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# Modelled approximations to the ideal filter with application to GDP and its components

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**Abstract**

This paper develops representations of the "ideal" band-pass filter for nonstationary time series. The approach ties together frequency domain perspectives that involve periodicity and gain functions with a statistical modelling framework. The approximating filter has several advantages compared to existing methods; it has a more attractive gain profile that more accurately matches the targeted pass-band of the "ideal" filter when this is the desired gain. Also, our proposed filter addresses the sample endpoint problem associated with previous representations and allows for evaluation of the "ideal" filter's implicit assumptions about trend-cycle dynamics. Further, it reveals how filtering errors can result from the indiscriminate use of the "ideal" filter and allows one to quantify such errors. A more flexible approach is to use a modelling framework to design band-pass filters that adapt to series' properties – consistent with how the trend and cycle components evolve and relate to each other – rather than emulating a given gain function. Computer code is freely available for implementing the methodology in a way that avoids the need for an expert operator. An application to cyclical fluctuations in macroeconomic time series is presented, showing how plausible and intuitive cycles are estimated via the ideal filter or with an adaptive framework.

KEYWORDS: Business cycles, Band pass filter, Ideal filter, signal extraction, stochastic cycles, unobserved components.

JEL classification: C22, E32.

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# 1 Introduction

Cyclical patterns of expansion-contraction occur in a broad range of time series, typically with varying intensity and with changing duration around some central period. For instance, in cycles in economic data, such movements typically recur with periodicity ranging from around 2 to 10 years. The analysis and monitoring of such persistent and repetitive fluctuations around long-run levels are of great importance to a large population of policymakers, economists, and forecasters. Researchers have devoted substantial efforts to studying economic cycles and their relation to key aspects like energy prices, policymaking, and international trade, see e.g. Cotis and Coppel (2005), Artis et al (2004), Filis (2010), Chen and Mills (2012) and Blonigen et al (2015). In addition to business cycles in economic activity, over the last several years, there has been growing awareness of cycles in financial and housing indicators, which are linked in various ways to business cycles and can have distinct dynamic properties; see for example recent contributions in Chen et al (2012), Claessens et al (2012), Borio (2014), Rünstler and Vlekke (2016), Bulligan et al (2017), and Gonzalez and Marinho (2017). Stochastic cycles have also been used in research in fields other than economics, e.g. Chow et al. (2009) focusses on their application and the use of model diagnostics in psychology.

The accurate estimation of such cyclical signals is often key for understanding basic phenomena and making predictions. Band-pass filters have been applied to this problem since they remove both low and high frequency movements and focus on the mid-range frequencies associated with the cyclical component; by virtue of noise elimination, they provide relatively smooth estimates and clear indications of major transitions in the cycle. Harvey and Trimbur (2003) develop a class of band-pass filters, which generalize Butterworth filters, to estimate cyclical as well as trend components. In particular, this parsimonious class has gain functions given by compact analytical expressions that depend on order indices, and the generalized Butterworth band-pass has parameters that control the location and sharpness of the band. The filters account for the presence of trend and noise components and for differences in the trend-cycle-noise properties across diverse macroeconomic indicators. The methodology also allows for forecasting cyclical positions in a way coherent with the filtering operations and data characteristics.

Here, we generalize the existing methodology by using a damping parameter for trend-growth. The model-based band pass filters from Harvey and Trimbur (2003) are employed in various studies of business cycle fluctuations and financial and housing cycles, such as Artis et al (2004), Chen et al (2012), Moës (2012), and Chen and Mills (2012). Chen et al (2012) explore leading indicator properties of US macroeconomic variables at business cycle frequencies, while Chen and Mills (2012) investigate estimates of the output gap, or cycle in Gross Domestic Product, for the Euro area. Busetti and Caivano (2016) study the Italian business cycle using band pass filtering; Rünstler and Vlekke (2016) examine cyclical components related to both housing and economic output; and recently, Gonzalez and Marinho (2017) analyze cyclical patterns in credit and capital with a Bayesian approach to stochastic cycles as in Harvey, Trimbur, and van Dijk (2007). A problem with Harvey and Trimbur (2003) is that the filters make use of trends with ever increasing orders of integration. Here, we extend the trend so that the component remains integrated of order one, even as the persistence increases; this helps avoid implausible features like high-degree polynomials in the long-run forecast path. Hence there can be stability in trend-growth while maintaining flexibility in the low-pass filters connected to the models.

An "ideal" filter, which has a block-like gain function, is a special case of the generalized Butterworth filters. In this paper we give explicit representations, using a more general filter class than in Harvey and Trimbur (2003), that are advantageous compared to previous work. Baxter and King (1999) and Christiano and Fitzgerald (2003) focus on finite sample approximations to a particular "ideal" filter. (Throughout this paper, we keep with standard terminology and refer to a block-like filter as the "ideal" filter, while recognizing that it may be far from ideal in reality.) To emulate such a filter with prespecified business cycle periods, Baxter and King (1999) present a time-invariant approximating filter that has become popular in economics. Their approach begins with the range of periods suggested for business cycle fluctuations in the early work of Burns and Mitchell (1946), which extends from 1 1/2 to 8 years. They impose this interval stringently on the pass-band of the filter; mandating that all relevant frequencies receive an equal weight of unity and that those outside the interval be completely annihilated.

In this paper, a primary goal is to improve on the "ideal" filtering methodology used in previous literature along three dimensions: the contour of the gain function, the treatment of end-point effects, and the availability of diagnostics for the filter's appropriateness. A related aim is to expand the available methodology and allow for adaptive measurement of cyclical movements in economic indicators that have different dynamic properties. To achieve these goals, we give various modelled forms of the "ideal" filter based on extensions (to account for more general trend dynamics) of the generalized Butterworth class in Harvey and Trimbur (2003). The modelled ideal (MI) filters are designed explicitly using various parameter combinations and have a direct basis in unobserved components models that generate approximations to block-like gain functions as their optimal cycle estimators.

The MI filter solves the three key deficiencies of the Baxter-King (BK) filter. First, while the BK gain displays substantial ripples in its variation over the desired band, the MI filters' typical gain has a significantly more stable pattern and gives a far closer match to the "ideal" filter's gain function. Second, the BK filter has the drawback of truncating the extracted cycle around the end of the sample, so it omits estimates at the most important times for analysis of recent and current conditions and for making informed policy decisions; in contrast, the asymmetric MI filters that give the observation weights for near-series-endpoint estimates are provided by standard algorithms or formulas. Third, the BK filter does not allow us to evaluate the viability of the ideal filter's underlying assumptions, whereas the MI filter directly makes available statistical evidence on its underlying models' performance.

In the following, in terms of applications we are primarily interested in macroeconomic indicators and in the consistent measurement of their cyclical fluctuations. For example, the assessment of the short- to medium-term performance and "health" of the macroeconomy depends on measuring the stochastic cycles in diverse economic indicators. In contrast to having period ranges like Burns-Mitchell (1946) refer to the filter, a more satisfactory approach is to instead let them refer directly to the cyclical component – in a series that also generally has trend and noise components. This direct specification lines up better with the original intentions of business cycle researchers; in noting their intention to follow Burns and Mitchell (1946), for instance, Baxter and King (1999) explicitly state that Burns and Mitchell (1946) specified *business cycles as cyclical components* of no less than six quarters in duration and that they typically last fewer than eight years. In a modelling framework, such conditions of intermediate frequency may be used in a more direct and plausible manner and may help guide parameter values that determine trend and cycle spectra. This contrasts with emulating a certain gain for the filter – which should instead adapt to how the series' components are related. Generally, the preferred gain functions have varying curvature at low and high frequencies and a location that fits with the properties of the input series, specifically with how the trend and noise component relate to the cyclical part.

As explored in this paper, the failure to adapt and modify a prespecified ideal filter entails risks to the most basic conclusions about the cyclical properties of a time series. For instance, with the indiscriminate application of the ideal filter, serially correlated output that appears to have some cyclicality is produced from pure random noise. As another example, an artificial cycle is erroneously generated from a pure random walk trend when applying an ideal band pass. Such spurious results and errors occur because, while starting with a plausible periodicity range, filters like that of BK skip crucial steps in jumping directly to the gain function. Clearly, there are advantages of simplicity and automatic application in the use of an ideal filter, and this provides a strong motivation for the current paper's examinations of it. However, as an offset to this benefit, there is a cost to using the ideal filter that can be substantial and that depends on the input series. These costs may be understood via the modelled version of the ideal filter set out below.

