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Inference About a Common Mean Vector from Several Independent Multinormal Populations with Unequal and Unknown Dispersion Matrices

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Inference about a common mean vector from several independent multinormal populations with unequal and unknown dispersion matrices

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Abstract

In this paper we consider the problem of drawing inference about a common mean vector based on data from several independent multivariate normal populations with unknown and unequal dispersion matrices. An unbiased estimate of the common mean vector with its asymptotic estimated variance is suggested to test a hypothesis about it and also to construct a confidence ellipsoid. Both are valid in large samples. Another approximate method based on the notion of generalized P-value is also mentioned. Exact test procedures and construction of exact confidence sets for the common mean vector are presented. A comparison of the exact tests based on their local power is carried out. Applications include a simulated data set and also data from Current Population Survey (CPS) Annual Social and Economic Supplement (ASEC) 2021, conducted by the US Bureau of the Census for the Bureau of Labor Statistics.

Keywords: Common Mean Vector; Confidence Set; Exact Test; Local Power; Meta-Analysis

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1. Introduction

The inferential problem of drawing inference about a common mean vector $\boldsymbol{\mu}$ of several independent normal populations with unequal and unknown dispersion matrices is considered in this paper. We treat the problems of 1) point estimation of $\boldsymbol{\mu}$, 2) test for $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ versus $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$, and 3) construction of confidence sets for $\boldsymbol{\mu}$.

Suppose there are $k(k \geq 2)$ p-variate normal populations with common mean vector $\boldsymbol{\mu}$ and unknown covariance matrices $\boldsymbol{\Sigma}_1, \ldots, \boldsymbol{\Sigma}_k$. Let $\boldsymbol{X}_{i1}, \ldots, \boldsymbol{X}_{in_i}$ be independent p-variate vector sample observations from the i^{th} population ($i = 1, \ldots, k$), and $\boldsymbol{X}_{ij} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}_i)$, $j = 1, \ldots, n_i$. For the i^{th} population, let

$$\bar{\boldsymbol{X}}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} \boldsymbol{X}_{ij} \quad \text{and} \quad \boldsymbol{S}_i = \sum_{j=1}^{n_i} (\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_i)(\boldsymbol{X}_{ij} - \bar{\boldsymbol{X}}_i)^t \tag{1}$$

be the sample mean vector and sample sum of squares and products matrix. Jointly, $\{\bar{\boldsymbol{X}}_i, \boldsymbol{S}_i, i = 1, \cdots, k\}$ provide minimal sufficient statistics for the unknown parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}_i$ $(i = 1, \cdots, k)$. It is well known that one can use the familiar Hotelling's T^2 test for H_0 and reject the null based on the i^{th} data set when $T_i^2 = n_i(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0)^t \boldsymbol{S}_i^{-1}(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0)$ is large. A confidence set for $\boldsymbol{\mu}$ based on the i^{th} data set is also readily obtained as $Pr\{\boldsymbol{\mu} : [\frac{n_i(n_i-p)}{p}](\bar{\boldsymbol{X}}_i - \boldsymbol{\mu})^t \boldsymbol{S}_i^{-1}(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}) \leq F_{\alpha,p,n_i-p}\} = 1 - \alpha, i = 1, \ldots, k.$

In Section 2 we provide an unbiased estimate of $\boldsymbol{\mu}$ based on the minimal sufficient statistics and provide an expression for its estimated asymptotic variance. A test procedure for H_0 and a confidence ellipsoid for $\boldsymbol{\mu}$ then readily follow. These results are asymptotic in nature.

An approximate procedure for test as well as confidence set for $\boldsymbol{\mu}$ in our context was suggested by Lin et al. (2007) based on the notion of generalized P-values (Tsui and Weerahandi, 1989; Weerahandi, 1993, 2003). This is briefly mentioned in Section 3. The authors clearly presented relevant algorithms to carry out the suggested procedures and also discussed their performance in terms of coverage probabilities and expected volumes in comparison with some existing methods. Notwithstanding their claim of anticipated better performance over existing procedures, the fact remains the generalized P-value based procedure is indeed approximate and may not work well in some situations in view of the unknown and unequal nature of the population dispersion matrices (see Tables 5 and 6 in their paper).

Exact tests for H_0 and exact confidence sets for $\boldsymbol{\mu}$ can be derived by efficiently combining the k independent Hotelling's T^2 statistics. Fortunately, several well-known exact procedures exist in the literature (Jordan and Krishnamoorthy, 1995; Hartung et al., 2008; Kifle et al., 2021). We provide in Section 4 a review of these exact test procedures and in Section 5 a review of exact confidence sets. Local powers of the exact tests are discussed in Section 6 based on a Taylor expansion of the power and these are compared in Section 7.

In Section 8, two applications are provided. First, we reproduce a data set from Jordan and Krishnamoorthy (1995) for $p = 2$, $k = 2$, $n_1 = n_2 = 12$, each from $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma_1})$, $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma_2})$, and use it to construct a confidence set for the bivariate common mean vector based on the above exact procedures. Plots showing the confidence sets appear in Figure 3. Our second example is based on data arising from Current Population Survey (CPS) conducted by the Bureau of the Census for the Bureau of Labor Statistics. It turns out that the sample sizes are large for the CPS data, thus enabling us to also include the large sample procedure described in Section 2. Plots showing the confidence sets appear in Figures 4 - 6. For the sake of completeness we have also plotted the confidence set derived from the generalized P-value based method (Lin et al., 2007) in both the applications. We conclude the paper with some conclusions in Section 9.

2. A large sample procedure

In this section we propose an unbiased estimate of $\boldsymbol{\mu}$ which is essentially a generalization of the familiar Graybill-Deal estimate (Graybill and Deal, 1959) of μ in case of univariate normal populations. This estimate and its estimated asymptotic variance in the univariate case are given by

$$\hat{\mu}_{GD} = \Big[\sum_{i=1}^{k} \frac{n_i}{S_i^2}\Big]^{-1} \Big[\sum_{i=1}^{k} \frac{n_i}{S_i^2}\bar{X}_i\Big] \quad \text{with} \quad V\hat{a}r(\hat{\mu}_{GD}) = \Big[\sum_{i=1}^{k} \frac{n_i}{S_i^2}\Big]^{-1} \tag{2}$$

A test for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ is based on the standard normal Z statistic defined as $Z = (\hat{\mu}_{GD} - \mu_0)/\sqrt{\widehat{Var}(\hat{\mu}_{GD})}$. An asymptotic confidence interval for μ is given by $\hat{\mu}_{GD} \mp Z_{\alpha/2}\sqrt{\widehat{Var}(\hat{\mu}_{GD})}$.

As a generalization to the multivariate case, we propose

$$\tilde{\boldsymbol{\mu}}_{GD} = \Big[\sum_{i=1}^{k} n_i \boldsymbol{S}_i^{-1}\Big]^{-1} \Big[\sum_{i=1}^{k} n_i \boldsymbol{S}_i^{-1}\bar{\boldsymbol{X}}_i\Big] \quad \text{with} \quad \widehat{Var}(\tilde{\boldsymbol{\mu}}_{GD}) = \Big[\sum_{i=1}^{k} n_i \boldsymbol{S}_i^{-1}\Big]^{-1} \tag{3}$$

An asymptotic test for $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ versus $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ can then be based on the χ_p^2 statistic

$$\chi_p^2 = (\tilde{\boldsymbol{\mu}}_{GD} - \boldsymbol{\mu})^t \big[\sum_{i=1}^{k} n_i \boldsymbol{S}_i^{-1}\big](\tilde{\boldsymbol{\mu}}_{GD} - \boldsymbol{\mu}) \tag{4}$$

A $(1-\alpha)100\%$ asymptotic ellipsoidal confidence set for $\boldsymbol{\mu}$ is provided by

$$Pr\big\{(\tilde{\boldsymbol{\mu}}_{GD} - \boldsymbol{\mu})^t \big[\sum_{i=1}^{k} n_i \boldsymbol{S}_i^{-1}\big](\tilde{\boldsymbol{\mu}}_{GD} - \boldsymbol{\mu}) \leq \chi_{p,\alpha}^2\big\} = 1 - \alpha \tag{5}$$

Our simulation studies (see Appendix A) demonstrate the robustness of the χ^2 cut-off point for variations in the unknown dispersion matrices. An applications of (5) for a real ASEC dataset appear in Section 8.

3. Confidence Set Based on Generalized *P*-value

Tsui and Weerahandi (1989) came up with a novel idea to deal with *uncommon* inference problems. Examples include the ANOVA problem under variance heteroscedasticity, test for treatment variance component in a one-way random effects model, test for reliability parameter $Pr(X > Y) = 1 - \Phi\big[\frac{\mu_y - \mu_x}{\sqrt{\sigma_x^2 + \sigma_y^2}}\big]$ when $X \sim N(\mu_x, \sigma_x^2)$, independent of $Y \sim N(\mu_y, \sigma_y^2)$, and so on. The method is based on a function $h(X; x, \theta, \eta)$ of underlying random variable X, its observed value x, θ, the parameter of interest, while η is a nuisance parameter. Under certain conditions on $h(\cdot)$, a test for θ and a confidence set for θ can be derived. Their method is referred to as generalized P-value based approach.

In our context, following Tsui and Weerahandi (1989), Lin et al. (2007) suggested the following algorithm to construct a confidence ellipsoid for $\boldsymbol{\mu}$. An extension of the generalized P-value method from a common univariate normal mean with unknown variances to the case of a common multivariate normal mean vector with unknown dispersion matrices is highly *nontrivial*, and the authors deserve a lot of credit to provide a solution. Starting with the basic ingredients, namely, $(n_1, \cdots, n_k; \bar{\boldsymbol{X}}_i, \ldots, \bar{\boldsymbol{X}}_k; \boldsymbol{S}_1, \ldots, \boldsymbol{S}_k)$, define $\boldsymbol{u}_i = n_i^{-1}\boldsymbol{S}_i$, $\boldsymbol{W}_i = \left[\boldsymbol{u}_i^{1/2}\tilde{\boldsymbol{R}}_i^{-1}\boldsymbol{u}_i^{1/2}\right]^{-1}$, $\boldsymbol{T}_i^* = \bar{\boldsymbol{X}}_i - \left[\boldsymbol{u}_i^{1/2}\boldsymbol{R}_i^{-1}\boldsymbol{u}_i^{1/2}\right]^{1/2}\boldsymbol{Z}_i$, and $\boldsymbol{W} = \sum_{i=1}^{k}\boldsymbol{W}_i$.

For $j = 1, \ldots, m$:

Generate $\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_k$ from $N_p(\boldsymbol{0}, \boldsymbol{I}_p)$.

Generate independent $\boldsymbol{R}_i$ and $\tilde{\boldsymbol{R}}_i$ from $W_p(n_i - 1, \boldsymbol{I}_p)$, $i = 1, \ldots, k$.

Compute $\boldsymbol{W}_1, \ldots, \boldsymbol{W}_k$ and $\boldsymbol{W}$.

Compute $\boldsymbol{T}_j = \boldsymbol{W}^{-1}\sum_{i=1}^{k}\boldsymbol{W}_i\boldsymbol{T}_i^*$.

(End j loop)

Compute $\hat{\boldsymbol{\mu}}_T = 1/m\sum_{j=1}^{m}\boldsymbol{T}_j$ and $\hat{\boldsymbol{\Sigma}}_T = 1/(m-1)\sum_{j=1}^{m}(\boldsymbol{T}_j - \hat{\boldsymbol{\mu}}_T)(\boldsymbol{T}_j - \hat{\boldsymbol{\mu}}_T)^t$.

Compute $||\tilde{\tilde{\boldsymbol{T}}}_j||$, where $\tilde{\tilde{\boldsymbol{T}}}_j = \hat{\boldsymbol{\Sigma}}_T^{-1/2}(\boldsymbol{T}_j - \hat{\boldsymbol{\mu}}_T)$, $j = 1, \ldots, m$.

Let $q_{\{||\tilde{\tilde{\boldsymbol{T}}}||;1-\alpha\}}$ be the $100(1-\alpha)^{th}$ percentile of $||\tilde{\tilde{\boldsymbol{T}}}_j||$, $j = 1, \ldots, m$, then the confidence ellipsoid of $\boldsymbol{\mu}$ can be obtained from the inequality

$$\left\{\boldsymbol{\mu} : (\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_T)^t\hat{\boldsymbol{\Sigma}}_T^{-1}(\boldsymbol{\mu} - \hat{\boldsymbol{\mu}}_T) \leq q^2_{\{||\tilde{\tilde{\boldsymbol{T}}}||;1-\alpha\}}\right\}. \tag{6}$$

We have used equation 6 in Section 8.

4. Exact tests for $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$

To develop the exact test for testing $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ versus $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ based on all the data sets, we proceed as follows. Recall that $T^2 = n(\bar{\boldsymbol{X}} - \boldsymbol{\mu})^t\boldsymbol{S}^{-1}(\bar{\boldsymbol{X}} - \boldsymbol{\mu})$ satisfies $T^2 = (\frac{p}{n-p})F_{p,n-p}$ where F_{ν_1,ν_2} follows an F-distribution with ν_1 and ν_2 degrees of freedom and our test procedure rejects H_0 when $F_{obs} > F_{\nu_1,\nu_2;\alpha}$, α being Type I error level and F_{obs} being the observed value of F under $\mu = \mu_0$. A test for H_0 based on a

P-value on the other hand is based on $P_{obs} = P[F_{\nu_1,\nu_2} > F_{obs}]$ and we reject H_0 at level α if $P_{obs} < \alpha$. It is easy to check that the two approaches are obviously equivalent.

