# Understanding Variance Estimator Bias in Stratified Two-Stage Sampling

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# Outline

1. Motivation
2. Overview of Current Population Survey (CPS)
3. Problem description
4. CPS variance estimation
5. Simulation results

# Motivation

- When estimating response rate $p$ and $var(p)$ for households in CPS **non-self-representing (NSR)** primary sampling units (PSU), we observed unusually high value of $var(p)$.
- We wanted to better understand the cause of this result.

# Current Population Survey

- One of the oldest surveys in the U.S. (in operation since 1942)
- Measuring national **unemployment rate**
- Monthly sample of ~72,000 households

# Primary Sampling Unit

- PSU - either a county or group of contiguous counties
- Two types of PSU:
  - Self-representing SR
  - Non-self-representing NSR

# CPS Sample Design

- Two-stage stratified sampling design for NSR PSUs:
  - **First stage**: select one PSU per stratum with probability proportional to size (civilian noninstitutionalized population 16+ = CNP 16+)
  - **Second stage**: do systematic sampling within selected PSUs
- Systematic sampling for SR PSUs

# CPS Sample Design

- Select PSUs once every 10 years
- 852 PSUs selected in first-stage (2010 design): 506 SR and 346 NSR
- Approximately 80% of CNP 16+ population in SR PSUs

| | Est. CNP 16+ (Sep 2017) |
|---|---|
| **SR** | 200,298,742 |
| **NSR** | 55,263,673 |