The paper proceeds as follow. In Section 2, we present the model-based representations of the ideal filter and compare them with the BK approach. With given sets of parameter values for different indices, the class of generalized Butterworth filters, along with a further generalization related to flexibility in the trend, is brought to bear on the design problem. Section 3 discusses and highlights the crucial limitations of the ideal filter with illustrations. Section 4 aims to clarify filter formation from underlying components, which can be used for a wide range of problems, and show how the component dynamics in the frequency domain feed into the design of filters. The decomposition model for macroeconomic data is based on prior notions of component behavior, that involve business cycle knowledge; the optimal *estimator* of the cycle then generates band-pass filter classes. In Section 5, we give an explicit framework with unobserved components models, which express the series' statistical behavior and give a quantitative lens for interpreting dynamics of trend-cycle-noise and their inter-relationships. Here, we make adaptive band-pass filtering operational through a choice of parametric models for stochastic trend, cycle, and noise components followed by optimal signal extraction. Parsimonious models are considered with standard models for stochastic trend, and with different variations of a stochastic cycle model. Analytical gain functions of the generalized Butterworth filters and the corresponding filters for related models follow from derivation of minimum mean-square error cycle estimates. Section 6 then compares the modelled ideal filter to that of the BK and adaptive filters in an application to extracting economic cycles. We consider a dozen time series of national accounts data, taken from the Bureau of Economic Analysis, that include Gross Domestic Product, Investment, Consumption, and other major categories of economic output. For certain series, it turns out that the cycle is either artificially dampened or incorrectly amplified by the ideal filter, whereas an adaptive filter accommodates the different cyclical intensities found among the series. Section 7 concludes.

# 2 Ideal filter approximations: representation and evaluation

The main goal of this Section is to form model-based versions of the ideal filter to more effectively handle situations when such a filter represents a tenable approach. Note that, as a general strategy, rather than impose conditions on the gain, the current paper suggests a different approach, where the knowledge about business cycles and their periodicity refer to the components and where the filter is adapted to the input series. Nevertheless, in certain cases, an ideal filter may be a reasonable notion to entertain. Therefore, instead of going from model to filter, in this Section we proceed in the opposite direction and study the model that implies a certain emulated gain function. In this case, the target is the ideal filter with gain indicated in figure 1 below. We first highlight the basic decomposition that is implicit in the use of all filtering techniques and that is made explicit by specification of models with stochastic trend and cyclical components. Next, the BK filter is reviewed and some major shortcomings are noted. The rest of the Section is then devoted to the development of modelled representations of the ideal filter.

## 2.1 Decomposition and unobserved components

Band-pass filters aim to remove both low and high frequency movements in a time series, and so can give smoother estimates and clearer indications of major transitions in the cycle. The basis for the design of a band-pass filter – its purpose and necessity – is to extract a cyclical component $\psi_t$, made up of a mid-range frequency spectrum, from a series $y_t$ that has other components:

$$y_t = \mu_t + \psi_t + \varepsilon_t, \quad t = 1, ..., T \tag{1}$$

where $\mu_t$ is composed of low-frequency parts and may be interpreted as stochastic trend and $\varepsilon_t$ captures any remaining noise.

In this paper, as empirical examples, we consider applications drawn from economics, for which the decomposition in (1) is quite natural and is connected with business cycle interpretations. Hence, the popularity of the ideal band pass filter of Baxter and King (1999) is not entirely surprising. Most major economic series such as real GDP have a trend dominated by frequencies at the low end of the spectrum. For an activity- or output-related variable, the trend component is dominated by forces such as changes in population and productivity; it is described by growth theory and evolves slowly over time. For other series, such as the unemployment rate, there may be other kinds of demographic factors behind the trend. The cycle component has rather different properties, representing the expansions and contractions that tend to recur around the trend as demand conditions adjust; these fluctuations have some central frequency in an intermediate (business cycle) range. Lastly, most economic data is subject to measurement error or is influenced by idiosyncratic effects unrelated to trend or cyclical movements as well; hence the need for the band pass to cut out the highest frequencies.

In a later section we consider an explicit class of stochastic models of the form (1), which gives a scientific and precise base for filtering methodology. This allows the use of statistical modelling principles and adaptive filter design. In the current section, we proceed directly to filtering in the frequency domain to discuss representations of the ideal filter.

## 2.2 The Baxter-King filter

Many researchers have applied a strategy of gain emulation and have in particular sought to reproduce the "ideal" filter[1], which has a perfectly block-like gain function, as closely as possible in a finite sample. The simple filter by Baxter and King (1999) [henceforth abbreviated as BK filter] represents the most popular representation for economic data. This sub-section compares an approximating modelled filter with the BK version.

In macroeconomic work, Baxter and King (1999) proposed using the interval from 6 to 32 quarters. The perfectly sharp gain then passes through this particular interval of periods associated with business cycles. The corresponding frequency range has lower limit $\lambda_l = \pi/16$ and upper limit $\lambda_u = \pi/3$. The filter has equivalent impact on the amplitudes of different frequency parts within that interval, as indicated by the dashed line in figure 1.

Given a truncation parameter $K$, the BK filter may be expressed in terms of the lag operator $L$ as

$$BK(L) = \sum_{j=-K}^{K} b_j L^j$$

where the coefficients are

$$\begin{aligned} b_j &= (1/|j|\,\pi)[\sin|j|\,\lambda_u - \sin|j|\,\lambda_l] + c_u - c_l, \quad j \neq 0 \\ b_0 &= c_u - c_l \end{aligned}$$

with the constants $c_u, c_l$ given by

$$c_u = \frac{1 - \sum_{j=-K}^{K} \sin|j|\,\lambda_u/(j\pi)}{2K+1}, \qquad c_l = \frac{1 - \sum_{j=-K}^{K} \sin|j|\,\lambda_l/(j\pi)}{2K+1}$$

The construction of the filter involves the choice of $K$; increasing this parameter gives a closer finite sample approximation but leads to the omission of $K$ estimates at both the beginning and end of the

[1] The use of the word "ideal" implies an innate preference for this sharp gain function. But such a rigid prior on periods is often inappropriate for the gain, as illustrated in later sections. It may be more ideal for the filter to account for the spectral shape of the cycle consistent with the input series and with its trend; in economics, we expect the cyclical part's spectral peak to lie within a vaguely defined business cycle range and which otherwise can have flexible characteristics. We use the standard terminology "ideal" in this paper only as an adjective describing the block gain shape.

sample. The authors suggest using $K = 12$ for quarterly data as a compromise between diminished accuracy in representation of the filter and loss of information near sample endpoints.

A highly unsatisfactory aspect of the BK filter is that, in the crucial pass-band region of intermediate frequencies, the gain function displays large, undesirable ripples. Figure 1 shows the gain of the BK filter (the dashed-dotted line) vs. the "ideal" gain (dashed line) with $K = 12$. As the frequency increases starting from the low end, the BK gain first rises to around 1.05, then falls back to below 0.95; finally, the gain rises to nearly 1.10 before finally tapering off toward zero at the right edge of the pass-band interval. Additional oscillations in the gain function of lesser consequence occur at higher frequencies.

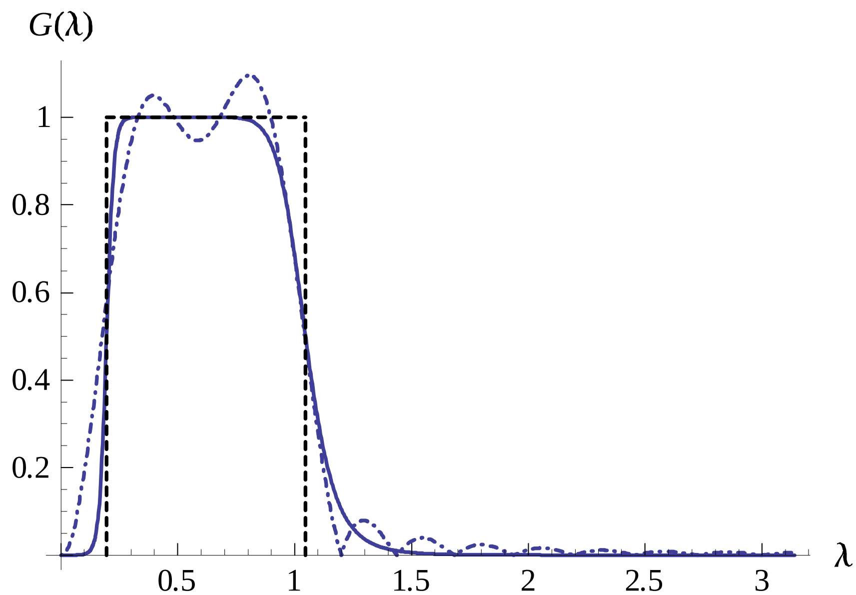


Figure 1: Gain functions for BK filter (truncation of twelve observations, as recommended for quarterly time series) shown as dotted-dashed line, ideal gain shown as dashed line, and model-based representation of ideal filter indicated by solid line.