A random P-value which has a Uniform(0,1) distribution under the null hypothesis is defined as $P_{ran} = P[F_{\nu_1,\nu_2} > F_{ran}]$, where $F_{ran} = \left[\frac{n(n-p)}{p}\right](\bar{\boldsymbol{X}} - \boldsymbol{\mu_0})^t \boldsymbol{S}^{-1}(\bar{\boldsymbol{X}} - \boldsymbol{\mu_0})$. All suggested exact tests for H_0 are based on P_{obs} and F_{obs} values, and their properties, including size and power, are studied under P_{ran} and F_{ran}. To simplify notations, we will denote P_{obs} by small p and P_{ran} by large P. Four exact tests based on p values and one exact test based on F_{obs} as available in the literature are listed below.

4.1. Tippett's test

This minimum P-value test was proposed by Tippett et al. (1931), who noted that, if $P_1, \cdots, P_k$ are independent p-values from continuous test statistics, then each has a uniform distribution under H_0. According to this method, the null hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu_0}$ is rejected at α level of significance if $P_{(1)} < \left[1 - (1-\alpha)^{\frac{1}{k}}\right]$ where $P_{(1)} = min\{P_1, \cdots, P_k\}$. Incidentally, this test is equivalent to the test based on $M_t = max_{1 \le i \le k}\{T_i^2\}$ suggested by Cohen and Sackrowitz (1984).

4.2. Wilkinson's test

This test statistic proposed by Wilkinson (1951) is a generalization of Tippett's test that uses the r^{th} smallest p-value ($P_{(r)}$) as a test statistic. The null hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu_0}$ will be rejected if $P_{(r)} < d_{r,\alpha}$, where $P_{(r)}$ follows a *Beta* distribution with parameters r and $(k-r+1)$ under H_0 and $d_{r,\alpha}$ satisfies $Pr\{P_{(r)} < d_{r,\alpha}|H_0\} = \alpha$. Obviously, this procedure generates a sequence of tests for different values of $r = 1, 2, \cdots, k$, and an attempt has been made to identify the best choice of r [Table 2].

4.3. Inverse normal test

This exact test procedure which involves transforming each p-value to the corresponding normal score was proposed independently by Stouffer et al. (1949) and Lipták (1958). Using this inverse normal method, the null hypothesis H_0 will be rejected at α

level of significance if $\left[\sum_{i=1}^{k}\Phi^{-1}(P_i)\right]\left[\sqrt{k}\right]^{-1} < -z_\alpha$, where Φ^{-1} denotes the inverse of the cdf of a standard normal distribution and z_α stands for the upper α level cutoff point of a standard normal distribution.

4.4. *Fisher's inverse χ^2-test*

This inverse χ^2-test is one of the most widely used exact test procedures for combining k independent p-values (Fisher, 1932). This procedure uses the $\prod_{i=1}^{k} P_i$ to combine the k independent p-values. Then, using the connection between uniform and χ^2 distributions, the null hypothesis H_0 is rejected if $-2\sum_{i=1}^{k}\ln(P_i) > \chi^2_{2k,\alpha}$, where $\chi^2_{2k,\alpha}$ denotes the upper α critical value of a χ^2-distribution with $2k$ degrees of freedom.

4.5. *Jordan-Kris test*

Jordan and Krishnamoorthy (1995) considered a weighted linear combination of the Hotelling's T^2 statistic, namely $T = \sum_{i=1}^{k} C_i T_i^2$, where $T_i^2 = n_i(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0)^t \boldsymbol{S}_i^{-1}(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0)$, and $C_i = \frac{[Var(T_i^2)]^{-1}}{\sum_{j=1}^{k}[Var(T_j^2)]^{-1}}$ with $Var(T_i^2) = \frac{2pm_i^2(m_i-1)}{(m_i-p-1)^2(m_i-p-3)}, m_i = n_i - 1, n_i > p+4, \forall_i, i = 1,\cdots,k$. The null hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu_0}$ will be rejected if $T > a$, where $Pr\{T > a|H_0\} = \alpha$. In applications a is computed by using the approximation $T \approx dF_{kp,\nu}$, where $\nu = \frac{4M_2kp - 2M_1^2(kp+2)}{M_2kp - M_1^2(kp+2)}$, $d = M_1(\frac{\nu-2}{\nu})$, $M_1 = p\sum_{i=1}^{k}\frac{C_i m_i}{m_i-p-1}$, and $M_2 = p(p+2)\sum_{i=1}^{k}\frac{C_i^2 m_i^2}{(m_i-p-1)(m_i-p-3)} + 2p^2\sum_{i>j}\frac{C_iC_jm_im_j}{(m_i-p-1)(m_j-p-1)}$.

5. Exact confidence sets for $\boldsymbol{\mu}$

In this section we present some exact confidence sets for $\boldsymbol{\mu}$, essentially based upon inverting the acceptance sets resulting from the discussion in Section 4.

5.1. *Confidence set based on Jordan-Kris method*

Following the method proposed in Jordan and Krishnamoorthy (1995) which is presented in section 4.5, a $100(1-\alpha)\%$ confidence ellipsoid for $\boldsymbol{\mu}$ is a set of values $\boldsymbol{\mu}$ satisfying the following inequality.

$$(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}})^t\boldsymbol{V}(\boldsymbol{\mu}-\hat{\boldsymbol{\mu}}) \le a - \sum_{i=1}^{k}\left(\sum_{j\neq i}^{k}(\bar{\boldsymbol{X}}_i-\bar{\boldsymbol{X}}_j)^t\boldsymbol{W}_j^{-1}\right)\boldsymbol{V}^{-1}\boldsymbol{W}_i^{-1}\boldsymbol{V}^{-1}\left(\sum_{j\neq i}^{k}\boldsymbol{W}_j^{-1}(\bar{\boldsymbol{X}}_i-\bar{\boldsymbol{X}}_j)\right) \quad (7)$$

where $\hat{\boldsymbol{\mu}} = \boldsymbol{V}^{-1}\sum_{j=1}^{k}\boldsymbol{W}_j^{-1}\bar{\boldsymbol{X}}_j$, $\boldsymbol{W}_i^{-1} = c_in_i\boldsymbol{S}_i^{-1}$, and $\boldsymbol{V} = \sum_{i=1}^{k}\boldsymbol{W}_i^{-1}$.

5.2. *P-value based confidence sets*

All the P-value based confidence sets are obtained by inverting the corresponding acceptance sets and here are the results. We refer to Yu et al. (1999) for results in case of univariate normals. We define $P_i(\boldsymbol{\mu}) = Pr\{F_{p,n_i-p} > \left[\frac{n_i(n_i-p)}{p}\right](\bar{\boldsymbol{X}}_i - \boldsymbol{\mu})^t \boldsymbol{S}_i^{-1}(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu})\}, i = 1, \ldots, k$.

5.2.1. *Confidence set based on Tippett's method*

A $100(1-\alpha)\%$ Tippett's confidence set for $\boldsymbol{\mu}$ is a set of values $\boldsymbol{\mu}$ satisfying $\{\boldsymbol{\mu} : P_{(1)}(\boldsymbol{\mu}) > 1 - [1-\alpha]^{1/k}\}$.

5.2.2. *Confidence set based on Wilkinson's method*

A $100(1-\alpha)\%$ Wilkinson's (order r) confidence set for $\boldsymbol{\mu}$ is a set of values $\boldsymbol{\mu}$ satisfying $\{\boldsymbol{\mu} : P_{(r)}(\boldsymbol{\mu}) > d_{r,\alpha}\}$.

5.2.3. *Confidence set based on INN method*

A $100(1-\alpha)\%$ confidence set for $\boldsymbol{\mu}$ based on INN is a set of values $\boldsymbol{\mu}$ satisfying $\{\boldsymbol{\mu} : \sum_{i=1}^{k} \frac{\Phi^{-1}(P_i(\boldsymbol{\mu}))}{\sqrt{k}} > -Z_\alpha\}$.

5.2.4. *Confidence set based on Fisher's method*

A $100(1-\alpha)\%$ confidence set for $\boldsymbol{\mu}$ based on Fisher's inverse χ^2-test is a set of values $\boldsymbol{\mu}$ satisfying $\{\boldsymbol{\mu} : -2\sum_{i=1}^{k} \ln(P_i(\boldsymbol{\mu})) < \chi^2_{2k,\alpha}\}$.

<u>*Remark:*</u> Unlike the large sample based confidence ellipsoid presented in Section 2, the generalized *P*-value based confidence ellipsoid presented in Section 3 and Jordan-Kris confidence ellipsoid presented in Section 4, the P-value based confidence sets described above may not always lead to confidence ellipsoids! In case of univariate normals, Yu et al. (1999) derived sufficient conditions which will guarantee ellipsoid shapes. Similar sufficient conditions can be derived in case of multinormal populations, but we have not pursued it here.

6. Expressions of local powers of proposed exact tests

In this section we provide the expressions of local powers of the suggested exact tests. A common premise is that we derive an expression of the power of a test under $\Delta^2 > 0$, and carry out its Taylor expansion around $\Delta^2 = 0$, where $\Delta^2 = n(\boldsymbol{\mu} - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \boldsymbol{\mu}_0)$, and retain terms of order $O(\Delta^2)$.

The pdfs of F statistic under the null and alternative hypotheses which will be required in the sequel are given below. $\Delta^2 = n(\boldsymbol{\mu_1} - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu_1} - \boldsymbol{\mu}_0)$ below stands for the non-centrality parameter when $\boldsymbol{\mu_1}$ is chosen as an alternative value.

$$f(F_{\nu_1,\nu_2}) = \frac{\left[\frac{\nu_1}{\nu_2}\right]^{\nu_1/2}}{B(\frac{\nu_1}{2}, \frac{\nu_2}{2})} F^{\nu_1/2-1}\left[1 + \frac{\nu_1}{\nu_2}F\right]^{-(\nu_1+\nu_2)/2} \tag{8}$$

$$f_{\Delta^2}(F_{\nu_1,\nu_2}) = \sum_{k=0}^{\infty} \frac{e^{-\Delta^2/2}[\frac{\Delta^2}{2}]^k}{B(\frac{\nu_2}{2}, \frac{\nu_1}{2} + k)k!}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}+k}\left[\frac{\nu_2}{\nu_2 + \nu_1 F}\right]^{\frac{\nu_1+\nu_2}{2}+k} F^{\nu_1/2-1+k} \tag{9}$$

$$\begin{aligned} f_{\Delta^2}(F_{\nu_1,\nu_2}) &\approx f_{\Delta^2=0}(F_{\nu_1,\nu_2}) + \frac{\Delta^2}{2} f_{\Delta^2=0}(F)\left[\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\right] \\ &= f_{\Delta^2=0}(F_{\nu_1,\nu_2})\left[1 + \frac{\Delta^2}{2}\left\{\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\right\}\right] \end{aligned} \tag{10}$$

The final expressions of the local powers of the proposed tests are given below in the general case and also in the special case when $n_1 = \cdots = n_k = n$. For detailed proofs of all technical results we refer to the Appendix-B section of this paper.

6.1. *Local power of Tippett's test* $[LP(T)]$

$$\begin{aligned} LP(T) &\approx \alpha + (1-\alpha)^{\frac{k-1}{k}} \sum_{i=1}^{k} \frac{\Delta_i^2}{2} \xi_{F_{c_{\alpha;\nu_1,\nu_{2i}}}} \\ &= \alpha + (1-\alpha)^{\frac{k-1}{k}} \xi_{F_{c_{\alpha;\nu_1,\nu_2}}} \left\{\sum_{i=1}^{k} \frac{\Delta_i^2}{2}\right\} \qquad \text{[special case]} \end{aligned} \tag{11}$$

where $\xi_{F_{c_\alpha;\nu_1,\nu_{2i}}} = \int_{c_\alpha;\nu_1,\nu_{2i}}^{\infty} f_0(F_{\nu_1,\nu_{2i}})\left[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i}}F}\right]dF$

6.2. Local power of Wilkinson's test $[LP(W_r)]$

$$\begin{aligned} LP(W_r) &\approx \alpha + \binom{k-1}{r-1} d_{r;\alpha}^{r-1}(1-d_{r;\alpha})^{k-r}\left[\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\xi_{F_{d_{r,\alpha;\nu_1,\nu_{2i}}}}\right] && (12)\\ &= \alpha + \binom{k-1}{r-1} d_{r;\alpha}^{r-1}(1-d_{r;\alpha})^{k-r}\xi_{F_{d_{r,\alpha,\nu_1,\nu_2}}}\left\{\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\right\} && [\text{special case}] \end{aligned}$$

where $\xi_{F_{d_{r,\alpha,\nu_1,\nu_2}}}$ is equivalent to $\xi_{F_{c_\alpha;\nu_1,\nu_2}}$ with $c_\alpha = d_{r;\alpha}$.