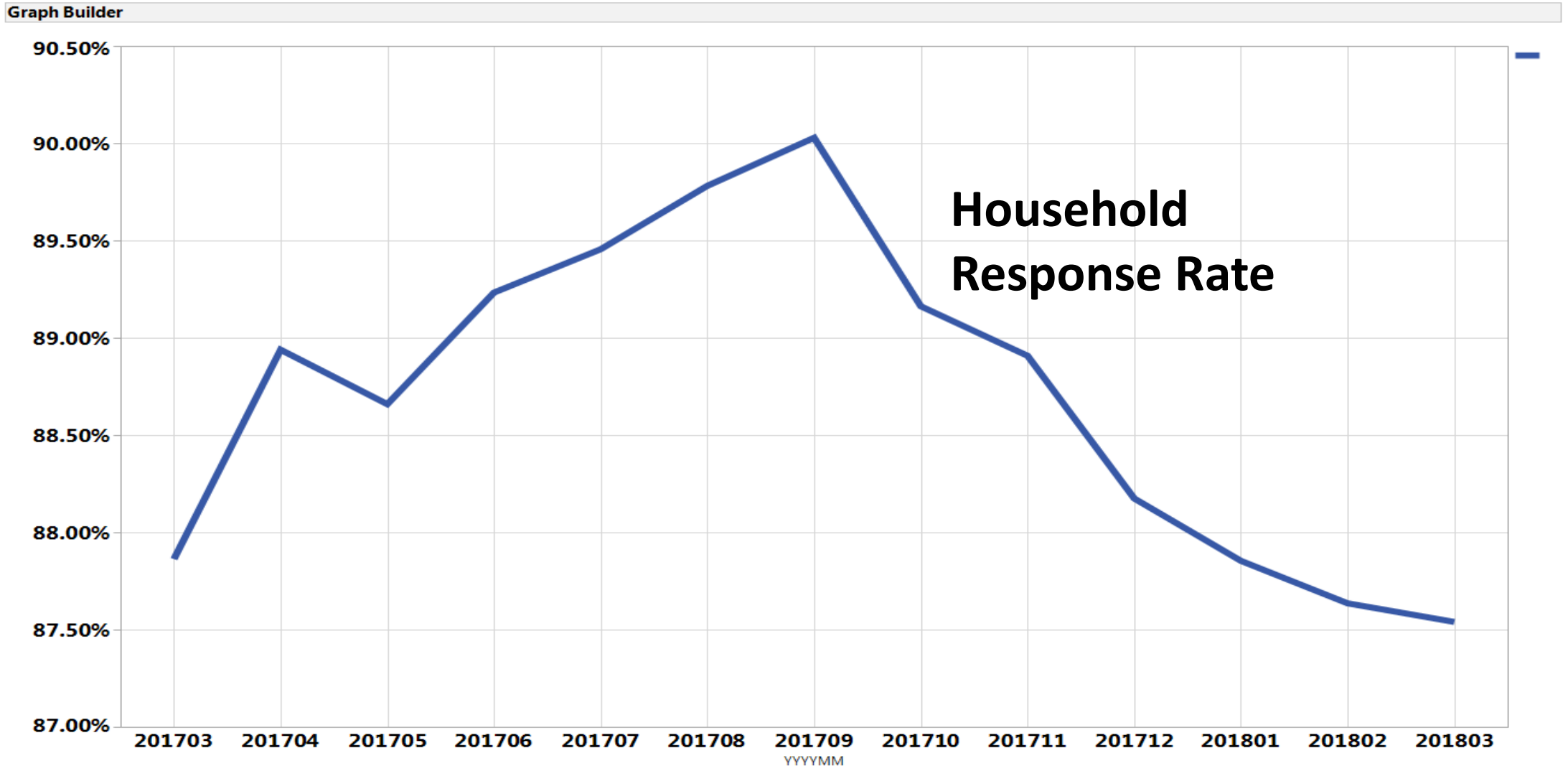
# Key Labor Force Estimates

- Noninstitutionalized civilian labor force statistics:
  - Unemployment/employment levels
  - Unemployment rate
  - Labor force participation rate