## 2.3 Ideal Filter – Modelled Versions

We consider generalizations of the Butterworth class of filters whose gain functions have the following compact forms:

$$GB^{lp}_{m,n}(\lambda;\phi,\lambda_c,\rho,q_\zeta,q_\kappa) = \frac{q_\zeta\left[\frac{1}{(2-2\cos\lambda)(1+\phi^2-2\phi\cos\lambda)^{m-1}}\right]}{q_\zeta\left[\frac{1}{(2-2\cos\lambda)(1+\phi^2-2\phi\cos\lambda)^{m-1}}\right] + q_\kappa\left[\frac{1+\rho^2\cos^2\lambda_c-2\rho\cos\lambda_c\cos\lambda}{1+\rho^4+4\rho^2\cos^2\lambda_c-4(\rho+\rho^3)\cos\lambda_c\cos\lambda+2\rho^2\cos 2\lambda}\right]^n + 1} \tag{2}$$

and

$$GB^{bp}_{m,n}(\lambda;\phi,\lambda_c,\rho,q_\zeta,q_\kappa)=\frac{q_\kappa\left[\frac{1+\rho^2\cos^2\lambda_c-2\rho\cos\lambda_c\cos\lambda}{1+\rho^4+4\rho^2\cos^2\lambda_c-4(\rho+\rho^3)\cos\lambda_c\cos\lambda+2\rho^2\cos 2\lambda}\right]^n}{q_\zeta\left[\frac{1}{(2-2\cos\lambda)(1+\phi^2-2\phi\cos\lambda)^{m-1}}\right]+q_\kappa\left[\frac{1+\rho^2\cos^2\lambda_c-2\rho\cos\lambda_c\cos\lambda}{1+\rho^4+4\rho^2\cos^2\lambda_c-4(\rho+\rho^3)\cos\lambda_c\cos\lambda+2\rho^2\cos 2\lambda}\right]^n+1} \tag{3}$$

where $0<\phi,\rho<1$ and $q_\zeta$, $q_\kappa>0$. The notation makes explicit the dependence on parameter values. The five parameters $\{\phi,\lambda_c,\rho,q_\zeta,q_\kappa\}$ all have interpretations related to implicit time series models of trends and cycles; in this Section we focus on their role in determining the form of the gain function. Note that this form is more general than that in Harvey and Trimbur (2003) since it allows for values of the trend-related parameter $\phi$ less than unity; the merits of this added flexibility are discussed with reference to underlying statistical models below.

Given positive integers $m$ and $n$, $\{GB^{lp}_{m,n}(\lambda;\lambda_c,\rho,q_\zeta,q_\kappa),GB^{bp}_{m,n}(\lambda;\lambda_c,\rho,q_\zeta,q_\kappa)\}$ stands for a low-pass and band-pass filter pair, each of order $m,n$ and defined mutually for internal consistency. The positive integer $m$ denotes the low-pass index and $n$ the band-pass index of the pair of filters; this terminology refers to the fact that $n$ primarily influences the band pass filter and $m$ the low pass filter of a given collective, though both $m$ and $n$ affect the precise forms of $GB^{lp}_{m,n}$ and of $GB^{bp}_{m,n}$ jointly. The parameter $\lambda_c$ is a major frequency that determines the location around which the band-pass filter is concentrated. The parameters $q_\zeta$ and $q_\kappa$ are referred to as "signal-noise" ratios for the trend and cycle, respectively, due to their connection with unobserved components models that is explained in more detail below; they influence the gains' location and spread. The parameters $\phi$ and $\rho$ mainly determine the filters' width and curvature.

Expression (3) encompasses a wide variety of gain shapes for the band-pass filter, which can be tuned to the desired shape by an appropriate setting of orders and parameters. The orders $m,n$ directly affect the sharpness of the filters, with higher $m$ being especially pertinent for more rectangular low-pass filters and higher $n$ for sharper band-passes. The two filters $\{GB^{lp}_{m,n}(\lambda;\lambda_c,\rho,q_\zeta,q_\kappa),GB^{bp}_{m,n}(\lambda;\lambda_c,\rho,q_\zeta,q_\kappa)\}$ in a given pair work in tandem and have complementary gain functions, which tend to focus on different areas of the frequency interval, while also sharing some overlap at contiguous frequencies. Here, we are primarily interested in the band-pass member of each pair; the band-pass filter is able to extract out the mid-range part of the spectrum in a way that concurs with how the low-frequency part is extracted by the complementary low-pass. For other purposes, note that if we were to focus attention on trend estimation, the low-pass filter in (2) improves on techniques like the HP filter (which is given by $GB^{lp}_{m,n}(\lambda;\lambda_c,\rho,q_\zeta,q_\kappa)$ by setting $m=2$, $\phi=1, q_\zeta=1/1600$, and $q_\kappa=0$), since it now adapts to the cyclical movements of the input series.

Using the parametric class given in equation (3), we can choose various combinations of parameter

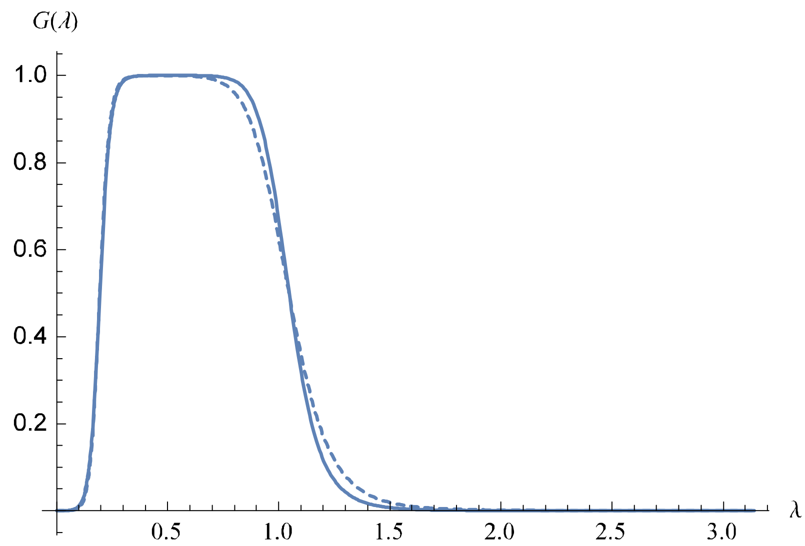


Figure 2: Gain functions for different ideal filter approximations with $n = 6$,both with $\rho = 0.8, \phi = 0.97$. For the solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.0124,\ 0.0322,\ 0.4910\}$, while for the dotted curve, $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{2.524,\ 0.279,\ 0.398\}$.

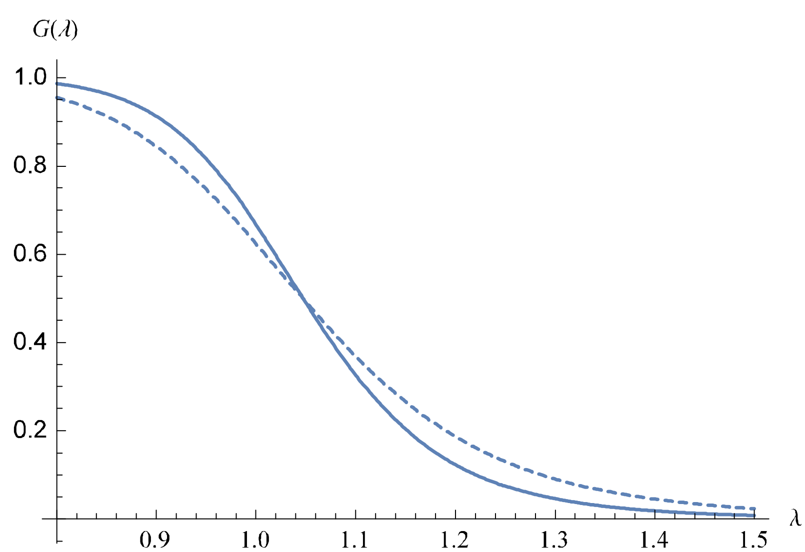


Figure 3: Close-up showing a high-frequency part – the right-side tail – of gain functions for different ideal filter approximations with $n = 6$,both with $\rho = 0.8, \phi = 0.97$. For the solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.0124,\ 0.0322,\ 0.4910\}$, while for the dotted curve, $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{2.524,\ 0.279,\ 0.398\}$.

values to design approximating filters for the desired gain profile, which in this case has the perfectly sharp profile with pre-specified boundaries related to business cycle periodicity. To achieve sufficient sharpness of the gain and emulate the block-like shape of the "ideal" filter, we require relatively high values for $n$. For the low-pass order, we set $m = 2$; higher values could be used, but they are not necessary for generating sharp band pass filters, and $m = 2$ is underpinned by a simple, plausible trend model as explained below. To explore the properties of different band-pass order representations, we use three alternative values given by $n = 4, 6$, and 8. The order $n$ represents a choice of trade-off between filter length in the time domain (which relates to achievable accuracy of finite sample estimates) and sharpness of gain function. Other aspects of the trade-off pertain to implied statistical models; the number of total processes needed in defining the model is linked to $n$, so that a sixth order case has 12 total cycle and auxiliary elements. Also, models with higher $n$ require more computing time for estimation and smoothing. In theory, arbitrary positive integers could be used for $m$ and $n$. However, very high values of $m$ would imply implausible implied trend models, as noted later on. We have experimented with filters having $m \leq 5$ without computational hurdles. In terms of the band-pass index $n$, values of at least 10 are feasible computationally; since the incremental differences in raising $n$ become very slight at this point, we did not investigate even higher values, though they remain a possibility that could be helpful depending on the application.