Remark: For the special case $r = 1$, $LP(W_r) = LP(T)$, as expected, because $d_{1;\alpha} = [1-(1-\alpha)^{\frac{1}{k}}]$, implying $(1-d_{1;\alpha})^{k-1} = (1-\alpha)^{\frac{k-1}{k}}$.

6.3. Local power of Inverse Normal test $[LP(INN)]$

$$\begin{aligned} LP(INN) &\approx \alpha + \frac{\phi(z_\alpha)}{2\sqrt{k}}\sum_{i=1}^{k}\Delta_i^2\left[\frac{z_\alpha}{2\sqrt{k}}B_{\nu_{1i},\nu_{2i}} - A_{\nu_{1i},\nu_{2i}}\right] && (13)\\ &= \alpha + \frac{\phi(z_\alpha)}{\sqrt{k}}\left[\frac{z_\alpha}{2\sqrt{k}}B_{\nu_1,\nu_2} - A_{\nu_1,\nu_2}\right]\left\{\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\right\} && [\text{special case}] \end{aligned}$$

where $A_{\nu_1,\nu_2} = \int_{-\infty}^{\infty} u\phi(u)Q_{\nu_1,\nu_2}(u)du$, $Q_{\nu_1,\nu_2}(u) = \left[\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\right]_{F=F_{\Phi(u);\nu_1,\nu_2}}$, $B_{\nu_1,\nu_2} = \int_{-\infty}^{\infty} u^2\phi(u)Q^*_{\nu_1,\nu_2}(u)du$, $Q^*_{\nu_1,\nu_2}(u) = \left\{Q_{\nu_1,\nu_2}(u) - E[Q_{\nu_1,\nu_2}(u)]\right\}$, $\phi(u)$ is standard normal pdf and $\Phi(u)$ is standard normal cdf.

6.4. Local power of Fisher's test $[LP(F)]$

$$\begin{aligned} LP(F) &\approx \alpha + \sum_{i=1}^{k}\frac{\Delta_i^2}{4}D_{\nu_{1i},\nu_{2i}}\left[E\left\{\left\{\ln(T/2)\right\}I_{\{T\geq\chi^2_{2k;\alpha}\}}\right\}_{T\sim\chi^2_{2k}} - \alpha D_0\right] && (14)\\ &= \alpha + \frac{D_{\nu_1,\nu_2}}{2}\left[E\left\{\left\{\ln(T/2)\right\}I_{\{T\geq\chi^2_{2k;\alpha}\}}\right\}_{T\sim\chi^2_{2k}} - \alpha D_0\right]\left\{\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\right\} && [\text{special case}] \end{aligned}$$

where $D_0 = E\left[\log(q)\right]$; $D_{\nu_1,\nu_2} = E\left[VQ^*_{\nu_1,\nu_2}(v)\right]$; $V \sim \exp[2]$; $q \sim$ gamma[1,k]; $T \sim gamma[2,k]$; $Q_{\nu_1,\nu_2}(v) = \left[\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\right]_{F=F_{\Phi(v);\nu_1,\nu_2}}$; $Q^*_{\nu_1,\nu_2}(u) = \left\{Q_{\nu_1,\nu_2}(u) - E\left[Q_{\nu_1,\nu_2}(u)\right]\right\}$.

6.5. Local power of Jordan-Kris test $[LP(JK)]$

$$
\begin{aligned}
LP(JK) &\approx \alpha + \sum_{i=1}^{k} \frac{\Delta_i^2}{2} E_{H_0}\left[\left\{\frac{F_i - 1}{1 + \frac{p}{n_i - p}F_i}\right\} I_{\{\sum_{i=1}^{k} C_i^* F_i > ak\}}\right] \qquad (15) \\
&= \alpha + E_{H_0}\left[\left\{\frac{F_i - 1}{1 + \frac{p}{n-p}F_i}\right\} I_{\{\sum_{i=1}^{k} F_i > ak\}}\right]\left\{\sum_{i=1}^{k} \frac{\Delta_i^2}{2}\right\} \qquad \left[\text{special case}\right]
\end{aligned}
$$

where $E_{H_0}[\cdot]$ stands for expectation w.r.t $F_1, \ldots, F_k$ under $H_0[F_i \sim F(\nu_1, \nu_{2i})]$.

7. Comparison of local powers

It is interesting to observe from the above expressions that in the special case of equal sample size, local powers can be readily compared, irrespective of the values of the unknown dispersion matrices $\mathbf{\Sigma}_i$, $i = 1, \cdots, k$.

Table 1 represents values of the second term of local power in case of equal sample size n given above in (11) - (15), apart from the common term $\left[\sum_{i=1}^{k} \Delta_i^2/2\right]$, for different values of k, p, and n. A comparison of the second term of local power of Wilkinson's test for different values of r $(\leq k)$ is provided in Table 2 for $n = 15$, $k \in \{2, 3, 5, 9, 10\}$ and $p \in \{2, 3, 4\}$. All throughout we have used $\alpha = 5\%$. It turns out that the exact tests based on Inverse Normal and Jordan-Kris methods perform the best. Figures 1 and 2 present local powers of Inverse Normal and Jordan-Kris methods as a function of Δ_1 and Δ_2 for the special case of $n_1 = n_2 = 15$, $k = 2$, and $p = 2$. It also turns out from Table 2 that an optimum choice of r for Wilkinson's method is nearly $\sqrt{k}$.

Table 1: Comparison of the 2^{nd} term of local powers [without $\sum_{i=1}^{k} \Delta_i^2/2$] of five exact tests for different values of k , p and n (equal sample size)

Exact Test	k=2								
	n=15			n=25			n=40		
	p=2	p=3	p=4	p=2	p=3	p=4	p=2	p=3	p=4
Tippett	0.0693	0.0490	0.0375	0.0777	0.0571	0.0455	0.0825	0.0618	0.0502
Wilkinson	0.0669	0.0514	0.0417	0.0777	0.0571	0.0463	0.0825	0.0618	0.0502
Inverse Normal	**0.0778**	**0.0585**	**0.0471**	**0.0833**	**0.0648**	**0.0529**	**0.0862**	**0.0672**	**0.0569**
Fisher	0.0596	0.0458	0.0355	0.0635	0.0493	0.0416	0.0667	0.0517	0.0445
Jordan-Kris	**0.0795**	**0.0598**	**0.0486**	**0.0863**	**0.0664**	**0.0552**	**0.0899**	**0.0697**	**0.0591**

Exact Test	k=3								
	n=15			n=25			n=40		
	p=2	p=3	p=4	p=2	p=3	p=4	p=2	p=3	p=4
Tippett	0.0496	0.0348	0.0264	0.0562	0.0410	0.0325	0.0601	0.0447	0.0361
Wilkinson	0.0545	0.0408	0.0325	0.0582	0.0449	0.0369	0.0601	0.0471	0.0393
Inverse Normal	**0.0615**	**0.0463**	**0.0372**	**0.0647**	**0.0509**	**0.0421**	**0.0681**	**0.0535**	**0.0455**
Fisher	0.0487	0.0375	0.0315	0.0526	0.0404	0.0345	0.0534	0.0426	0.0352
Jordan-Kris	**0.0621**	**0.0472**	**0.0392**	**0.0668**	**0.0521**	**0.0433**	**0.0701**	**0.0547**	**0.0462**

Exact Test	k=5								
	n=15			n=25			n=40		
	p=2	p=3	p=4	p=2	p=3	p=4	p=2	p=3	p=4
Tippett	0.0322	0.0223	0.0168	0.0370	0.0268	0.0211	0.0399	0.0294	0.0236
Wilkinson	0.0393	0.0299	0.0241	0.0424	0.0324	0.0270	0.0443	0.0340	0.0285
Inverse Normal	**0.0459**	**0.0349**	**0.0283**	**0.0487**	**0.0376**	**0.0314**	**0.0495**	**0.0395**	**0.0336**
Fisher	0.0373	0.0299	0.0242	0.0418	0.0334	0.0239	0.0403	0.0331	0.0283
Jordan-Kris	**0.0469**	**0.0351**	**0.0295**	**0.0499**	**0.0397**	**0.0331**	**0.0516**	**0.0419**	**0.0344**

8. Applications

8.1. Confidence set comparison using simulated data

In this section we follow the framework in Jordan and Krishnamoorthy (1995) who simulated bivariate samples of 12 vectors (equal sample size $n_1 = n_2 = 12$) each from two bivariate normal distributions, $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}_1)$ and $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}_2)$ with $\boldsymbol{\mu}' = (5, 8)$,

Table 2: Comparison of the 2^{nd} term of local powers [without $\sum_{i=1}^{k} \Delta_i^2/2$] of Wilkinson's exact test for $n = 15$ (equal sample size) and different values of k , p and $r\,(\leq k)$

r	k=2			k=3			k=5			k=9			k=10		
	p=2	p=3	p=4	p=2	p=3	p=4	p=2	p=3	p=4	p=2	p=3	p=4	p=2	p=3	p=4
1	**0.0693**	0.0490	0.0375	0.0496	0.0348	0.0264	0.0322	0.0223	0.0168	0.0194	0.0133	0.0099	0.0177	0.0121	0.009
2	0.0669	**0.0514**	**0.0417**	**0.0545**	**0.0408**	**0.0325**	0.0391	0.0286	0.0224	0.0253	0.0182	0.014	0.0234	0.0167	0.0129
3				0.0463	0.0369	0.0306	**0.0393**	**0.0299**	**0.0241**	0.0276	0.0204	0.0161	0.0257	0.0189	0.0148
4							0.0358	0.0283	0.0234	**0.0281**	0.0212	0.0170	**0.0264**	0.0199	0.0159
5							0.0286	0.0238	0.0203	0.0275	**0.0213**	**0.0173**	0.0262	**0.0201**	**0.0163**
6										0.026	0.0206	0.0171	0.0253	0.0198	0.0163
7										0.0239	0.0194	0.0163	0.0238	0.019	0.0159
8										0.0208	0.0174	0.0149	0.0217	0.0178	0.015
9										0.0162	0.0141	0.0123	0.0188	0.0158	0.0136
10													0.0146	0.0128	0.0113

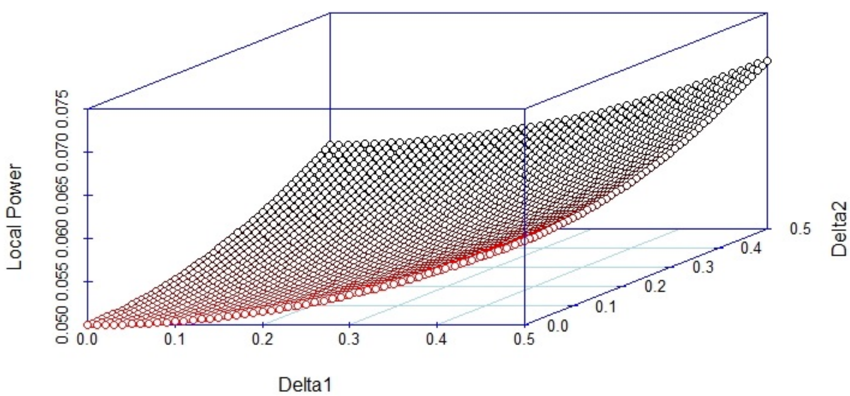

Figure 1: Local Power of Inverse Normal Method for n1=n2=15, k=2, p=2

$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 4.5 & 3.0 \\ 3.0 & 7.9 \end{bmatrix}$ and $\boldsymbol{\Sigma}_2 = \begin{bmatrix} 5.5 & 3.8 \\ 3.8 & 6.3 \end{bmatrix}$. Summary statistics based on these two simulated data sets are: $\bar{\boldsymbol{X}}_1' = (4.73, 7.93)$, $\bar{\boldsymbol{X}}_2' = (5.21, 8.89)$, $\boldsymbol{S}_1 = \begin{bmatrix} 2.71 & 4.46 \\ 4.46 & 10.91 \end{bmatrix}$, $\boldsymbol{S}_2 = \begin{bmatrix} 5.39 & 1.37 \\ 1.37 & 2.84 \end{bmatrix}$, and $\boldsymbol{V} = \begin{bmatrix} 8.07 & -3.39 \\ -3.39 & 4.10 \end{bmatrix}$.

Based on the above simulated data, we present below the 95% confidence sets for $\boldsymbol{\mu}$ resulting from the five exact methods (Figure 3) and the method based on the generalized

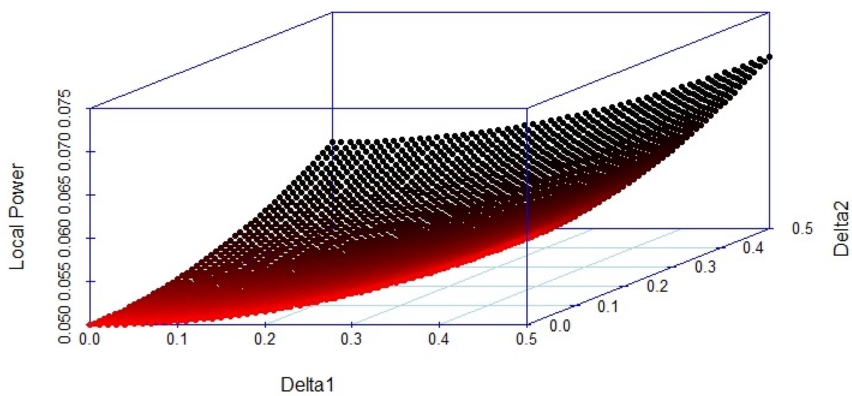

Figure 2: Local Power of Jordan-Kris Method for n1=n2=15, k=2, p=2

P-value. It turns out that INN method followed by Jordan-Kris and Fisher methods yield smaller observed confidence sets than Tippett and Wilkinson ($r = 2$) methods. As remarked earlier, the confidence ellipsoid based on the generalized P-value method although seems to have smaller volume, its coverage probability can not be guaranteed.