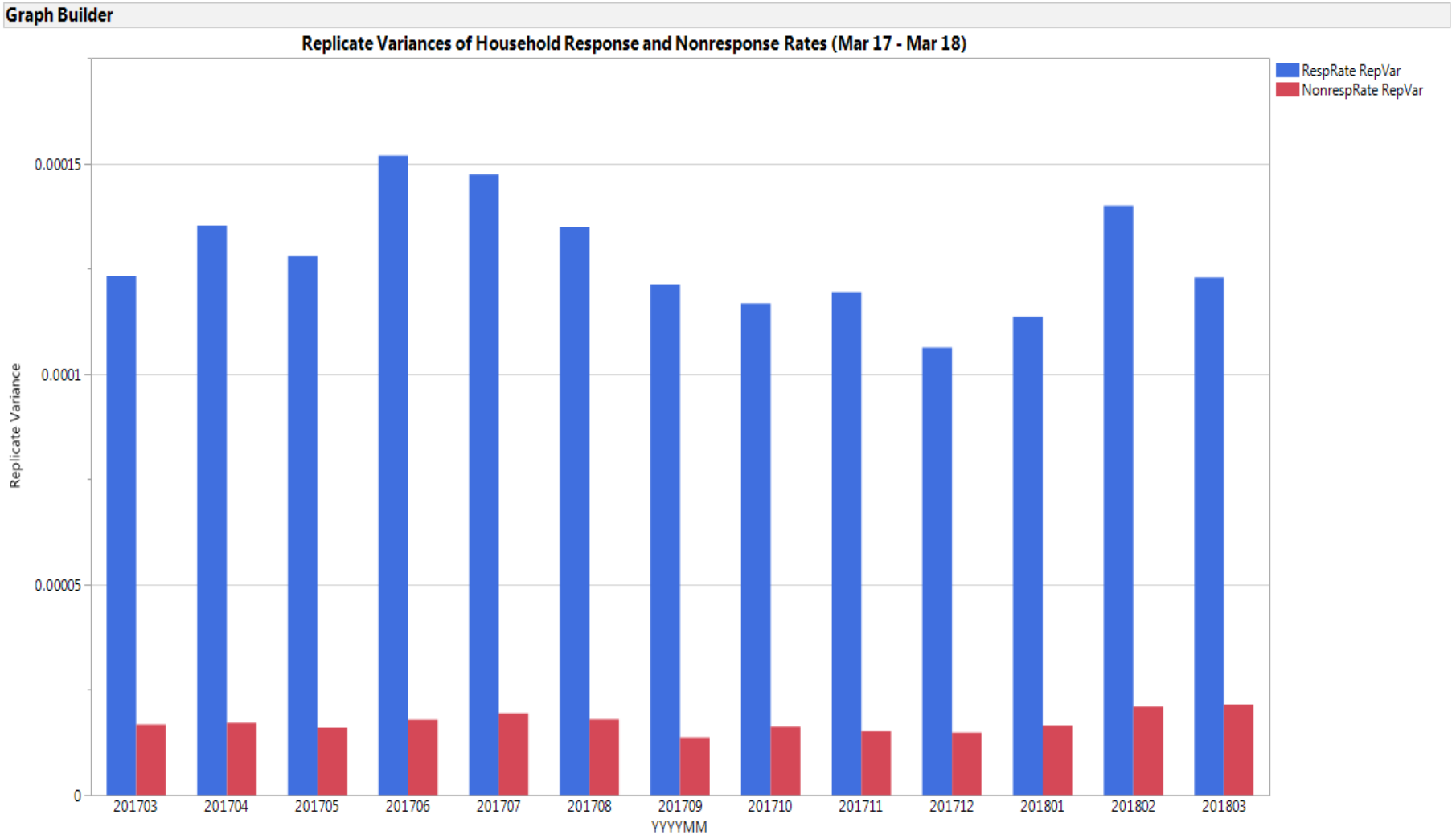
# Problem

- Estimate monthly response rate $p$, variance $var(p)$ for CPS households **in NSR PSUs**.
- The sample is at household level: 1 record for each sampled household in each month.
- Response $y_i$ has binary outcome: 1--response and 0--nonresponse.
- Time period: March 17 – March 18

# NSR Household Response Rates March 17 – March 18



# Estimated Variance for Response and Nonresponse Rates Mar 17 – Mar 18




- **Expect to see $var(p) = var(1-p)$, but they are NOT.**
- **Our chosen variance estimator introduces bias in some way.**

# CPS Variance Estimation

- Due to CPS sample design, there is no direct variance estimator formula:
  - Select only **one** PSU per NSR stratum
  - **Systematic sampling** within PSU
- Currently use **balanced-repeated replication (BRR)** method for **NSR PSUs**.

# CPS Variance Estimation

- BRR variance estimator:

$$var(\hat{Y}) = \frac{1}{R(1-K)^2} \sum_{r=1}^{R} (\hat{Y}_r - \hat{Y})^2$$

where

$\hat{Y}_r$ = the $r$-th replicate estimate of $Y$

$\hat{Y}$ = the full sample estimate of $Y$

$R$ = number of replicates

$K$ = perturbation factor; $0 \leq K < 1$

- BRR requires selecting **two** PSUs per stratum, but CPS selects only one PSU per stratum → collapse strata to make pseudo-strata.
- These pseudo-strata should **ideally contain exactly 2 perfectly matched strata**.

# BRR with Pseudo-Strata

- Suppose we want to estimate a population total $Y$ using

$$\hat{Y} = \sum_{h=1}^{L} \hat{Y}_h$$

where $L$ denotes the number of strata.

- Consider the simple case when $L$ is even, we estimate the variance of $\hat{Y}$ by combining the $L$ strata into $G$ groups of two strata each ($L = 2G$).

# BRR with Pseudo-Strata

- Hence,

$$\hat{Y} = \sum_{g=1}^{G} \hat{Y}_g = \sum_{g=1}^{G} (\hat{Y}_{g1} + \hat{Y}_{g2})$$

$$\boldsymbol{Var(\hat{Y}) = \sum_{g=1}^{G} Var(\hat{Y}_{g1}) + Var(\hat{Y}_{g2}) = \sum_{g=1}^{G} (\sigma_{g1}^2 + \sigma_{g2}^2)}$$

# BRR with Pseudo-Strata

- The $r$-th replicate estimate of $Y$:

$$\hat{Y}_r = \sum_{g=1}^{G} (1 + (1 - K)\delta_{gr})\,\hat{Y}_{g1} + (1 - (1 - K)\delta_{gr})\hat{Y}_{g2}$$

where $\delta_{gr} = 1$ if the **first** stratum in $g$-th group is selected and $\delta_{gr} = -1$ if the **second** stratum in $g$-th group is selected.