Given the three candidate values of the order $n = \{4, 6, 8\}$, substantial sharpness in the gain also necessitates a sufficiently large damping factor; for the current application, the value $\rho = 0.8$ is chosen. Lastly, a high value of $\phi$ (as discussed below, this relates to the persistennce of trend-growth) is needed, so the value 0.97 (between the lower bound of 0.95 used for quarterly data applications and unity) is used. Using the orders {4,6,8} gives a broad range of gain sharpness - orders less than 4 are unlikely to produce decent approximations, while increasing the order to above 8 involves some undesirable features with excessively intense cyclical patterns in end of series estimates (this implies an extremely resonant cycle that has implausible forecasts). Furthermore, in the evolution toward a block-like filter, as $n$ rises, the benefits begin to reach a limit at very high orders, i.e. the advances from 4 to 6 are more noticeable than those from order 6 to 8, whereas incremental improvements in gain sharpness become rather small after order 8.

For a given order, $GB^{bp}_{m,n}(\lambda)$ also depends on the primary frequency parameter and two major q-ratios, which represent the three remaining filter parameters: $q_\zeta$, $q_\kappa$, and $\lambda_c$. The flexibility in the approximating gain is achieved – apart from changing $n$ – by using different combinations $\{q_\zeta, q_\kappa, \lambda_c\}$, which are obtained as the solutions to three equations. The first two equations are composed by setting the gain equal to one-half at the business cycle boundaries, that is $GB^{bp}_{m,n}(\lambda) = 1/2$ for $\lambda = \pi/16$, $\pi/3$

corresponding to periods of 1 1/2 and 8 years. A third equation is formed by setting the gain equal to a high value very close to unity (the gain will never be exactly one though it may get very close) at a mid-range frequency, that is $GB^{bp}_{m,n}(\lambda) = 1 - \varepsilon$ when $\lambda = 0.55$ for some small $\varepsilon > 0$. This constitutes three equations in three unknowns that may be solved numerically. In practice, this was achieved using a program written in the Mathematica language of Wolfram (2003); this approach is both simple and very fast to execute. By varying $\varepsilon$ monotonically and obtaining the solutions $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\}$, the designed gain is gradually adjusted in rectangular resolution and fidelity to the ideal gain.

For each $n$, there is a certain span in the right tail's sharpness that can be attained, which translates into different qualities of noise removal. To illustrate this, for $n = 6$ twelve representative combinations for a certain set of $\varepsilon$'s, are selected among the various solutions – there is a single solution for each specific $\varepsilon$, so that by varying $\varepsilon$ different parameter triplets are obtained. As $\varepsilon$ is adjusted monotonically, $\overline{q}_\zeta$ and $\overline{q}_\kappa$ also vary monotonically. Figure 2 displays the two extreme cases considered among the twelve possibilities, in the sense of being the sharpest vs. the least sharp. Ten other combinations with $\overline{q}_\zeta$ and $\overline{q}_\kappa$ lying between the values in the extreme cases are also considered. In figure 2, the dotted line represents the approximation with the least noise elimination and corresponds to $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{2.524, 0.279, 0.398\}$, while the solid line gives the sharpest gain obtained for $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.0124, 0.0322, 0.4910\}$.

The close-up in figure 3 reveals the contrast in the right-side tail in greater detail and clearly indicates how the filter with smaller values of $\overline{q}_\zeta$, $\overline{q}_\kappa$ eliminates more high-frequency parts. The discrepancy in low-frequency removal is more modest and is detected through examination of the close-up in figure B.1 in Appendix B. Between these two extremes is a sequence of filters with gains enclosed by the two shown in the figures, where in each case the gain equals one-half for periods of 1 1/2 and 8 years.

A similar type of filter range occurs for the order set to 4, and the gain is less block-like than for sixth order. Figures B.2, B.3, and B.4 in Appendix B show how the span between the gains, with the slowest relative to the most rapid descent at higher frequencies, appears somewhat larger. On the other hand, for $n = 8$ there is less apparent variability in the frequency response's contour among the possibilities entertained.

## 2.4 Determination of ideal filter representations

Note that the suitability of the ideal filter representation depends on the type of series in question. Hence in Section 6, a number of major macroeconomic time series representing Gross Domestic Product and its components are analyzed to explore patterns in the appropriateness of various approximations and variation in results across data series. These 12 time series are chosen because they indicate the most important measure of activity in the domestic economy, and its major components have different dynamic properties, making it fruitful to analyze the different major components separately in order to understand overall activity. For a given series, each representation (12 combinations for each of $n = 4, 6$, and 8 so 36 in total) of the ideal filter is fitted by maximizing the log-likelihood subject to the constraints implied by the filter parameters, which consist of variance ratios. The full parameter vector includes all variance parameters and slope constants that describe the series' dynamics, while the specification of $q's$ in the filter's formula omits information about a scale factor and about trend growth rates. Note that the differential behavior in the right tail of the gain function across different approximations means that different quantities of noise will be removed from an input series, which will affect the smoothness of the extracted cyclical signal. The estimated noise variance depends on the $q$ values, already specified in the filter form, and relates to the observed process's scale.

For order 6, the parameter estimates are contained in table set D3 in Appendix D, while the corresponding diagnostics are reported in table set D4. For each series, various fit statistics help determine the best representation from among the 12 possibilities. Comparing these choices across the 12 national accounts variables leads to the selection of a specific representation from among the 12 candidates. For each of the three band pass orders, a representation was selected in this way, where the aim is to focus on approximate ideal filters modelled for quarterly US series related to economic activity. Details are provided below in the Application Section.

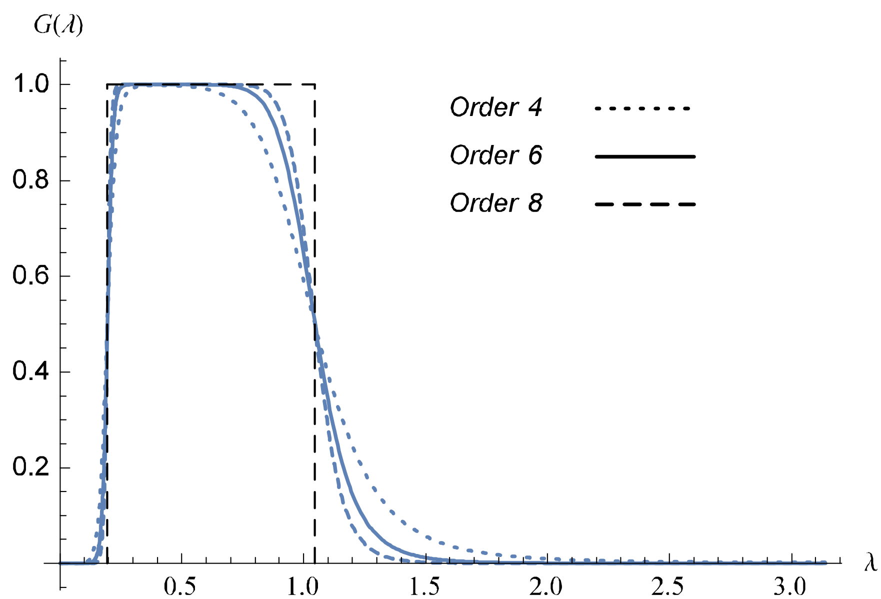


Figure 4: Three modelled representations of the ideal filter for various orders.