8.2. Data Analysis: Current Population Survey (CPS) Annual Social and Economic Supplement (ASEC)

In this section we provide a statistical analysis of data arising from Current Population Survey (CPS) Annual Social and Economic Supplement (ASEC) 2021, conducted by the Bureau of the Census for the Bureau of Labor Statistics.

Under CPS typically some 70,000 housing units or other living quarters are assigned for interview each month; about 50,000 of them containing approximately 100,000 persons 15 years old and over are interviewed. The universe in this survey is the civilian noninstitutional population of the United States living in housing units and members of the Armed Forces living off post or living with their families on post. Sampling units are scientifically selected (based on a probability sample) on the basis of area of residence to represent the nation as a whole, individual states, and other specified areas.

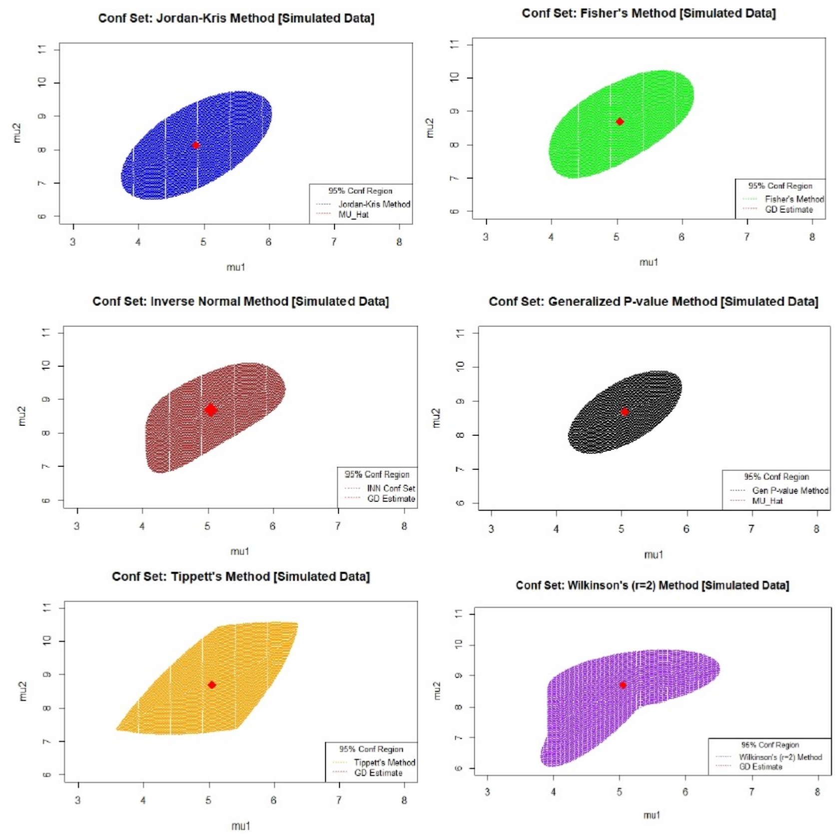

Figure 3: 95% confidence sets based on a simulated bivariate data

Although the main purpose of the survey is to collect information on the employment situation, a very important secondary purpose is to collect information on the demographic status of the population, information such as age, sex, race, marital status, educational attainment, and family structure. The statistics resulting from these questions serve to update similar information collected once every 10 years through the decennial census and are used by government policymakers and legislators as important indicators of our nation's economic situation and for planning and evaluating many government programs. CPS is the only source of monthly estimates of total employment (both farm and nonfarm); nonfarm self-employed persons, domestics, and unpaid workers in nonfarm family enterprises; wage and salary employees; and, finally, estimates of total unemployment.

The Annual Social and Economic (ASEC) Supplement contains the basic monthly demographic and labor force data described above, plus additional data on work experience, income, noncash benefits, health insurance coverage, and migration. Since 1976, the survey has been supplemented with about 6000 Hispanic households from which at least 4500 are interviewed. But in 2002 another sample expansion occurred to help improved states estimates of Children's Health Insurance (CHIP) resulting in the addition of 19000 households and raising up the total sample size for the ASEC to about 95000 households. All Current Population Reports are available online at `https://www.census.gov/library/publications.html`.

For the purpose of our data analysis, we mainly consider two variables out of a wide range of available data: total income and income components covering nine noncash income sources: food stamps, school lunch program, employer-provided group health insurance plan, employer-provided pension plan, personal health insurance, Medicaid, Medicare, or military health care, and energy assistance. Characteristics such as age, sex, race, household relationship, and Hispanic origin are shown for each person in the household enumerated. Although above type of data is available for all the 50 states, we focus on the data from California (CA) as it is argued that this data source is the most reliable and the sample sizes are fairly large. There are 58 counties in CA and the table below shows a summary of the bivariate sample mean vectors and 2x2 sample variance-covariance matrices for 13 selected counties in CA, divided into three groups (Table 3). We observe that the simple mean vectors in each group are fairly close, suggesting a common population mean vector within the chosen counties of a group. The 95% confidence sets of the common mean vector based on the methods discussed in this paper appear in Figures 4 - 6. The location of the well-known Graybill-Deal estimate of the common mean vector is shown in each ellipsoid. As an illustration, we have also added a figure (Figure 7) depicting three confidence sets under Fisher method for the three groups for the sake of comparison.

Table 3: Summary of the bivariate sample mean vectors and 2×2 sample variance-covariance matrices for 13 selected counties in CA, divided into three groups

		Group-I [K=5 Counties]		
Butte County	**Kings County**	**Shasta County**	**Tulare County**	**Stanislaus County**
n=32	n=49	n=45	n=57	n=67
$\bar{X}_{11} = \begin{bmatrix} 46.8048 \\ 63.8037 \end{bmatrix}$	$\bar{X}_{12} = \begin{bmatrix} 45.6735 \\ 64.3907 \end{bmatrix}$	$\bar{X}_{13} = \begin{bmatrix} 48.9864 \\ 71.6568 \end{bmatrix}$	$\bar{X}_{14} = \begin{bmatrix} 53.5000 \\ 73.6172 \end{bmatrix}$	$\bar{X}_{15} = \begin{bmatrix} 49.2051 \\ 79.8356 \end{bmatrix}$
$S_{11} = \begin{bmatrix} 2255.263 & 1820.256 \\ 1820.256 & 1851.509 \end{bmatrix}$	$S_{12} = \begin{bmatrix} 2249.200 & 1958.085 \\ 1958.085 & 2223.194 \end{bmatrix}$	$S_{13} = \begin{bmatrix} 2290.233 & 1670.224 \\ 1670.224 & 2172.094 \end{bmatrix}$	$S_{14} = \begin{bmatrix} 4702.451 & 4576.456 \\ 4576.456 & 5330.648 \end{bmatrix}$	$S_{15} = \begin{bmatrix} 3298.961 & 2527.469 \\ 2527.469 & 2828.913 \end{bmatrix}$
		Group-II [K=5 Counties]		
Monterey County	**Los Angeles County**	**Sacramento County**	**Santa Cruz County**	**San Luis Obispo County**
n=62	n=1548	n=216	n=50	n=43
$\bar{X}_{21} = \begin{bmatrix} 80.5238 \\ 96.1007 \end{bmatrix}$	$\bar{X}_{22} = \begin{bmatrix} 80.8304 \\ 101.0140 \end{bmatrix}$	$\bar{X}_{23} = \begin{bmatrix} 75.0044 \\ 99.6372 \end{bmatrix}$	$\bar{X}_{24} = \begin{bmatrix} 77.1391 \\ 100.2768 \end{bmatrix}$	$\bar{X}_{25} = \begin{bmatrix} 76.0411 \\ 103.1578 \end{bmatrix}$
$S_{21} = \begin{bmatrix} 7888.208 & 8107.861 \\ 8107.861 & 8989.313 \end{bmatrix}$	$S_{22} = \begin{bmatrix} 10992.71 & 11338.02 \\ 11338.02 & 12834.65 \end{bmatrix}$	$S_{23} = \begin{bmatrix} 6417.046 & 6005.736 \\ 6005.736 & 7266.494 \end{bmatrix}$	$S_{24} = \begin{bmatrix} 7108.753 & 7022.462 \\ 7022.462 & 7855.648 \end{bmatrix}$	$S_{25} = \begin{bmatrix} 5310.268 & 4276.638 \\ 4276.638 & 4593.337 \end{bmatrix}$
		Group-III [K=3 Counties]		
	Alameda County	**San Francisco County**	**Sonoma County**	
	n=247	n=90	n=50	
	$\bar{X}_{31} = \begin{bmatrix} 126.3072 \\ 147.3478 \end{bmatrix}$	$\bar{X}_{32} = \begin{bmatrix} 127.4039 \\ 155.6503 \end{bmatrix}$	$\bar{X}_{33} = \begin{bmatrix} 122.2838 \\ 166.5137 \end{bmatrix}$	
	$S_{31} = \begin{bmatrix} 17857.07 & 17442.44 \\ 17442.44 & 19388.81 \end{bmatrix}$	$S_{32} = \begin{bmatrix} 19639.91 & 18455.13 \\ 18455.13 & 19352.74 \end{bmatrix}$	$S_{33} = \begin{bmatrix} 16522.68 & 17558.55 \\ 17558.55 & 23443.13 \end{bmatrix}$	

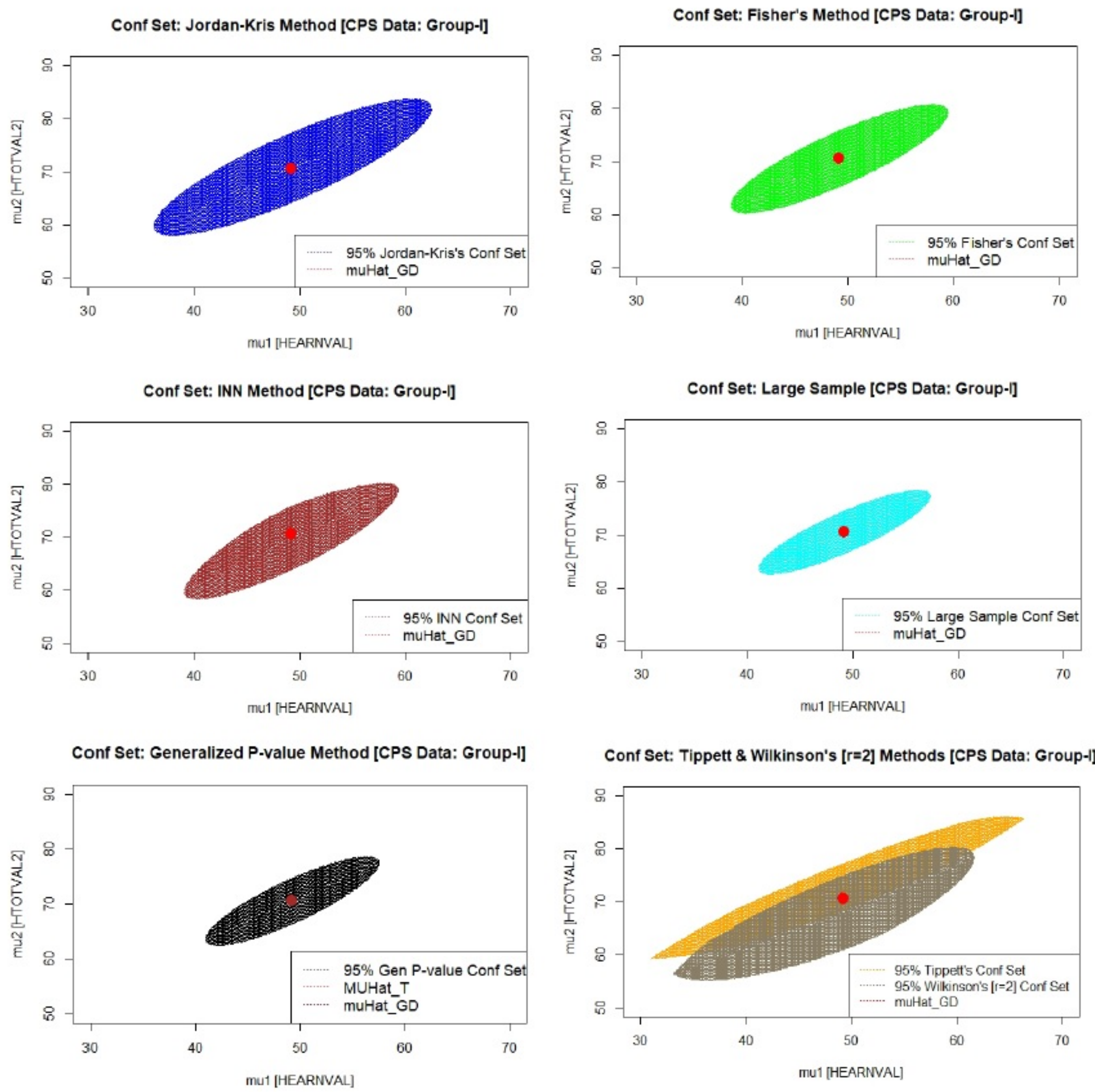

Figure 4: 95% confidence sets based on CPS bivariate data [Group-I]