- $\delta_{gr}$ are chosen from entries of a Hadamard matrix.
- Rows of a Hadamard matrix are mutually orthogonal:

$$\sum_{r=1}^{R} \delta_{gr}\delta_{kr} = 0\ (\forall\ g \neq k)$$

# BRR with Pseudo-Strata

$$\hat{Y}_r - \hat{Y} = \sum_{g=1}^{G} (1 - K)\, \delta_{gr} \left( \hat{Y}_{g1} - \hat{Y}_{g2} \right)$$

$$(\hat{Y}_r - \hat{Y})^2 = \sum_{g=1}^{G} (1 - K)^2\, \delta_{gr}^2 \left( \hat{Y}_{g1} - \hat{Y}_{g2} \right)^2 + \sum_{g=1}^{G} \sum_{k \neq g}^{G} (1 - K)^2 \delta_{gr} \delta_{kr} \left( \hat{Y}_{g1} - \hat{Y}_{g2} \right) \left( \hat{Y}_{k1} - \hat{Y}_{k2} \right)$$

# BRR with Pseudo-Strata

$$\frac{1}{R(1-K)^2}\sum_{r=1}^{R}(\hat{Y}_r-\hat{Y})^2=\frac{1}{R(1-K)^2}\sum_{r=1}^{R}\sum_{g=1}^{G}(1-K)^2\left(\hat{Y}_{g1}-\hat{Y}_{g2}\right)^2$$

$$+\frac{1}{R(1-K)^2}\sum_{g=1}^{G}\sum_{k\neq g}^{G}(1-K)^2\left(\hat{Y}_{g1}-\hat{Y}_{g2}\right)\left(\hat{Y}_{k1}-\hat{Y}_{k2}\right)\sum_{r=1}^{R}\delta_{gr}\delta_{kr}$$

- Therefore,

$$var\left(\hat{Y}\right)=\sum_{g=1}^{G}(\hat{Y}_{g1}-\hat{Y}_{g2})^2=\sum_{g=1}^{G}(\hat{Y}_{g1}^2+\hat{Y}_{g2}^2-2\hat{Y}_{g1}\hat{Y}_{g2})$$

# Bias in BRR with Pseudo-Strata

- Taking expectation:

$$E\left\{\sum_{g=1}^{G}(\hat{Y}_{g1}^2+\hat{Y}_{g2}^2-2\hat{Y}_{g1}\hat{Y}_{g2})\right\} = Var(\hat{Y}_{g1}) + \mu_{g1}^2 + Var(\hat{Y}_{g2}) + \mu_{g2}^2 - 2\mu_{g1}\mu_{g2}$$

$$= \sum_{g=1}^{G}(\sigma_{g1}^2 + \sigma_{g2}^2) + \sum_{g=1}^{G}(\mu_{g1} - \mu_{g2})^2$$

$$= Var(\hat{Y}) + Bias^2$$

where $\sigma_{gh}^2 = \text{Var}\{\hat{Y}_{gh}\}$ and $\mu_{gh} = E\{\hat{Y}_{gh}\}$.

- Bias squared term is positive and would ADD to variance estimate.
- Bias squared term would be zero if the pair of PSUs in each group were perfectly matched.

# How are Strata Collapsed ?

- In CPS, the objective function is a function of:
  - Unemployment
  - Civilian labor force
  - Children 0-17 at or below $200\%$ poverty level

# Simulation Overview

- Use one month CPS data (Mar 18) which has pseudo-strata information.
- For each household, generate $y_i$ responses iid from Bernoulli distribution with various $p = 0.03, 0.06, \ldots, 0.99$.
- For each $p$:
  - Run 5,000 sims.
  - Compute true variance and BRR variance.
  - Compute bias squared term $\sum_{g=1}^{G}(\mu_{g1} - \mu_{g2})^2$
  - Compare true variance with BRR variance after adjusting for bias.

# Simulation Computation

- Total number of households: $\widehat{N} = \sum_{i=1}^{n} w_i$
- Full sample estimated response count: $\hat{Y} = \sum_{i=1}^{n} w_i y_i$
- Replicate $r$ estimated response count: $\hat{Y}_r = \sum_{i=1}^{n} w_i y_i f_{ir}$ where $f_{ir}$ is either 1.5 or 0.5.