Figure 4 shows the three gain functions corresponding to representations of order 4, 6, and 8. The

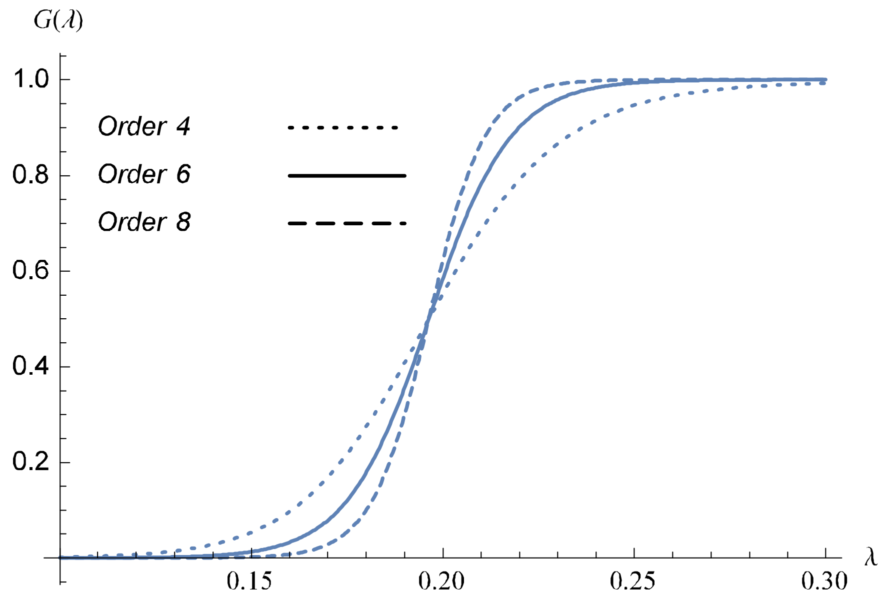


Figure 5: Three modelled representations of the ideal filter for various orders, with a focus on the low frequency region.

parameter settings for $\{\overline{q}_{\zeta}, \overline{q}_{\kappa}, \overline{\lambda}_c\}$, as determined based on diagnostics and fit statistics for a set of U.S. macroeconomic time series, are {0.04946, 0.04589, 0.4611} for $n = 6$, {0.05722, 0.174, 0.4146} for $n = 4$, and {0.05188, 0.01226, 0.4815} for $n = 8$. In terms of fidelity to the ideal gain, there is a noticeable improvement in the approximation in going from fourth to sixth order, in that the right side of the gain around the higher frequency cutoff is more cleanly captured; components with larger frequencies are more effectively removed. Increasing the cycle index further to $n = 8$ also leads to an improved approximation, though the changes are more modest. There are similar alterations in the left-hand side of the gain, as illustrated in the close-up plot in figure 5, which focusses on a sub-interval of low frequencies. These differences appear small in figure 4 but are brought out more clearly in figure 5 once the focus rests on the more limited span of frequencies relegated primarily to trend. Similar to the high frequency part of the spectrum, in the low frequency area there is a significant adjustment toward a block-like gain shape as $n$ increases from 4 to 6, and further moderate changes in moving to order 8 in figure 5.

Figure 1 above shows the preferred ideal filter representation for order $n = 6$; recall the most notable contrasts with the BK filter: $GB^{bp}_{m,n}(\lambda)$ cuts out more low-frequency content and is far smoother than BKs. In applying any given filter, the effect of the operation is to take a weighted average of the observations. We compute the implied weights for the model-based version of the "ideal" filter using the "zero-one" method; these results are compared to the BK weights in figure B.5. The patterns for the two filters appear close for many separations; however, there are some differences in weights that can be gleaned from the plot, in addition to non-zero weights at lags beyond the 12-period truncation for the modelled version. This divergence in the weights at bigger separations is reflected in more apparent discrepancies in the gain function, where the longer kernel of the model-based filter leads to the more attractive frequency profile displayed in figure B.5. Appendix B gives some additional details on the design of suitable approximations to the "ideal" filter.

Note that this analysis of gain functions and filter design has been conducted with economic data in mind. In a different type of application, say with data related to the solar cycle – manifested in sun-spot numbers or intensity for instance – the boundaries of the ideal filter could be changed to more appropriate values, perhaps surrounding 8 to 14 years or so. (Likewise, with respect to the modelling strategy described below, the interval restriction for the central frequency parameter could be adjusted – or simply discarded if the likelihood function is sufficiently well-defined around a maximum for some plausible period). The same approach to approximation may then be used to design suitable filters for the modified range.

# 3 Flexibility in filter design and problems of the "ideal" filter

In certain applications in engineering (from which field it originated), the "ideal" filter may indeed represent the choice strategy; for time series in economics and the social sciences, however, this is typically not the case. The entire point of using a band-pass in the first place is that the cycle component combines with other components. We must apply some operator, or weighting function, to the observations to extract out the fluctuations of interest in the presence of these additional components. Below, we show how the gain functions of optimal filters depend on, for instance, the typical strength of cyclical movements, the type of trend, and the degree of variability of trend shocks and irregular movements.

In general, the stochastic nature of fluctuations and the composition of the chief dynamics warrants a more nuanced and careful treatment to avoid the kind of pitfalls illustrated in the next sub-section. In particular, for two simulated series, we examine the output from the empirical filter of Baxter and King (1999); though they proposed a specific simple and fixed approximation to the "ideal" filter, the conclusions hold generally for any reasonably close approximation. For the first case, we use an elementary stochastic trend - the simple random walk case; for the second case, we filter a purely random process. Both examples are devoid of cyclical dynamics, and they provide fundamental and poignant illustrations of the substantial risk of fallacious results with a non-adaptive filter and how, at a minimum, it is preferable to consider a broader array of gain functions that have some degree of flexibility.

This section illustrates two basic examples of spurious results coming from the automatic use of the "ideal" filter as motivation for our modelling methodology. We start with a series given solely by a stochastic trend. For this case, with all changes permanent, when extracting a cycle, it is preferred that the filter produce a series of negligible values and attribute most of the movements to trend. If however, a fixed and pre-specified band-pass filter is applied, such as the "ideal" filter, then a cycle is effectively created by the filtering. For the second example, with series equated to white noise, the spurious generation of cycle by the "ideal" filter is also a severe problem; our adaptive design method evades this pitfall.


Figure 6: Simulated random walk (dotted green line) and estimated trend (solid blue line) using model-based approach.

## 3.1 Simulated illustration I: Random walk trend

Consider as a first case a time series that has only a trend component, $y_t = \mu_t$. This could be viewed as a limiting situation of a variable that has a very weak cycle and very little additional noise, such as a core inflation series with volatile price components removed. The simplest stochastic trends, random walks for I(1) series or smooth trend models for I(2) series, are widely used in time series modelling.

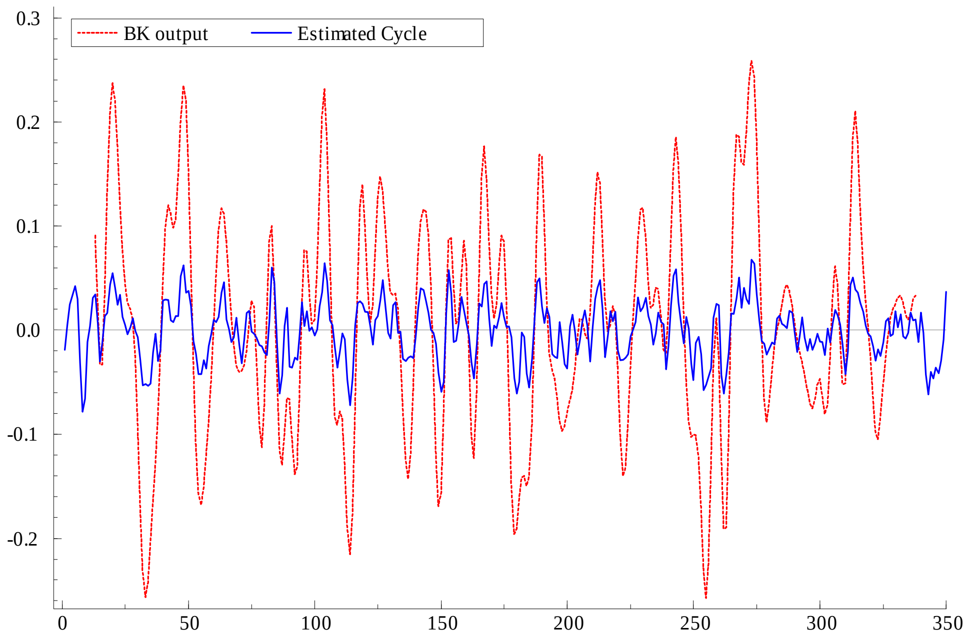


Figure 7: Estimated cycle (solid blue line) using model-based approach, along with output from the BK filter (dotted red line), for a simulated random walk.

Here, we consider for the nonstationary part a random walk $\mu_t$ defined by

$$\mu_t = \mu_{t-1} + \zeta_t, \quad \zeta_t \sim WN(0, \sigma_\zeta^2) \tag{4}$$

Suppose we have a simulated series for $\mu_t$ based on disturbance variance of $\sigma_\zeta^2 = 0.01$ and a starting value of $\mu_0 = 3.00$ with $T = 350$, as displayed in figure 6. The trend generally meanders down over the sample period, with the pace quickening in the latter part. Figure 6 also shows the estimated trend from fitting a time series model according to the method described in Sections 4 and 5. Figure 7 displays the BK output, along with the estimated cycle. The span of the BK output, which might be erroneously associated with a cyclical amplitude, is far greater than that of the model-fitted cycle. Additionally, there is a good deal more persistence in the BK-generated series, and it appears to possess peaks and troughs and spurious qualities that would be expected from the estimation of a veritable cyclical process. Now, in this case where the series contains only a trend, it is desirable to reach a conclusion as close as possible to the non-existence of a cycle or a very weak one at most. It is clear from figure 7 that the BK filter fails in this regard.