9. Concluding remarks

In the spirit of statistical meta-analysis, this paper discusses asymptotic and exact methods for efficiently combining data from several independent multinormal populations with a common mean vector $\boldsymbol{\mu}$ to draw inference upon $\boldsymbol{\mu}$. It turns out that, in

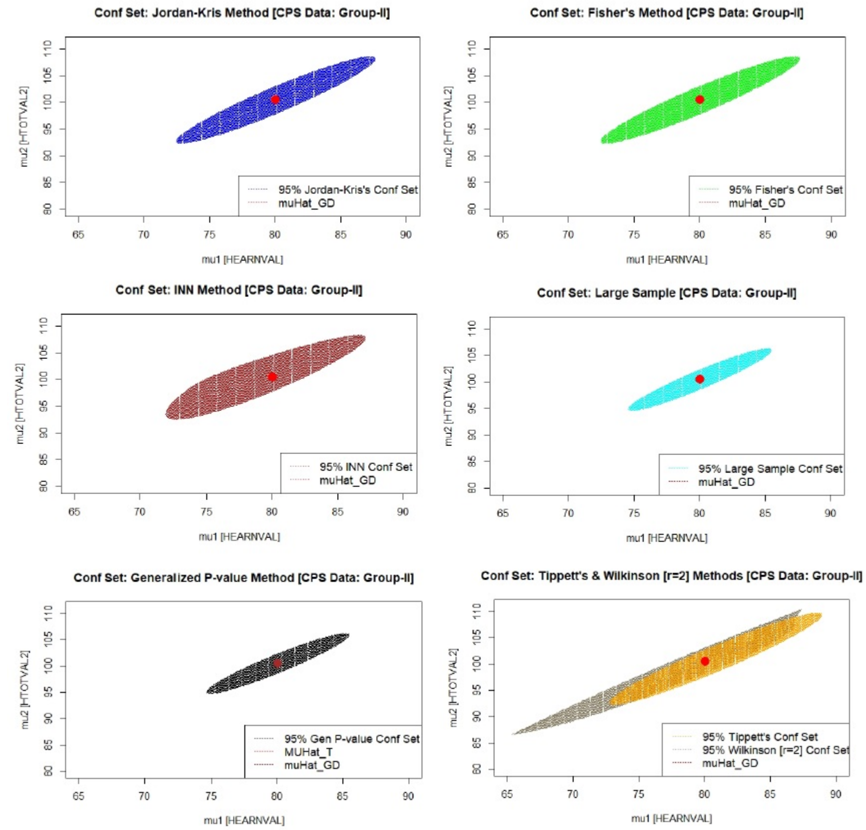

Figure 5: 95% confidence sets based on CPS bivariate data [Group-II]

large samples, a procedure based on a standardized Graybill-Deal estimate of $\boldsymbol{\mu}$ is quite satisfactory and easy to carry out. In small samples, however, several exact procedures with good frequentist properties exist. We should point out that although the plots of the confidence ellipsoids based on the approximate generalized P-value method suggested in Lin et al. (2007) appear to be quite satisfactory, there is no guarantee that the coverage level is maintained. We hope that the methods of data analysis developed and discussed in this paper will be used in applications whenever warranted.

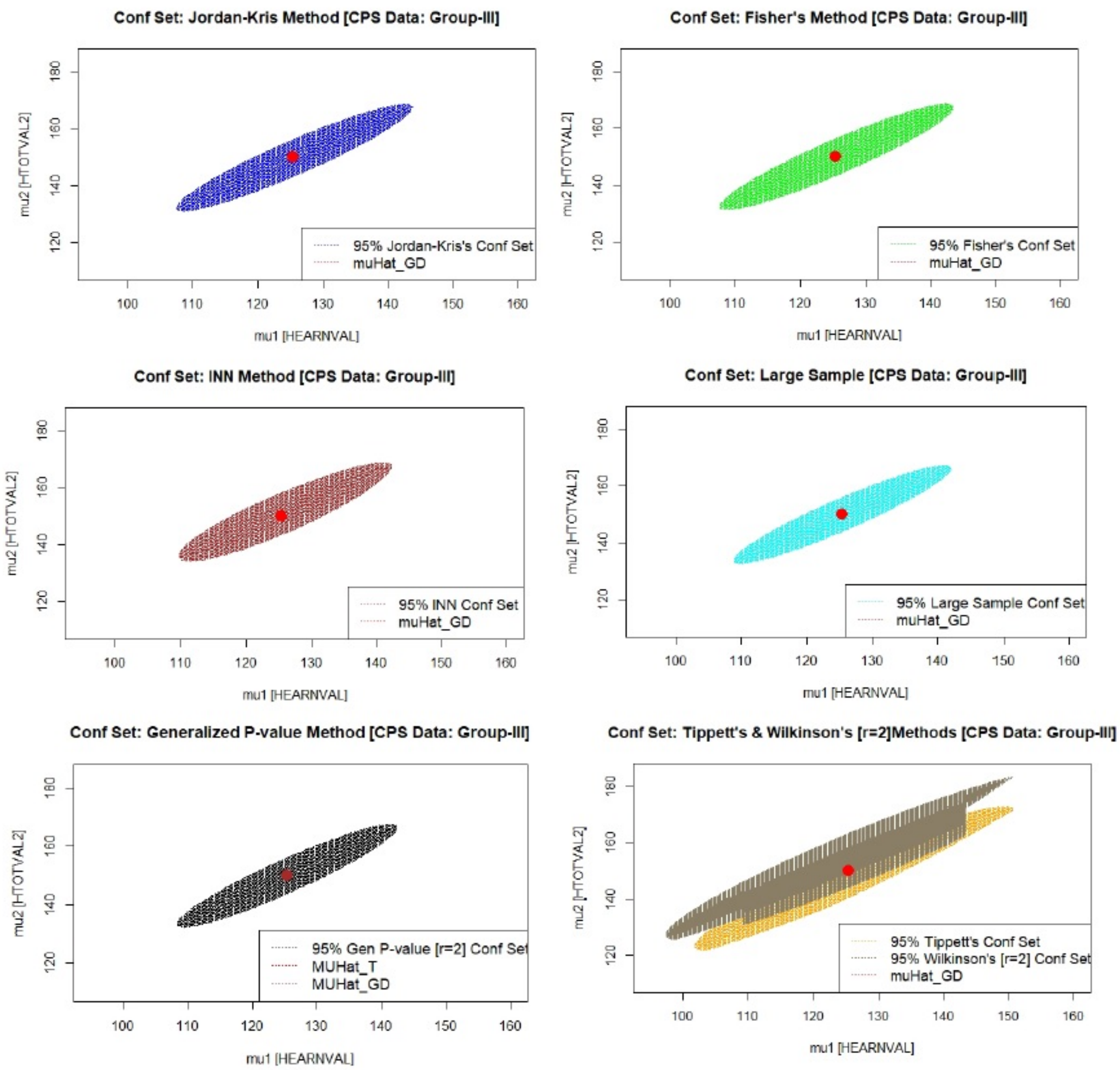

Figure 6: 95% confidence sets based on CPS bivariate data [Group-III]

Acknowledgments

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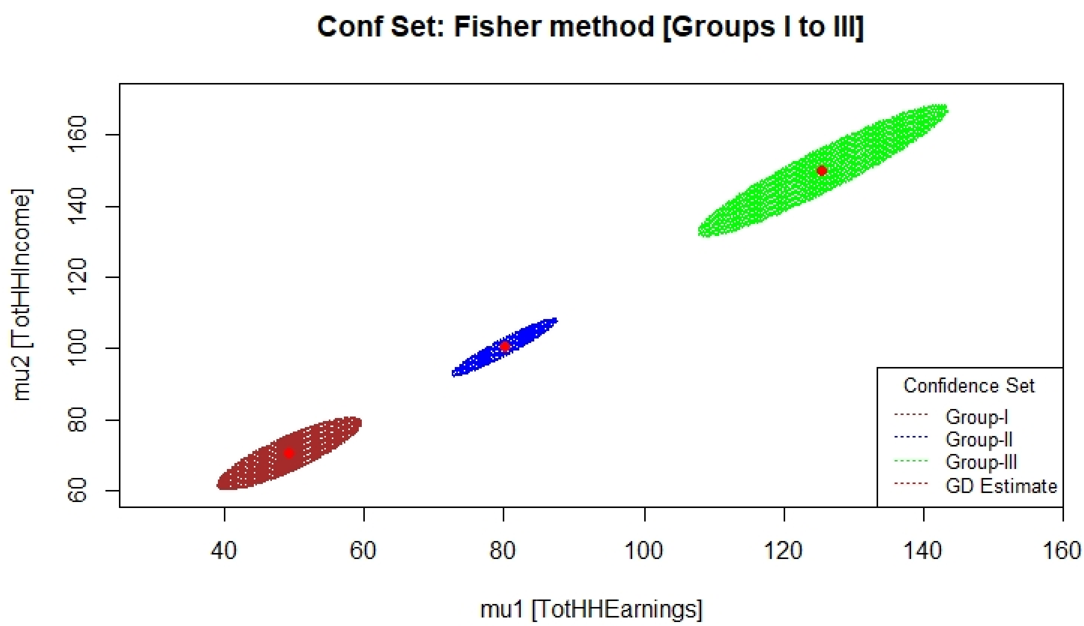

Figure 7: 95% confidence sets based on CPS bivariate data [three groups]

Appendix A: Robustness of the χ^2 cut-off point

We present here the results of our simulation based on $N = 10,000$ replications related to the cut-off point of the test statistic given by (4). We have taken several scenarios of dispersion matrices and $n = 50, 100$. Our simulation studies demonstrate the robustness of the χ^2 cut-off point for variations in the unknown dispersion matrices (Table 4 and Figure 8).

Table 4: 95% Cut-off points for different sample sizes and dispersion matrices

Dispersion matrices		95% Cut-off points	
		n=50	n=100
$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 4.5 & 3.0 \\ 3.0 & 7.9 \end{bmatrix}$	$\boldsymbol{\Sigma}_2 = \begin{bmatrix} 5.5 & 3.8 \\ 3.8 & 6.3 \end{bmatrix}$	6.63	6.29
$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$	$\boldsymbol{\Sigma}_2 = \begin{bmatrix} 1.0 & 0.6 \\ 0.6 & 1.0 \end{bmatrix}$	6.64	6.21
$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$	$\boldsymbol{\Sigma}_2 = \begin{bmatrix} 25.0 & 1.6 \\ 1.6 & 1.0 \end{bmatrix}$	6.62	6.24
$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 10.0 & 2.0 \\ 2.0 & 1.0 \end{bmatrix}$	$\boldsymbol{\Sigma}_2 = \begin{bmatrix} 1.0 & 0.6 \\ 0.6 & 1.0 \end{bmatrix}$	6.62	6.24
$\boldsymbol{\Sigma}_1 = \begin{bmatrix} 20.0 & 1.5 \\ 1.5 & 2.0 \end{bmatrix}$	$\boldsymbol{\Sigma}_2 = \begin{bmatrix} 5.0 & 0.5 \\ 0.5 & 1.0 \end{bmatrix}$	6.62	6.26

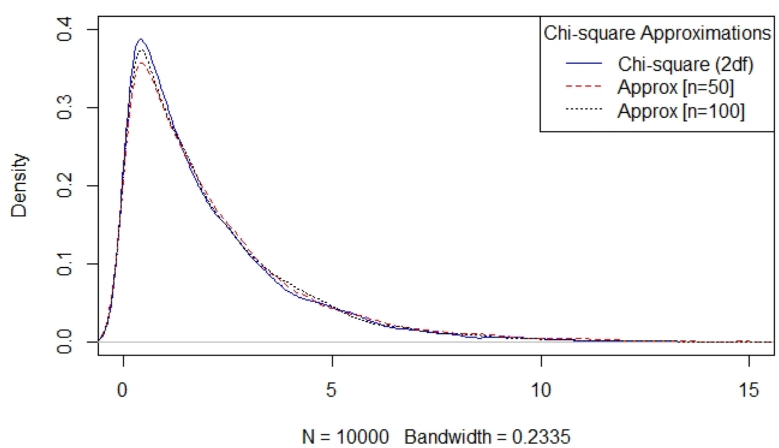

Figure 8: Chi-square with 2 degrees of freedom and large sample distributions with $n = 50$ and $n = 100$

Appendix B: Proofs of local powers of exact tests

We begin by stating a result which will be crucial for providing the main results on local power of all P-value based exact tests. We denote $F_{\nu_1,\nu_2}(\cdot)$ to represent the cdf of a central F-distribution with ν_1 and ν_2 degrees of freedom.