# Simulation Computation

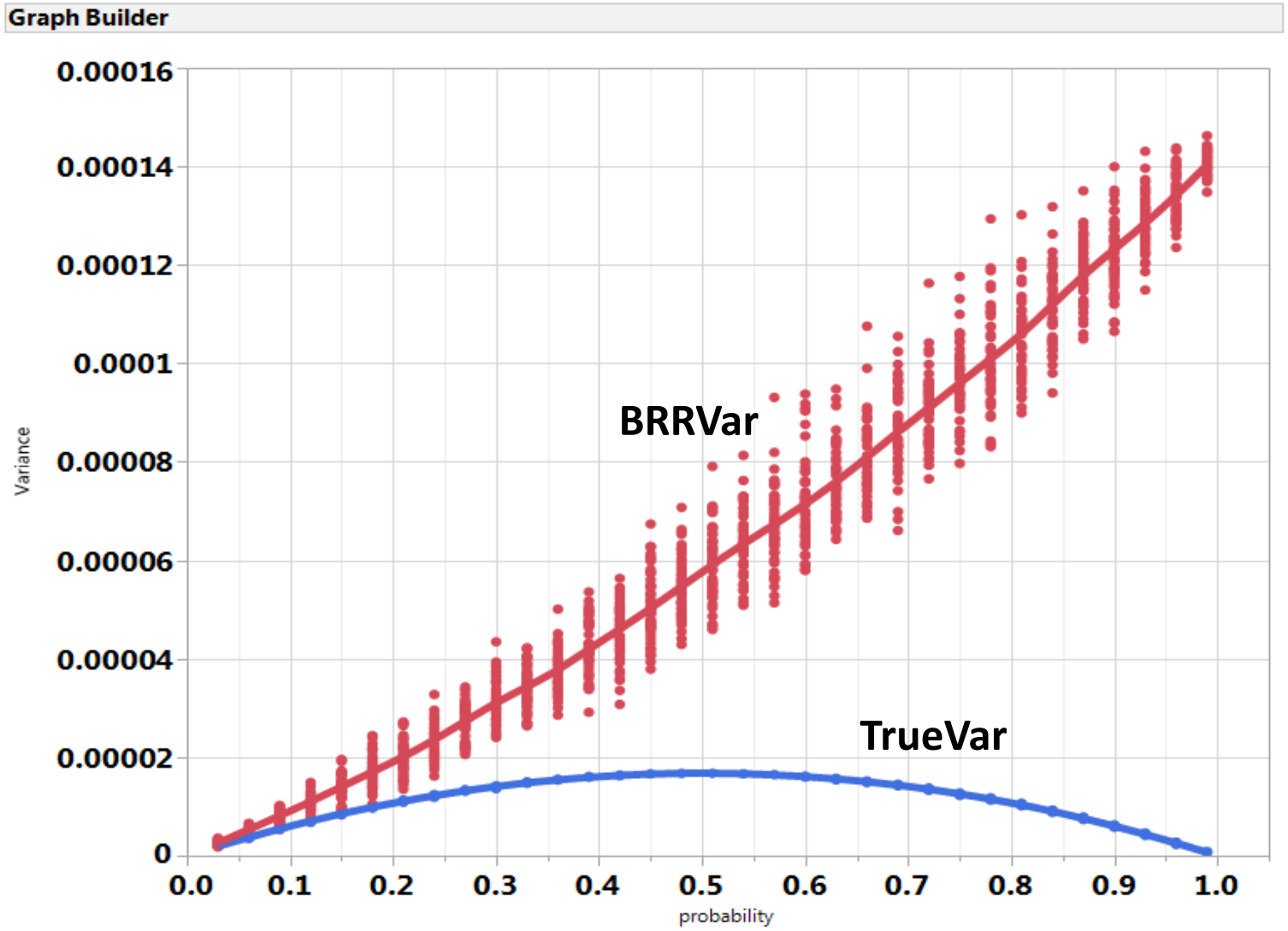
- BRR variance of $\hat{Y}$: $BRRVar(\hat{Y}) = \frac{4}{160}\sum_{r=1}^{160}(\hat{Y}_r - \hat{Y})^2$
- BRR variance of response rate $p$:

$$BRRVar(p) = \left(\frac{1}{\widehat{N}}\right)^2 BRRVar(\hat{Y})$$

assuming $N = \widehat{N}$ is fixed from outside knowledge.