The previous example was based on a nonstationary trend component. It might seem that there are special implications of applying a band-pass filter in the presence of nonstationarity given the theoretical dichotomy between stationary and nonstationary dynamics. This point is raised for the HP filter in

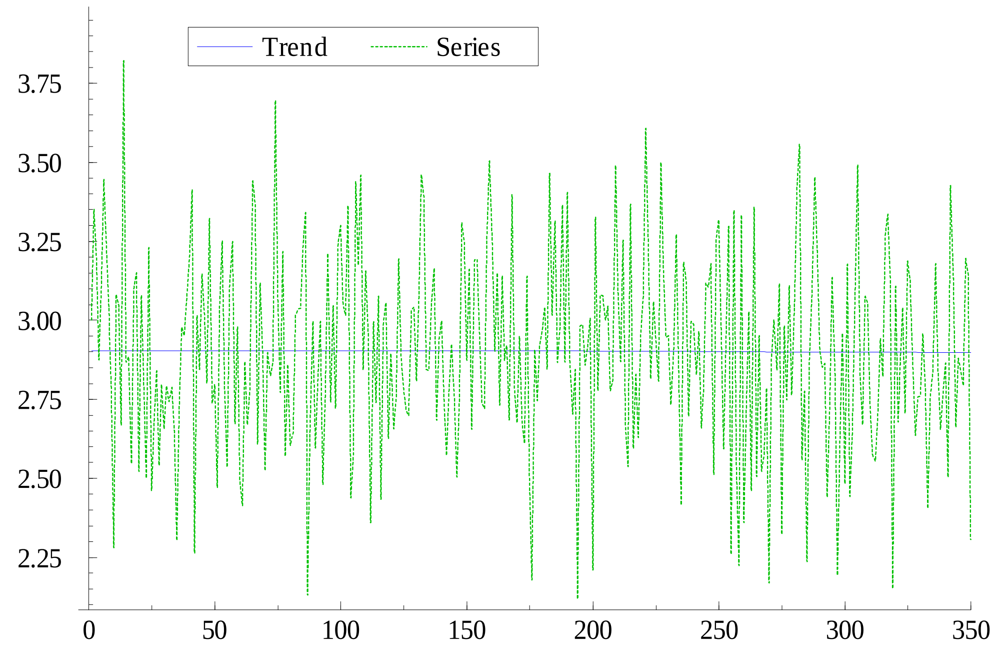


Figure 8: Simulated Gaussian white noise (green, dotted line) and estimated trend (blue, solid line) using model-based approach.

Harvey and Jaeger (1993), while Murray (2003) considers the "ideal" filter in this manner. However, as discussed in Appendix C, the root cause of the filtering problem is the mere presence of other components with spectral power in the band of interest, whether stationary or nonstationary.

## 3.2 Simulated illustration II: White noise irregular

To give a different representative example and to illustrate that filtering distortions do not critically depend on nonstationarity, we now consider a second case where adaptive filtering may help avoid serious errors in analysis. Suppose the pre-specified band-pass filter is mistakenly applied to a pure irregular process. The simulated series is now $y_t = \mu_0 + \varepsilon_t$, where

$$\varepsilon_t \sim NID(0, \sigma_\varepsilon^2), \quad t = 1, ..., T$$

Figure 8 shows such a series that has no stochastic trend or cycle present; the simulation is based on an irregular variance of 0.1 and a constant mean of $\mu_0$ =3.00 again with $T = 350$. The results of using the Baxter-King band-pass are shown (as the dotted line), along with the adaptive results (as the solid line). The adaptive results were again obtained by using a model-based approach and first estimating a general model with all three components prior to filtering. In figure 9, it can be seen that the BK filter gives rise to an apparent cycle that has more persistence than a white noise process and that has

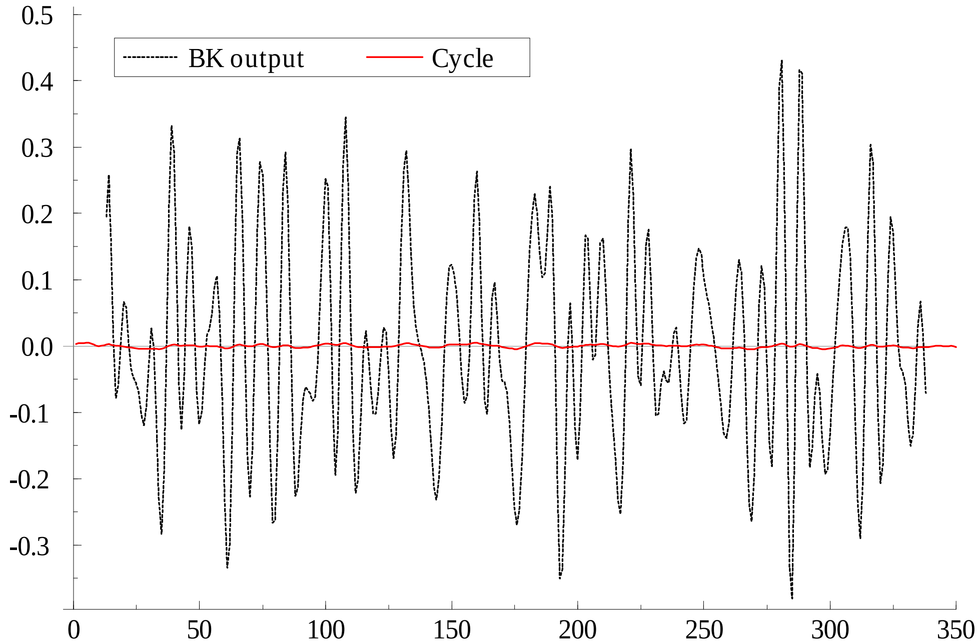


Figure 9: Estimated cycle (red, solid) using model-based approach and output from BK filter (black, dotted) for simulated Gaussian white noise.

peaks and troughs at short intervals; this result constitutes a completely spurious cycle being created out of pure noise. Using such a false discovery for policy or other major decisions could prove extremely detrimental. In contrast, the modeling approach gives the correct finding of the absence of cyclical components in the series.

# 4 Cyclical movements and adaptive gain functions for diverse time series: basic considerations

This Section sets out the framework underpinning the adaptive estimation of cyclical movements. This general formulation allows us to develop ideal filtering methodology that has several advantageous aspects and that gives a way to understand and evaluate the implicit assumptions, on the modelling side, involved in the application of an ideal filter approximation. We consider how an appropriate band-pass filter seeks to extract a cyclical component from an observed series *in the presence of other components*; then the defining characteristic of the band-pass means having its gain concentrated around an adjustable span of intermediate frequencies – in a malleable fashion.

We first discuss the flexible expression of cyclical periodicity that is designed to let the data speak and account for different quasi-periodic properties across series. These moldable cyclical notions are used in combination with trend and noise, and directly expressed in unobserved components form. Then, signal extraction methods can be employed to decipher the precise structure of the gain function of the optimal band-pass.

At the core of the problem, a given set of observations must be filtered to estimate cyclical components when additional components exist that can be confounded. Most macroeconomic series are nonstationary and usually contain a prominent stochastic trend as well as being subject to short-lived or temporary factors. Indeed, the premise for using band-pass filters is to extract the cyclical component from a time series by removing such low- and high-frequency components. From a time series perspective, we can express this structure with trend-cycle-noise dynamics as in (1), where $\psi_t$ is a stationary component having cyclical properties[2] and $\varepsilon_t$ is higher-frequency noise. We stress that such decompositions are already implicit in the gain of a band-pass filters; an econometric specification of the model makes explicit their stochastic form of components and provides a theoretical basis and quantitative approach for estimation of cycles. In this way, the unobserved components approach with structural trends and cycles provides a bridge between frequency-domain ideas and strategies on the one hand, and statistical and econometric modelling on the other.

## 4.1 The expression of periodicity through cyclical spectra

From the frequency domain perspective, we use a versatile concept of cyclical process as reflected in a spectral peak at some mid-range period. This general definition allows for diverse cases; three spectral examples are shown in figure 10. A certain region of central frequencies receives the most emphasis, and in having a gradual decline in the spectrum about the peak, there is generally a continuous change in weights. This means that nearby frequencies are weighted similarly, rather than going from a significant positive weighting to being suddenly and sharply deleted when the frequency changes by very small amounts. In contrast the ideal filter's definition via the gain function involves discontinuity at the high and low cutoff frequencies.

In figure 10, the dashed line represents a sharply defined cycle, as the amplitude of frequency parts decreases rapidly as the distance increases from the central frequency of maximal spectral power. The dotted line indicates a cyclical process of higher frequency at its peak and with a larger spread; compared to the sharp apex case, its movements can be expected to show more times of either very persistent or very noisy stretches within a general upswing or downturn; essentially the character of fluctuations shows

[2] If a band-pass filter is used to extract the stationary component, then clearly we expect the stationary component to have strong mid-range components characteristic of a stochastic cycle.

more variety relative to a simple fixed cycle. Finally, the spectrum of a longer periodicity stochastic cycle is shown by the solid line. In addition to having a lower central frequency, the process also has substantial power around the lowest frequencies near the origin, reflecting significant persistence in its fluctuations. Overall, these three examples are intended to illustrate the generality of the cases considered, which aims to cover the wide array of stochastic cycles that may be encountered in practice.