Lemma 1. *Let T be a random variable with pdf $h(t)$ and CDF $H(t)$, $-\infty < t < \infty$. Define*

$$I(t) = \int_{F_{H(t);\nu_1,\nu_2}}^{\infty} f_0(F)\Big\{\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\Big\}dF \tag{16}$$

where $f_0(F)$ follows an F distribution with ν_1 and ν_2 degrees of freedom (F_{ν_1,ν_2}) and $F_{H(t);\nu_1,\nu_2}$ satisfies

$$H(t) = Pr[F_{\nu_1,\nu_2} > F_{H(t);\nu_1,\nu_2}]. \tag{17}$$

Then

$$\frac{d}{dt}\{I(t)\} = h(t)\left\{\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\right\}_{F=F_{H(t);\nu_1,\nu_2}} \tag{18}$$

Proof. Obviously,

$$\frac{d}{dt}\{I(t)\} = \Big[-\frac{d}{dt}\{F_{H(t);\nu_1,\nu_2}\}\Big]\left[f_0(F)\Big\{\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\Big\}\right]_{F=F_{H(t);\nu_1,\nu_2}} \tag{19}$$

From equation (17), differentiating both sides with respect to t, we get

$$h(t) = \Big[-\frac{d}{dt}\{F_{H(t);\nu_1,\nu_2}\}\Big]f_0(F)\Big|_{F=F_{H(t);\nu_1,\nu_2}} \tag{20}$$

Implies

$$-\frac{d}{dt}\{F_{H(t);\nu_1,\nu_2}\} = \frac{h(t)}{f_0(F)\big|_{F=F_{H(t);\nu_1,\nu_2}}} \tag{21}$$

Lemma 1 (which is equation 18) follows upon combining equations (19) and (21). □

I. Local power of Tippett's test $[LP(T)]$

Recall that Tippett's exact test rejects the null hypothesis if $P_{(1)} < \left[1-(1-\alpha)^{\frac{1}{k}}\right] = c_{\alpha}$, where $c_{\alpha} = 1 - [1-\alpha]^{1/k}$. This leads to

$$
\begin{aligned}
\text{Power} &= Pr\{P_{(1)} < c_{\alpha}|H_1\} \\
&= 1 - Pr\{P_{(1)} > c_{\alpha}|H_1\} \\
&= 1 - Pr\{P_i > c_{\alpha}, \forall_i|H_1\} \\
&= 1 - \prod_{i=1}^{k} Pr\{P_i > c_{\alpha}|H_1\} \\
&= 1 - \prod_{i=1}^{k} Pr\{F_{p,n_i-p} < F_{c_{\alpha};p,n_i-p}|H_1\} \\
&= 1 - \prod_{i=1}^{k} \left[1 - Pr\{F_{\nu_1,\nu_{2i}} > F_{c_{\alpha};\nu_1,\nu_{2i}}|H_1\}\right] \qquad [\nu_1 = p, \nu_{2i} = n_i - p] \\
&= 1 - \prod_{i=1}^{k} \left[1 - \int_{F_{c_{\alpha};\nu_1,\nu_{2i}}}^{\infty} f_{\Delta^2}(F_{\nu_1,\nu_2}) dF\right]
\end{aligned}
$$

Applying the approximate distribution of $F_{\nu_1,\nu_2}(\cdot)$ under the alternative hypothesis following its Taylor expansion around $\Delta^2 = 0$, the local power of Tippett's test is calculated as follows:

$$
\begin{aligned}
\text{Local power} &\approx 1 - \prod_{i=1}^{k} \left\{1 - \int_{F_{c_{\alpha};\nu_1,\nu_{2i}}}^{\infty} \left[f_{\Delta^2=0}(F_{\nu_1,\nu_{2i}}) + \frac{\Delta_i^2}{2} \int_{F_{c_{\alpha};\nu_1,\nu_{2i}}}^{\infty} f_{\Delta^2=0}(F)\left[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i}}F}\right]\right] dF\right\} \\
&\quad \text{where } \Delta_i^2 = n_i(\boldsymbol{\mu} - \boldsymbol{\mu}_0)^t \boldsymbol{\Sigma}_i^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_0) \\
&= 1 - \prod_{i=1}^{k} \left\{1 - \int_{F_{c_{\alpha};\nu_1,\nu_{2i}}}^{\infty} f_{\Delta^2=0}(F_{\nu_1,\nu_{2i}}) dF - \frac{\Delta_i^2}{2} \int_{F_{c_{\alpha};\nu_1,\nu_{2i}}}^{\infty} f_{\Delta^2=0}(F)\left[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i}}F}\right] dF\right\} \\
&= 1 - \prod_{i=1}^{k} \left\{(1 - c_{\alpha;\nu_1,\nu_{2i}}) - \frac{\Delta_i^2}{2} \xi_{F_{c_{\alpha;\nu_1,\nu_{2i}}}}\right\} \qquad [\xi_{F_{c_{\alpha;\nu_1,\nu_{2i}}}} > 0] \\
&\quad \text{where } \xi_{F_{c_{\alpha;\nu_1,\nu_{2i}}}} = \int_{F_{c_{\alpha};\nu_1,\nu_{2i}}}^{\infty} f_0(F_{\nu_1,\nu_{2i}})\left[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i}}F}\right] dF \\
&= 1 - \left\{[1 - c_{\alpha}]^k - [1 - c_{\alpha}]^{k-1} \sum_{i=1}^{k} \frac{\Delta_i^2}{2} \xi_{F_{c_{\alpha;\nu_1,\nu_{2i}}}}\right\} \\
&= \alpha + (1-\alpha)^{\frac{k-1}{k}} \sum_{i=1}^{k} \frac{\Delta_i^2}{2} \xi_{F_{c_{\alpha;\nu_1,\nu_{2i}}}} \qquad (22)
\end{aligned}
$$

For the special case $n_1 = \cdots = n_k = n$ and $\xi_{F_{c\alpha;\nu_1,\nu_{21}}}, = \cdots = \xi_{F_{c\alpha;\nu_1,\nu_{2k}}} = \xi_{F_{c\alpha;\nu_1,\nu_2}}$, the local power of Tippett's test reduces to:

$$\text{LP(T)} = \alpha + (1-\alpha)^{\frac{k-1}{k}} \xi_{F_{c\alpha;\nu_1,\nu_2}} \left\{ \sum_{i=1}^{k} \frac{\Delta_i^2}{2} \right\}. \tag{23}$$

II. Local power of Wilkinson's test $[LP(W_r)]$

Using r^{th} smallest p-value $P_{(r)}$ as a test statistic, the null hypothesis will be rejected if $P_{(r)} < d_{r,\alpha}$, where $P_{(r)} \sim Beta[r, k-r+1]$ under H_0 and $d_{r,\alpha}$ satisfies

$$\begin{aligned} \alpha &= Pr\{P_{(r)} < d_{r,\alpha}|H_0\} \\ &= \int_0^{d_{r,\alpha}} \frac{u^{r-1}(1-u)^{k-r}}{B[r,k-r+1]} du \\ &= \sum_{l=1}^{k} \binom{k}{l} d_{r,\alpha}^{l} [1-d_{r,\alpha}]^{k-l}. \end{aligned} \tag{24}$$

This leads to

$$\begin{aligned} \text{Power} &= Pr[P_{(r)} < d_{r,\alpha}|H_1] \\ &= \sum_{l=r}^{k} Pr\{P_{i_1}, \ldots, P_{i_l} < d_{r,\alpha} < P_{i_{l+1}}, \ldots, P_{i_k}|H_1\} \end{aligned}$$

where $(i_1, \cdots, i_l, i_{l+1}, \cdots, i_k)$ is a permutation of $(1, \cdots, k)$. Applying Lemma 1, we get

$$\begin{aligned} Pr\{P_{i_1}, \quad \ldots \quad &, P_{i_l} < d_{r,\alpha} < P_{i_{l+1}}, \ldots, P_{i_k}|H_1\} \\ &\approx \left\{ \prod_{j=1}^{l} Pr\big(P_{i_j} < d_{r,\alpha}\big) \right\} \left\{ \prod_{j=l+1}^{k} \Pr\big(P_{i_j} > d_{r,\alpha}\big) \right\} \\ &= \left\{ \prod_{j=1}^{l} Pr\big(F_{\nu_1,\nu_{2i_j}} > F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}|H_1\big) \right\} \left\{ \prod_{j=l+1}^{k} \Pr\big(F_{\nu_1,\nu_{2i_j}} < F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}\big) \right\} \\ &= \left\{ \prod_{j=1}^{l} Pr\big(F_{\nu_1,\nu_{2i_j}} > F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}|H_1\big) \right\} \left\{ \prod_{j=l+1}^{k} \big[1 - \Pr\big(F_{\nu_1,\nu_{2i_j}} > F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}\big)\big] \right\} \\ &= \left\{ \prod_{j=1}^{l} \int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} F_{\Delta_{i_j}^2} dF \right\} \left\{ \prod_{j=l+1}^{k} \big[1 - \int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_{\Delta_{i_j}^2}(F_{\nu_1,\nu_{2i_j}}) dF\big] \right\} \end{aligned} \tag{25}$$

Let's apply now the following fact in 25

$$\int_{F_{d_{r,\alpha},\nu_1,\nu_{2i}}}^{\infty} F_{\Delta_i^2} dF \approx d_{r,\alpha} + \frac{\Delta_i^2}{2} \int_{F_{d_{r,\alpha},\nu_1,\nu_{2i}}}^{\infty} f_0(F_{\nu_1,\nu_{2i}}) \Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i}}F} \Big] dF$$

$$
\begin{aligned}
LP(W_r) &\approx \prod_{j=1}^{l}\left\{d_{r,\alpha} + \frac{\Delta_{i_j}^2}{2}\int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_0(F_{\nu_1,\nu_{2i_j}})\Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i_j}}F}\Big]dF\right\} \\
&\quad \times \prod_{j=l+1}^{k}\left\{(1-d_{r,\alpha}) - \frac{\Delta_{i_j}^2}{2}\int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_0(F_{\nu_1,\nu_{2i_j}})\Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i_j}}F}\Big]dF\right\} \\
&= \left\{d_{r,\alpha}^{l} + d_{r,\alpha}^{l-1}\sum_{j=1}^{l}\frac{\Delta_{i_j}^2}{2}\int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_0(F_{\nu_1,\nu_{2i_j}})\Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i_j}}F}\Big]dF\right\} \\
&\quad \times\left\{(1-d_{r,\alpha})^{k-l} - (1-d_{r,\alpha})^{k-l-1}\sum_{j=l+1}^{k}\frac{\Delta_{i_j}^2}{2}\int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_0(F_{\nu_1,\nu_{2i_j}})\Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i_j}}F}\Big]dF\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= d_{r,\alpha}^{l}(1-d_{r,\alpha})^{k-l} - d_{r,\alpha}^{l}(1-d_{r,\alpha})^{k-l-1}\left\{\sum_{j=l+1}^{k}\frac{\Delta_{i_j}^2}{2}\int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_0(F_{\nu_1,\nu_{2i_j}})\Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i_j}}F}\Big]dF\right\} \\
&\quad + d_{r,\alpha}^{l-1}(1-d_{r,\alpha})^{k-l}\left\{\sum_{j=1}^{l}\frac{\Delta_{i_j}^2}{2}\int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_0(F_{\nu_1,\nu_{2i_j}})\Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i_j}}F}\Big]dF\right\} \\
&= d_{r,\alpha}^{l}(1-d_{r,\alpha})^{k-l} - d_{r,\alpha}^{l}(1-d_{r,\alpha})^{k-l-1}\left\{\sum_{j=l+1}^{k}\frac{\Delta_{i_j}^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}\right\} \\
&\quad + d_{r,\alpha}^{l-1}(1-d_{r,\alpha})^{k-l}\left\{\sum_{j=1}^{l}\frac{\Delta_{i_j}^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}\right\} \quad \text{where } \xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}} = \int_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}^{\infty} f_0(F_{\nu_1,\nu_{2i_j}})\Big[\frac{F-1}{1+\frac{\nu_1}{\nu_{2i_j}}F}\Big]dF
\end{aligned}
$$

Permuting $(i_1,\ldots,i_k)$ over $(1,\ldots,k)$, we get for any fixed l $(r \le l \le k)$,

$$
\begin{aligned}
\text{1st term} &= \binom{k}{l} d_{r,\alpha}^{l}(1-d_{r,\alpha})^{k-l} \\
\text{2nd term} &= -d_{r,\alpha}^{l}(1-d_{r,\alpha})^{k-l-1}\left\{\binom{k-1}{k-l-1}\left(\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i}}}\right)\right\} \\
\text{3rd term} &= d_{r,\alpha}^{l-1}(1-d_{r,\alpha})^{k-l}\left\{\binom{k-1}{l-1}\left(\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i}}}\right)\right\}.
\end{aligned}
$$

The 2nd term above follows upon noting that when $\big[\sum_{j=l+1}^{k}\frac{\Delta_{i_j}^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}\big]$ is permuted over $(i_{l+1} < \cdots < i_k) \subset (1,\ldots,k)$, each term $\frac{\Delta_{i_j}^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}$ appears exactly $\binom{k-1}{k-l-1}$ times, for each $i = 1,\cdots,k$. The 3rd term, likewise, follows upon noting that when $\big[\sum_{j=1}^{l}\frac{\Delta_{i_j}^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}\big]$ is permuted over $(i_1 < \cdots < i_l) \subset (1,\ldots,k)$, each term

$\frac{\Delta_{i_j}^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i_j}}}$ appears exactly $\binom{k-1}{l-1}$ times, for each $i = 1, \cdots, k$.