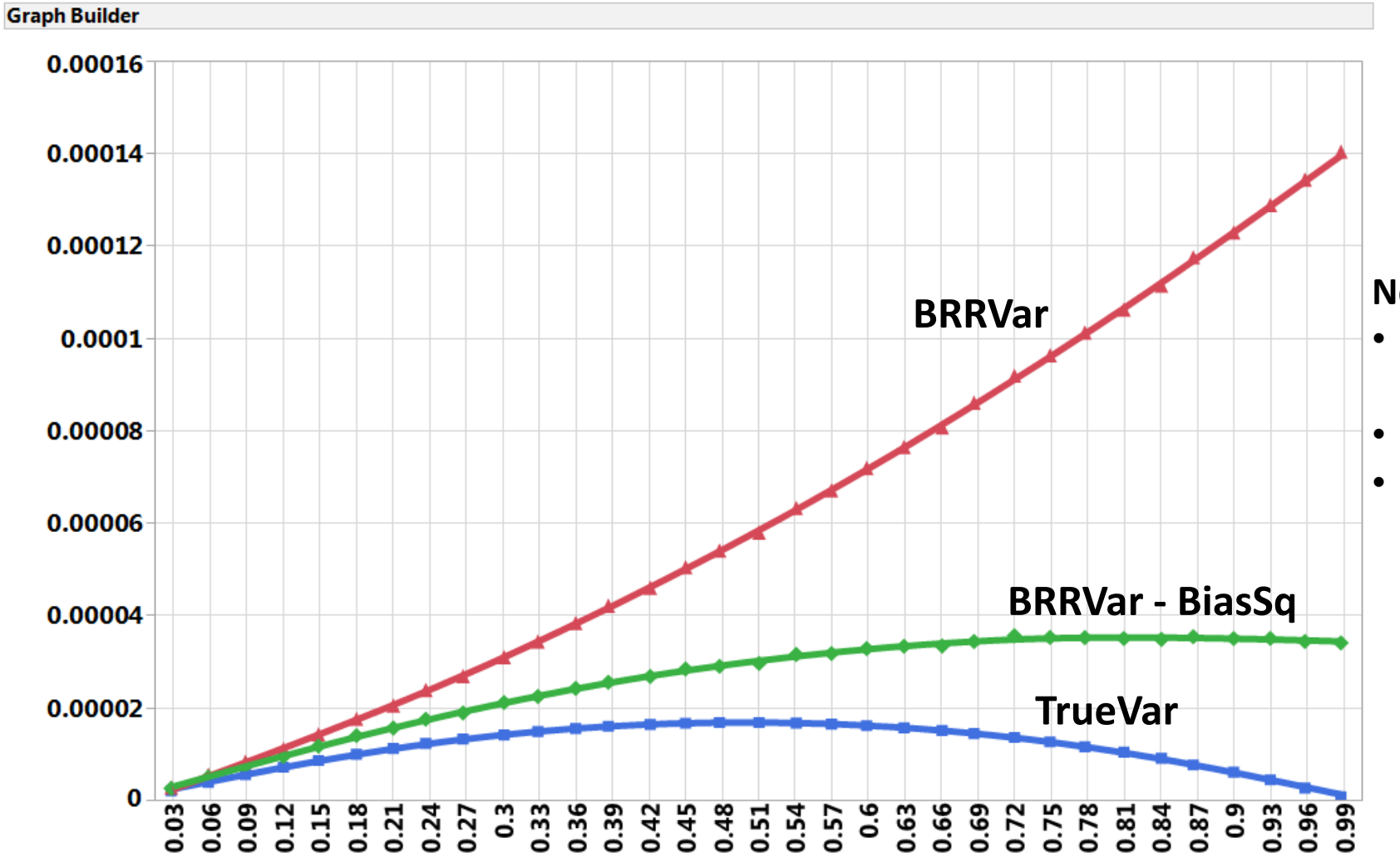
- $\mu_{gh} = p \times N_{gh}$

# BRR Variance vs. True Variance




As $p$ gets close to 1, BRR variance estimate is far different from true variance.

# BRR Variance vs. True Variance




**Not all of bias can be explained due to:**

- **Strata collapsed based on different set of covariates**
- **Use AHS MOS instead of CPS**
- **AHS MOS not currently updated (from 2010 design)**

# Summary

- For NSR component:
  - CPS collapses strata to make pseudo-strata.
  - There is no perfect matching of strata → bias in variance estimator.
  - Bias gets significantly large when $p$ gets close to 1.
  - Quick fix is to use $var(1-p)$ for large $p$.
- CPS is designed for civilian labor force statistics. Expect more bias when estimating variance of other statistics.

# Questions?

# Thank You!

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