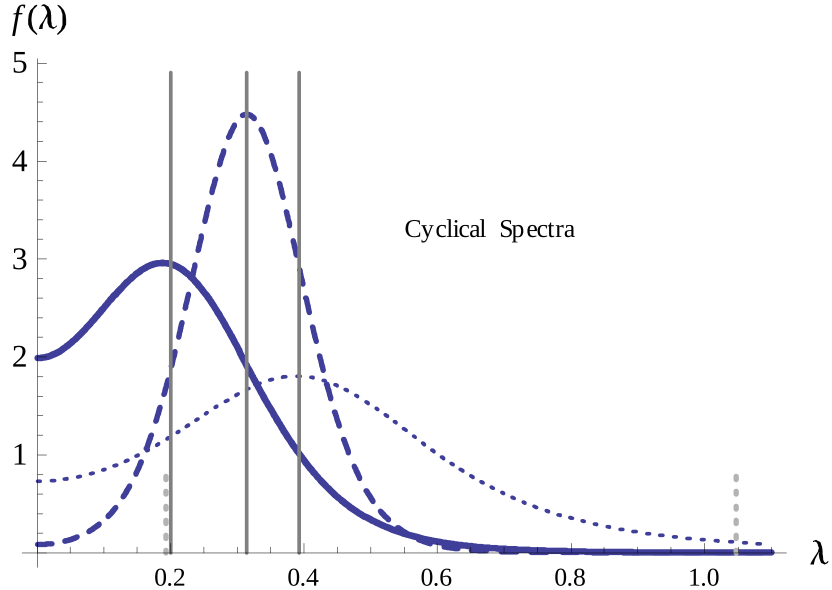


Figure 10: Three examples of a stochastic cyclical process, represented in the frequency domain, that have a peak at a mid-range period. The solid curve is the spectrum of a highly persistence cycle. The dotted curve represents a weakly defined cycle. The dashed line is the spectrum of a pronounced stochastic cycle. The vertical solid lines mark their central frequencies. The vertical dotted segments mark the business cycle interval end-points with periods of 8 (left) and 1 1/2 (right).

In macroeconomics the basis for business cycle analysis is our knowledge about previous upturns and downturns in the U.S. economy, which have had an average period of around 5 to 8 years; this is suggestive for the spectral peak location. The cyclical movements have displayed a range of different realized durations and intensities across specific historical episodes. Previous patterns in macroeconomic growth suggest that a full swing in the business cycle – a sequence of expansion followed by contraction – of about six to eight years is more likely than a short boom-bust pattern of around two years or than an extra long swing of say, twelve years or more. This suggests for the fluctuations a range of possible periodicity – so the appropriate spectrum may have a good deal of breadth around the peak that occurs around some intermediate period – and with declining power away from the peak. We recognize that such notions must be applied only loosely for economic time series, since each individual time series will have its own idiosyncrasies. Therefore the range, location, and other aspects of cycle spectrum naturally differ across indicators having their own individual properties.

Further, we don't know in advance about the exact properties of the cycles before the analysis; indeed, the modelling of business cycles remains subject to uncertainty, and one goal of this paper is to shed new light on this area. We allow for flexibility in cyclical spectra across different economic series, which then implies malleable filters for business cycle movements, with the detailed structures being guided primarily by the data. There is also the consideration that business cycles may have evolved and may show longer typical duration in the post-WWII period; the typical period may have changed from say 5 years to as high as 8 years, which means higher probabilities of episodes lasting, say 10 or 12 years. As evidenced in recent decades, somewhat more extreme episodes are possible; the occurrence of rather long expansions supports the occasional presence of significant momentum or longer periodicity in the business cycle.

## 4.2 Spectral shapes for economic series and estimation of cyclical components

For economic time series (and data in numerous other fields), the frequent presence of trend has been well-known for some time; this motivated for instance Granger's (1966) discussion of the "typical spectral shape" as having most of its power concentrated toward low-end frequencies. Additionally, as discussed above, the movements in macroeconomic indicators (measuring aggregate activity), relative to trend or long-run level, tend to recur with some regularity and in periodic fashion. This suggests a generalization of Granger's spectral shape – an extension of the kind of (pseudo-)spectrum common for an observed process in economics – to incorporate additional power around some intermediate vicinity as in figure 10. The use of $\psi_t$ in (1) is tantamount to such an expansion of form in the frequency domain.

An illustration is given in figure 11, which builds the overall spectrum, $f_y(\lambda)$, from component spectra (assuming uncorrelated components). The spectra are based on models (7) and (13), set out below, with certain parameter values. Starting from the frequency origin and moving to the right, the trend pseudospectrum, $f_\mu(\lambda)$, tapers off while the cycle spectrum, $f_\psi(\lambda)$, gradually rises at higher frequencies. The sum of their spectra gives the spectrum of the observed series, $f_y(\lambda)$, which has the same pattern as $f_\mu(\lambda)$ around low frequencies but also has a local spectral peak from $f_\psi(\lambda)$.

As the frequency transitions into the mid-range, the trend's pseudospectrum tapers off as the cycle spectrum rises, and the two eventually intersect at some point, with this particular frequency equally associated with trend and cycle. As the frequency increases further, the cyclical spectrum dominates as its apex is approached. At still higher frequencies, $f_\psi(\lambda)$ falls off and crosses the irregular spectrum,

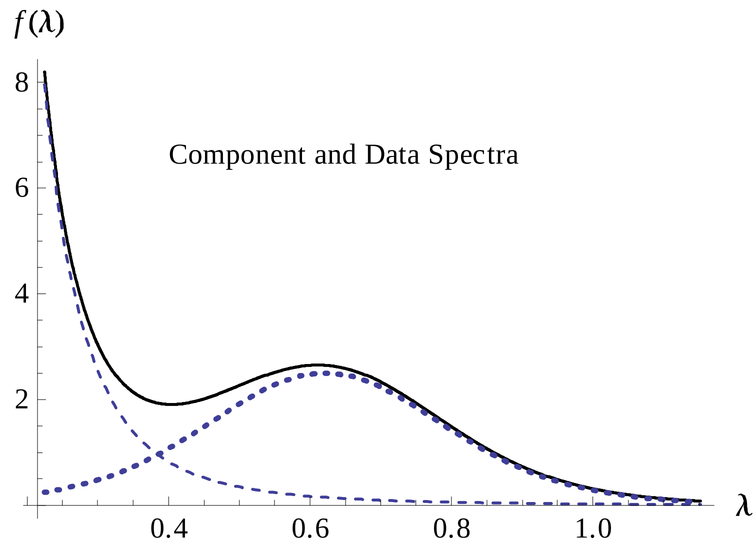


Figure 11: Illustration of trend pseudospectrum and stochastic cycle spectrum.

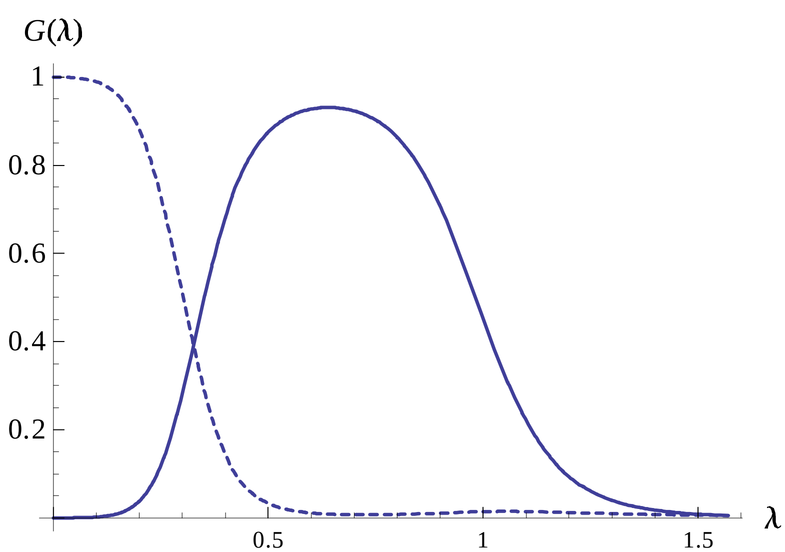


Figure 12: Illustration of trend (low-pass) and cycle (band-pass) filters formed from model.

$f_\varepsilon(\lambda)$, which is flat.