Adding the above three terms and applying 24, we get

$$LP(W_r) \approx \alpha + \binom{k-1}{r-1} d_{r;\alpha}^{r-1}(1-d_{r;\alpha})^{k-r}\left[\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\xi_{F_{d_{r,\alpha},\nu_1,\nu_{2i}}}\right] \quad (26)$$
$$\text{where } \Delta_i^2 = n_i(\boldsymbol{\mu}-\boldsymbol{\mu_0})^t\boldsymbol{\Sigma}_i^{-1}(\boldsymbol{\mu}-\boldsymbol{\mu_0})$$

For the special case $n_1 = \cdots = n_k = n$, and $\xi_{F_{d_{r,\alpha},\nu_1,\nu_{21}}} = \cdots = \xi_{F_{d_{r,\alpha},\nu_1,\nu_{2k}}} = \xi_{F_{d_{r,\alpha},\nu_1,\nu_2}}$, the local power of Wilkinson's test reduces to:

$$LP(W_r) \approx \alpha + \binom{k-1}{r-1} d_{r;\alpha}^{r-1}(1-d_{r;\alpha})^{k-r}\xi_{F_{d_{r,\alpha},\nu_1,\nu_2}}\left\{\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\right\}. \quad (27)$$

III. Local power of Inverse Normal test $[LP(INN)]$

Under this test, the null hypothesis will be rejected if $\frac{1}{\sqrt{k}}\sum_{i=1}^{k} U_i < -z_\alpha$, where $U_i = \Phi^{-1}(P_i)$, Φ^{-1} is the inverse cdf and z_α is the upper α level critical value of a standard normal distribution. This leads to

$$\text{Power} = Pr\left\{\frac{1}{\sqrt{k}}\sum_{i=1}^{k} U_i < -z_\alpha | H_1 : \Delta_i^2 > 0, \forall_i\right\}.$$

First, let us determine the pdf of U under H_1, $f_{H_1}(u)$, via its cdf $F_{H_1}(u) = Pr\{U \leq u|H_1\}$.

$$\begin{aligned}
Pr\{U \leq u|H_1\} &= Pr\{\Phi(U) \leq \Phi(u)|H_1\} \\
&= Pr\{P \leq \Phi(u)|H_1\} \qquad \left[U = \Phi^{-1}(P) \implies P = \Phi(U)\right] \\
&= 1 - Pr\{P > \Phi(u)|H_1\} \\
&= 1 - Pr\{F_{p,n-p} < F_{\Phi(u);p,n-p}|H_1\} \\
&= Pr\{F_{p,n-p} > F_{\Phi(u);p,n-p}|H_1\} \\
&= \int_{F_{\Phi(u);p,n-p}}^{\infty} f_{\Delta^2}(F)dF \\
&\approx \int_{F_{\Phi(u);p,n-p}}^{\infty} f_{\Delta^2=0}(F) + \frac{\Delta^2}{2} f_{\Delta^2=0}(F)\left[\frac{F-1}{1+\frac{p}{n-p}F}\right]dF \\
&\approx \Phi(u) + \frac{\Delta^2}{2}\int_{F_{\Phi(u);p,n-p}}^{\infty} f_{\Delta^2=0}(F)\left[\frac{F-1}{1+\frac{p}{n-p}F}\right]dF
\end{aligned} \quad (28)$$

This implies

$$
\begin{aligned}
f_{H_1}(u) &\approx \frac{d}{du}\Big[\Phi(u) + \frac{\Delta^2}{2}\int_{F_{\Phi(u);p,n-p}}^{\infty} f_{\Delta^2=0}(F)\big[\frac{F-1}{1+\frac{p}{n-p}F}\big]dF\Big] \\
&\approx \phi(u) + \frac{\Delta^2}{2}\Big\{\frac{d}{du}\int_{F_{\Phi(u);p,n-p}}^{\infty} f_{\Delta^2=0}(F)\big[\frac{F-1}{1+\frac{p}{n-p}F}\big]dF\Big\} \\
&\quad \text{Applying Lemma 1} \\
&\approx \frac{\phi(u)\big[1+\frac{\Delta^2}{2}Q_{\nu_1,\nu_2}(u)\big]}{1+\frac{\Delta^2}{2}\int_{-\infty}^{\infty}\phi(u)Q_{\nu_1,\nu_2}(u)du}, \qquad Q_{\nu_1,\nu_2}(u) = \big[\frac{F-1}{1+\frac{p}{n-p}F}\big]_{F=F_{\Phi(u);\nu_1,\nu_2}} \\
&\approx \phi(u)\big[1+\frac{\Delta^2}{2}\{Q_{\nu_1,\nu_2}(u) - E[Q_{\nu_1,\nu_2}(u)]\}\big] \qquad (29) \\
&\quad \text{where } E\big[Q_{\nu_1,\nu_2}(u)\big] = \int_{-\infty}^{\infty} Q_{\nu_1,\nu_2}(u)\phi(u)du
\end{aligned}
$$

Let us define $Q^*_{\nu_1,\nu_2}(u)$, A_{ν_1,ν_2}, and B_{ν_1,ν_2} as $Q^*_{\nu_1,\nu_2}(u) = \{Q_{\nu_1,\nu_2}(u) - E[Q_{\nu_1,\nu_2}(u)]\}$, $A_{\nu_1,\nu_2} = \int_{-\infty}^{\infty} u\phi(u)Q_{\nu_1,\nu_2}(u)du$, and $B_{\nu_1,\nu_2} = \int_{-\infty}^{\infty} u^2\phi(u)Q^*_{\nu_1,\nu_2}(u)du$. Using these three quantities, we now approximate the distribution of U as:

$$
\begin{aligned}
U \ &\sim\ N[E_{H_1}(U), Var_{H_1}(U)] \quad \text{where} \quad E_{H_1}(U) = \int_{-\infty}^{\infty} u f_{H_1}(u)du \approx \frac{\Delta^2}{2}A_{\nu_1,\nu_2} \quad \text{and} \\
&\quad Var_{H_1}(U) = \int_{-\infty}^{\infty} u^2 f_{H_1}(u)du \approx 1 + \frac{\Delta^2}{2}B_{\nu_1,\nu_2}.
\end{aligned}
$$

This leads to:

$$
\begin{aligned}
\frac{1}{\sqrt{k}}\sum_{i=1}^{k} U_i \ &\sim\ N\Big[\frac{1}{\sqrt{k}}\sum_{i=1}^{k} E(U_i), \frac{1}{k}\sum_{i=1}^{k} Var(U_i)\Big] \\
&\sim\ N\Big[\frac{1}{\sqrt{k}}\delta_1, 1 + \frac{1}{k}\delta_2\Big] \\
\text{where} \quad \delta_1 &= \sum_{i=1}^{k}\frac{\Delta_i^2}{2}A_{\nu_{1i},\nu_{2i}} \quad \text{and} \quad \delta_2 = \sum_{i=1}^{k}\frac{\Delta_i^2}{2}B_{\nu_{1i},\nu_{2i}}.
\end{aligned}
$$

Using the above result, the local power of inverse normal test is obtained by approximating its power which is $Pr\{\frac{1}{\sqrt{k}}\sum_{i=1}^{k} U_i < -z_\alpha | H_1\}$ as

$$\begin{aligned}
\text{Local power (INN)} &\approx \Phi\left[\frac{-z_\alpha - \frac{1}{\sqrt{k}}\delta_1}{\sqrt{1+\frac{1}{k}\delta_2}}\right] \\
&\approx \Phi\left[-z_\alpha - \frac{1}{\sqrt{k}}\delta_1 + \frac{z_\alpha}{2k}\delta_2\right] \\
&\approx \Phi\left[-z_\alpha + \frac{1}{\sqrt{k}}\left(\frac{z_\alpha}{2\sqrt{k}}\delta_2 - \delta_1\right)\right] \\
&\approx \Phi(-z_\alpha) + \frac{\phi(z_\alpha)}{\sqrt{k}}\left[\frac{z_\alpha}{2\sqrt{k}}\delta_2 - \delta_1\right] \\
&\approx \alpha + \frac{\phi(z_\alpha)}{\sqrt{k}}\left[\frac{z_\alpha}{2\sqrt{k}}\delta_2 - \delta_1\right].
\end{aligned}$$

Substituting back the expressions for δ_1 and δ_2 results in:

$$LP(INN) \approx \alpha + \frac{\phi(z_\alpha)}{2\sqrt{k}}\sum_{i=1}^{k}\Delta_i^2\left[\frac{z_\alpha}{2\sqrt{k}}B_{\nu_{1i},\nu_{2i}} - A_{\nu_{1i},\nu_{2i}}\right].$$

For the special case $n_1 = \cdots = n_k = n$, the local power of *Inverse Normal test* reduces to:

$$\begin{aligned}
LP(INN) &\approx \alpha + \frac{\phi(z_\alpha)}{2\sqrt{k}}\left(\sum_{i=1}^{k}\Delta_i^2\right)\left[\frac{z_\alpha}{2\sqrt{k}}B_{\nu_1,\nu_2} - A_{\nu_1,\nu_2}\right] \\
&= \alpha + \frac{\phi(z_\alpha)}{\sqrt{k}}\left[\frac{z_\alpha}{2\sqrt{k}}B_{\nu_1,\nu_2} - A_{\nu_1,\nu_2}\right]\left\{\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\right\}.
\end{aligned}$$

IV. Local power of Fisher's test $[LP(F)]$

According to Fisher's exact test, the null hypothesis will be rejected if $\sum_{i=1}^{k} V_i > \chi^2_{2k;\alpha}$, where $V_i = -2\ln(P_i)$, and $\chi^2_{2k;\alpha}$ is the upper α level critical value of a χ^2-distribution with $2k$ degrees of freedom. This leads to

$$\text{Power} = Pr\left\{ \sum_{i=1}^{k} V_i > \chi^2_{2k;\alpha} | H_1 \right\}.$$

In a similar way to the inverse normal test in Appendix-B section III, first let us determine the pdf of V under H_1, $g_{H_1}(v)$, via its cdf $G_{H_1}(v) = Pr\{V \le v|H_1\}$.

$$\begin{aligned}
Pr\{V \le v|H_1\} &= Pr\{-2\ln(P) \le v|H_1\} \\
&= Pr\{\ln(P) > -v/2|H_1\} \\
&= Pr\{P > e^{-v/2}|H_1\} \\
&= Pr\{F_{p,n-p} < F_{e^{-v/2};p,n-p}|H_1\} \\
&= 1 - Pr\{F_{p,n-p} > F_{e^{-v/2};p,n-p}|H_1\} \\
&= 1 - \int_{F_{e^{-v/2};p,n-p}}^{\infty} f_{\Delta^2}(F)dF \\
&\approx 1 - \int_{F_{e^{-v/2};p,n-p}}^{\infty} f_{\Delta^2=0}(F) + \frac{\Delta^2}{2} f_{\Delta^2=0}(F)\Big[\frac{F-1}{1+\frac{p}{n-p}F}\Big]dF \\
&\approx (1-e^{-v/2}) - \frac{\Delta^2}{2}\int_{F_{e^{-v/2};p,n-p}}^{\infty} f_{\Delta^2=0}(F)\Big[\frac{F-1}{1+\frac{p}{n-p}F}\Big]dF
\end{aligned} \tag{30}$$

This implies

$$
\begin{aligned}
g_{H_1}(v) &\approx \frac{d}{dv}\left[(1-e^{-v/2}) - \frac{\Delta^2}{2}\int_{F_{e^{-v/2};p,n-p}}^{\infty} f_{\Delta^2=0}(F)\big[\frac{F-1}{1+\frac{p}{n-p}F}\big]dF\right] \\
&\approx \frac{1}{2}e^{-v/2} - \frac{\Delta^2}{2}\left\{\frac{d}{du}\int_{F_{e^{-v/2};p,n-p}}^{\infty} f_{\Delta^2=0}(F)\big[\frac{F-1}{1+\frac{p}{n-p}F}\big]dF\right\} \\
&\approx \frac{1}{2}e^{-v/2} - \frac{\Delta^2}{2}\left\{-\frac{e^{-v/2}}{2}\big[\frac{F-1}{1+\frac{p}{n-p}F}\big]_{F=F_{e^{-v/2};\nu_1,\nu_2}}\right\} \qquad \text{[Applying Lemma 1]} \\
&\approx \frac{e^{-v/2}}{2}\big[1+\frac{\Delta^2}{2}Q_{\nu_1,\nu_2}(v)\big], \qquad Q_{\nu_1,\nu_2}(v) = \big[\frac{F-1}{1+\frac{p}{n-p}F}\big]_{F=F_{e^{-v/2};\nu_1,\nu_2}} \\
&\approx \frac{\frac{e^{-v/2}}{2}\big[1+\frac{\Delta^2}{2}Q_{\nu_1,\nu_2}(v)\big]}{\int_0^{\infty}\frac{e^{-v/2}}{2}\big[1+\frac{\Delta^2}{2}Q_{\nu_1,\nu_2}(v)\big]dv} \\
&\approx \frac{\frac{e^{-v/2}}{2}\big[1+\frac{\Delta^2}{2}Q_{\nu_1,\nu_2}(v)\big]}{1+\frac{\Delta^2}{2}\big[E\big(Q_{\nu_1,\nu_2}(v)\big)\big]}, \qquad E\big[Q_{\nu_1,\nu_2}(v)\big] = \int_0^{\infty}\frac{e^{-v/2}}{2}Q_{\nu_1,\nu_2}(v)dv \\
&\approx \frac{e^{-v/2}}{2}\left[1+\frac{\Delta^2}{2}Q^*_{\nu_1,\nu_2}(v)\right] \qquad (31)
\end{aligned}
$$

where $Q^*_{\nu_1,\nu_2}(v) = Q_{\nu_1,\nu_2}(v) - \int_0^{\infty}\frac{e^{-v/2}}{2}Q_{\nu_1,\nu_2}(v)dv$

Expectation of V can now be obtained as

$$
E_{H_1}(V) = 2 + \frac{\Delta^2}{2}\int_0^{\infty} v\frac{e^{-v/2}}{2}Q^*_{\nu_1,\nu_2}(v)dv = 2 + \frac{\Delta^2}{2}D_{\nu_1,\nu_2} \qquad (32)
$$

$$
\text{where } D_{\nu_1,\nu_2} = \int_0^{\infty} v\frac{e^{-v/2}}{2}Q^*_{\nu_1,\nu_2}(v)dv
$$

Let's approximate the distribution of V under the alternative using the method of moments, which implies $E(V) = 2d = 2 + \frac{\Delta^2}{2}D_{\nu_1,\nu_2}$, and hence $d = 1 + \frac{\Delta^2}{4}D_{\nu_1,\nu_2}$. We can now approximate the distribution of V under H_1 as:

$$
V \sim Gamma[\beta = 2, \gamma_{\nu_1,\nu_2}] \quad \text{where} \quad \gamma_{\nu_1,\nu_2} = \big[1 + \frac{\Delta^2}{4}D_{\nu_1,\nu_2}\big].
$$

Here Gamma$[\beta, \gamma_{\nu_1,\nu_2}]$ stands for a Gamma random variable with scale parameter β and shape parameter γ_{ν_1,ν_2} with the pdf $f(x) = [e^{-x/\beta}x^{\gamma_{\nu_1,\nu_2}-1}]/[\beta^{\gamma_{\nu_1,\nu_2}}\Gamma(\gamma_{\nu_1,\nu_2})]$. By the additive property of independent $Gamma[\beta = 2, \gamma_{\nu_{11},\nu_{21}}], \cdots, Gamma[\beta = 2, \gamma_{\nu_{1k},\nu_{2k}}]$ corresponding to $V_1, \cdots, V_k$, we readily get the approximate distribution of $(V_1 + \cdots + V_k)$ as:

$$
\sum_{i=1}^{k} V_i \sim Gamma\big[\beta = 2, k + A\Delta^2\big] \quad \text{where} \quad A = \frac{1}{4}\sum_{i=1}^{k} D_{\nu_{1i},\nu_{2i}}.
$$

The local power of Fisher's test under H_1 is then obtained as follows:

$$\begin{aligned}\text{Local power (F)} &\approx \int_{\chi^2_{2k;\alpha}}^{\infty} \frac{\exp(-t/2)t^{k+A\Delta^2-1}}{2^{k+A\Delta^2}\Gamma(k+A\Delta^2)}dt \qquad \left[\text{since } \sum_{i=1}^{k} V_i \sim Gamma\left[\beta=2, k+\Delta^2 A\right]\right]\\ &= Q(\Delta^2).\end{aligned}$$

We now expand $Q(\Delta^2)$ around $\Delta^2 = 0$ to get

$$\begin{aligned}\text{Local power (F)} &\approx \alpha + \Delta^2 \int_{\chi^2_{2k;\alpha}}^{\infty} \frac{\exp(-t/2)t^{k-1}}{2^k}\left[\frac{d}{d\Delta^2}\left(\frac{(t/2)^{A\Delta^2}}{\Gamma(k+A\Delta^2)}\right)_{\Delta^2=0}\right]dt\\ &\approx \alpha + \Delta^2 \int_{\chi^2_{2k;\alpha}}^{\infty} \frac{\exp(-t/2)t^{k-1}}{2^k}\left[\frac{A\ln(t/2)\Gamma(k) - A\int_0^\infty \exp(-v)v^{k-1}\ln(v)dv}{[\Gamma(k)]^2}\right]dt\\ &\approx \alpha + \Delta^2 \int_{\chi^2_{2k;\alpha}}^{\infty} \frac{\exp(-t/2)t^{k-1}}{2^k}\left[\frac{A\ln(t/2)}{\Gamma(k)} - \frac{A\int_0^\infty \exp(-v)v^{k-1}\ln(v)dv}{[\Gamma(k)]^2}\right]dt\\ &\approx \alpha + \Delta^2 A \int_{\chi^2_{2k;\alpha}}^{\infty} \frac{\exp(-t/2)t^{k-1}}{2^k\Gamma(k)}\left[\ln(t/2) - \int_0^\infty \frac{1}{\Gamma(k)}\exp(-v)v^{k-1}\ln(v)dv\right]dt\\ &\approx \alpha + \Delta^2 A\left[E\left\{\left\{\ln(T/2)\right\}I_{\{T\geq\chi^2_{2k;\alpha}\}}\right\}_{T\sim\chi^2_{2k}} - \alpha D_0\right] \qquad (33)\\ &\quad \text{where} \quad D_0 = \int_0^\infty \frac{1}{\Gamma(k)}\exp(-v)v^{k-1}\ln(v)dv.\end{aligned}$$

Here we have used the fact that 1) $\frac{d}{dx}\left[c^{\xi(x)}\right] = (\xi'(x))(\ln(x))c^{\xi(x)}$, 2) $\frac{d}{dx}\left[\Gamma(\alpha + \beta\xi(x))\right] = \frac{d}{dx}\left[\int_0^\infty e^{-t}t^{\alpha+\beta\xi(x)-1}dt\right] = \int_0^\infty e^{-t}t^{\alpha-1}\beta\xi'(x)t^{\beta\xi(x)}\ln(t)dt$, and 3) $\frac{d}{dx}\Gamma(\alpha + \beta\xi(x))\Big|_{x=0} = \int_0^\infty e^{-t}t^{\alpha-1}\ln(t)\left\{\beta\xi'(0)t^{\beta\xi(0)}\right\}dt$, where $\beta\xi'(0)t^{\beta\xi(0)}$ is a constant. Therefore, in our context, $\frac{d}{dx}\Gamma(k+A\Delta^2)\Big|_{\Delta^2=0} = A\int_0^\infty e^{-t}t^{k-1}\ln(t)dt$. Now substituting back the expressions for A in (33) results in:

$$LP(F) \approx \alpha + \sum_{i=1}^{k}\frac{\Delta_i^2}{4}D_{\nu_{1i},\nu_{2i}}\left[E\left\{\left\{\ln(T/2)\right\}I_{\{T\geq\chi^2_{2k;\alpha}\}}\right\}_{T\sim\chi^2_{2k}} - \alpha D_0\right]. \qquad (34)$$

For the special case $n_1 = \cdots = n_k = n$ and $\nu_{21} = \cdots = \nu_{2k} = \nu_2 = n-1$, the local power of *Fisher's* test reduces to:

$$\begin{aligned}LP(F) &\approx \alpha + \frac{D_{\nu_1,\nu_2}}{2}\left[\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\right]\left[E\left\{\left\{\ln(T/2)\right\}I_{\{T\geq\chi^2_{2k;\alpha}\}}\right\}_{T\sim\chi^2_{2k}} - \alpha D_0\right]\\ &= \alpha + \frac{D_{\nu_1,\nu_2}}{2}\left[E\left\{\left\{\ln(T/2)\right\}I_{\{T\geq\chi^2_{2k;\alpha}\}}\right\}_{T\sim\chi^2_{2k}} - \alpha D_0\right]\left\{\sum_{i=1}^{k}\frac{\Delta_i^2}{2}\right\}. \qquad (35)\end{aligned}$$

V. Local power of a Jordan-Kris test $[LP(JK)]$

According to this test based on a weighted linear combination of the Hotelling's T^2, the null hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu_0}$ will be rejected if $T > a$, where $T = \sum_{i=1}^{k} C_i T_i^2$, $C_i \propto [Var(T_i^2)]^{-1}$, and $Pr\{T \approx dF_{kp,\nu} > a|H_0\} = \alpha$. In applications a is computed by using the approximation $T \approx dF_{kp,\nu}$, where $\nu = \frac{4M_2kp-2M_1^2(kp+2)}{M_2kp-M_1^2(kp+2)}$, $d = M_1(\frac{\nu-2}{\nu})$, $M_1 = p\sum_{i=1}^{k}\frac{C_i m_i}{m_i-p-1}$, and $M_2 = p(p+2)\sum_{i=1}^{k}\frac{C_i^2 m_i^2}{(m_i-p-1)(m_i-p-3)} + 2p^2\sum_{i>j}\frac{C_iC_jm_im_j}{(m_i-p-1)(m_j-p-1)}$.

$$
\begin{aligned}
\text{Power of } JK &= Pr\Big\{\sum_{i=1}^{k} C_i T_i^2 > a|H_1\Big\} \qquad [T_i^2 = n_i(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0)^t \boldsymbol{S}_i^{-1}(\bar{\boldsymbol{X}}_i - \boldsymbol{\mu}_0)] \\
&= Pr\Big\{\sum_{i=1}^{k} C_i \frac{(n_i-1)p}{n_i-p} F_i > a|H_1\Big\} \qquad [F_i \sim F(p, n_i-p)] \\
&= Pr\Big\{\sum_{i=1}^{k} C_i^* F_i > a|H_1\Big\} \qquad \big[C_i^* = \frac{C_i(n_i-1)p}{n_i-p}\big] \\
&= \int_{\sum_{i=1}^{k} C_i^* F_i > a} \cdots \int \prod_{i=1}^{k} f_{H_1}(F_i)\mathrm{d}F_i
\end{aligned}
$$

Note that $f_{H_1}(F)$ and its local expansion around $\Delta^2 = 0$ are give by

$$
f_{H_1}(F) \approx f_{\Delta^2=0}(F_{\nu_1,\nu_2})\left[1 + \frac{\Delta^2}{2}\left\{\frac{F-1}{1+\frac{\nu_1}{\nu_2}F}\right\}\right] \tag{36}
$$

Using the above first order expansion of $f_{H_1}(F)$ leads to the following local power of T.

$$
\begin{aligned}
LP(JK) &\approx \int_{\sum_{i=1}^{k} C_i^* F_i > a} \cdots \int \prod_{i=1}^{k}\left(f_{\Delta^2=0}(F_{\nu_1,\nu_{2i}})\left[1 + \frac{\Delta_i^2}{2}\left\{\frac{F_i-1}{1+\frac{\nu_1}{\nu_{2i}}F_i}\right\}\right]\right)\prod_{i=1}^{k}\mathrm{d}F_i \\
&= \int_{\sum_{i=1}^{k} C_i^* F_i > a} \cdots \int \left(\prod_{i=1}^{k} f_{\Delta^2=0}(F_{\nu_1,\nu_{2i}})\right)\left[1 + \sum_{i=1}^{k}\frac{\Delta_i^2}{2}\left[\frac{F_j-1}{1+\frac{\nu_1}{\nu_{2j}}F_j}\right]\right]\prod_{i=1}^{k}\mathrm{d}F_i \\
&= \alpha + \sum_{i=1}^{k}\frac{\Delta_i^2}{2}\int_{\sum_{i=1}^{k} C_i^* F_i > a} \cdots \int \left(\prod_{i=1}^{k} f_{\Delta^2=0}(F_{\nu_1,\nu_{2i}})\right)\left[\frac{F_j-1}{1+\frac{\nu_1}{\nu_{2j}}F_j}\right]\prod_{i=1}^{k}\mathrm{d}F_i \\
&= \alpha + \sum_{i=1}^{k}\frac{\Delta_i^2}{2}E_{H_0}\left[\left\{\frac{F_j-1}{1+\frac{\nu_1}{\nu_{2j}}F_j}\right\}I_{\{\sum_{i=1}^{k} C_i^* F_i > a\}}\right]
\end{aligned} \tag{37}
$$

where E_{H_0} stands for expectation w.r.t $F_1, \ldots, F_k$ under $H_0[F_j \sim F(\nu_1, \nu_{2j})]$.

For the special case $n_1 = \cdots = n_k = n$, the local power of this test based on a weighted linear combination of the Hotelling's T^2 reduces to:

$$\begin{aligned} LP(JK) &\approx \alpha + \sum_{i=1}^{k} \frac{\Delta_i^2}{2} E_{H_0}\left[\left\{\frac{F_j - 1}{1 + \frac{p}{n-p}F_j}\right\} I_{\{\sum_{i=1}^{k} F_i > ak\}}\right] \\ &= \alpha + E_{H_0}\left[\left\{\frac{F_j - 1}{1 + \frac{p}{n-p}F_j}\right\} I_{\{\sum_{i=1}^{k} F_i > ak\}}\right]\left\{\sum_{i=1}^{k} \frac{\Delta_i^2}{2}\right\} \end{aligned}$$

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