Starting with the base decomposition in (1), the optimal filter's gain function accounts for the relationship between the components as expressed in their spectra. The gain of the low-pass filter that gives the optimal estimator of the trend may be expressed as

$$F_M^{lp}(\lambda) = \frac{f_\mu(\lambda)}{f_\mu(\lambda) + f_\psi(\lambda) + f_\varepsilon(\lambda)} \tag{5}$$

Likewise, the gain of the band-pass filter that gives the optimal estimator of the cycle is

$$F_M^{bp}(\lambda) = \frac{f_\psi(\lambda)}{f_\mu(\lambda) + f_\psi(\lambda) + f_\varepsilon(\lambda)} \tag{6}$$

where *lp* denotes low-pass and *bp* stands for band-pass. This gives the frequency domain expression of the Wiener-Kolmogorov filters, which minimize the mean-squared error of each component (see Bell [1984]).

The filter gains for the trend and cycle depend on the interactions among $f_\mu(\lambda)$, $f_\psi(\lambda)$, and $f_\varepsilon(\lambda)$. Consider the band-pass implied by the trend-cycle components of figure 11; the gain is shown in figure 12. It starts out near zero for low frequencies and then gradually rises. The gain reaches one-half where the spectra cross, and continues increasing as we move toward the peak of the cycle spectrum. The gain never reaches exactly unity, though if the trend spectrum falls away rapidly enough as frequency increases[3] and the irregular variance is relatively small, then the gain may come close. Finally, at higher frequencies, the gain fades back toward zero. Hence, in general the band-pass gain has some curvature, where the flexibility emerges from the combination of the components with various characteristics (and with shared frequency regions in their spectra).

# 5 Adaptive estimation of cyclical components: explicit methodology

Above, the goal was to explain filter formation and give illustrations to clarify the method. Here, we make the method operational by considering a certain class of models for (1). We then consider examples of band-pass filters, which illustrate how one may design a specific gain profile.

[3] For purpose of illustration, I have selected an example where the trend pseudospectrum falls away gradually enough so that the effects of the crossing with the cycle spectra can be seen graphically. In practice, the trend pseudospectrum may decay more or less rapidly.

## 5.1 Modelling economic fluctuations with stochastic trends and cycles

Explicit decomposition models are reviewed here for the essential trend-cycle-noise components. These models have simple interpretations, they serve to make the adaptive filtering method operational, and they also provide all the advantages of time series models in general - in giving statistical summaries and parameter estimates (in this case, with intuitive quantities such as cycle period) useful for interpretations and descriptions, whose explanatory performance can be measured via fit and diagnostic evaluations, and allowing for forecasts of future values.

Here, we consider a damped smooth trend:

$$\begin{aligned} \mu_t &= \mu_{t-1} + \beta_{t-1}, \\ \beta_t &= (1-\phi)\overline{\beta} + \phi\beta_{t-1} + \zeta_t, \qquad \zeta_t \sim WN(0, \sigma^2_\zeta) \end{aligned} \tag{7}$$

where $\beta_t$ is the slope, and the coefficient $\phi$ satisfies the relation $0 < \phi \leq 1$. Often, $\phi$ has been set to unity, which gives the integrated random walk or smooth trend model.

A more flexible trend results from allowing this parameter to have values less than one, in which case $\beta_t$ has unconditional mean $\overline{\beta}$. This specification describes the tendency for growth rates well above or below $\overline{\beta}$ to revert back toward $\overline{\beta}$ over time. This seems plausible for economic series like real GDP; following unusually high $\beta_t$, the smooth trend form with $\phi = 1$ would imply that further positive and negative changes of a given magnitude are equally probable. In contrast, for potential or trend output, one would actually expect relatively rapid growth to more likely be followed by some moderation at some stage. Regarding the mean, for some series, $\overline{\beta} = 0$ is a plausible assumption. For instance, for an inflation measure or the unemployment rate, it seems unreasonable that the series tends to increase indefinitely moving forward, as policy responses would keep such advances in check. For GDP in contrast, and associated component series like investment, there tends to be a positive long-term rate of growth.

A more general trend is obtained as

$$\begin{aligned} \mu_{m,t} &= \mu_{m,t-1} + \mu_{m-1,t-1}, \\ \mu_{i,t} &= \phi\mu_{i,t-1} + \mu_{i-1,t-1}, \qquad i = 2, ..., m-1 \end{aligned} \tag{8}$$

with $\mu_{1,t} = \beta_t$ as given in (7). This model underpins the generalized Butterworth filter whose gain has the form in (2) and (3). Here we focus attention on an order $m = 2$ trend given in (7) because it works well empirically, is relatively simple, and allows us to focus on the band-pass aspect.

In building structural time series processes for overall series or as components, we start with linear difference equations and intertwine them with stochastic shocks. In the case of a cyclical component, we first consider a simple fixed cycle

$$\psi_t = A\cos(\lambda_c t - \omega) \tag{9}$$

with period $2\pi/\lambda_c$, amplitude $A$, and phase $\omega$ being fixed parameters to start with. This initially departs from a linear construction, but we can design a linear difference equation form by re-expressing and transforming (9).

First, write

$$\psi_t = (\psi_0 \cos\lambda_c t + \psi_0^* \sin\lambda_c t) \tag{10}$$

which is equivalent to (9) with $A = \sqrt{\psi_0^2 + \psi_0^{*2}}$ and $\omega = \tan^{-1}(\psi_0^*/\psi_0)$. Next, let the amplitude vary over time according to $A_t = \rho^t\sqrt{\psi_0^2 + \psi_0^{*2}}$ where $\rho$ is a damping parameter:

$$\psi_t = \rho^t(\psi_0 \cos\lambda_c t + \psi_0^* \sin\lambda_c t) \tag{11}$$

with $0 < \rho \leq 1$ and $0 < \lambda_c < \pi$. The scale of oscillations decays by $\rho$ each period.

We convert this to linear difference form by augmenting the cycle with a companion process $\psi_t^*$ that also evolves over time, and then writing a recursion:

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} \tag{12}$$

where the constants $\psi_0$ and $\psi_0^*$ become the starting values of the vector in the above equation. The linear difference equation (12) generates $\psi_t$, where the form including the process $\psi_t^*$ enables us to express the cycle as part of a linear bivariate equation.

The definition of a stochastic cycle $\psi_t$ uses the above recursion and augments equation (12) by adding a vector of shock variables to the right hand side. The process evolves as

$$\begin{bmatrix} \psi_t \\ \psi_t^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{t-1} \\ \psi_{t-1}^* \end{bmatrix} + \begin{bmatrix} s_t \\ s_t^* \end{bmatrix} \tag{13}$$

where $s_t$ and $s_t^*$ are the shocks feeding into the cycle and auxiliary process respectively. Then the cyclical fluctuations tend to recur with frequency $\lambda_c$, which may be interpreted as a central frequency, with corresponding period of $2\pi/\lambda_c$.

A time-varying amplitude can be defined as $A_t = \sqrt{(\psi_t^2 + \psi_t^{*2})}$ , which would be the initial scale of the cycle if, starting at time $t$, the cycle were projected forward in time in the absence of shocks. Similarly, a phase can be computed via $\omega_t = \tan^{-1}(\psi_t^*/\psi_t)$. Proceeding to $t+1$, according to (12), at first the position of the cycle advances by $\lambda_c$, the amplitude is multiplied by $\rho$, and the phase stays the same. However, with random shocks added to $\psi_{t+1}$ and $\psi_{t+1}^*$, the amplitude and phase change stochastically over time. The particular manner in which they evolve is determined by the nature of the shocks. In this way, stochastic properties of $\psi_t$ depend on the specification of the shock vector.

Now in the simplest case, $s_t$ and $s_t^*$ are uncorrelated white noise, that is, $[s_t, s_t^*] = [\kappa_t, \kappa_t^*] \sim WN(0, \sigma_\kappa^2)$ and the error $\kappa_s$ is uncorrelated with $\kappa_t^*$ for all $s, t$. This case is a first order cycle, $\psi_{1,t}$. As another case, the shocks can themselves be periodic; this form is connected with the idea of a resonant process, where cyclical forces feeding into a process accentuate the overall oscillatory behavior of the system. A second order cycle has stochastic shocks given by a first order cycle, i.e., $\psi_{2,t}$ is subject to $[s_t, s_t^*] = [\psi_{1,t}, \psi_{1,t}^*]$. $\psi_t$ can be constructed to have even more resonance by taking the shocks in (13) to be of second order, which gives a third order cycle, and so forth.

Generally, a class of models is built recursively, for $n > 1$, by making the shocks to $\psi_{n,t}$ periodic of order $n-1$. In each case, the model in (13) guarantees a peaked spectral shape reaching its maximum around $\lambda_c$. As $n$ rises, the spectrum increasingly concentrates in a band around $\lambda_c$. Since a given value of $\rho$ is tied to more persistence as $n$ increases, estimates of $\rho$ for a given series usually show a pattern of decline with rising order. This form with both $\kappa_t$ and $\kappa_t^*$ present, along with both $s_t$ and $s_t^*$, is referred to as the "Balanced" model type. An alternative specification with $\kappa_t^*$ and all $s_t^*$ set to zero is called the "Butterworth" form. These two models are discussed further in Appendix E.

Trimbur (2006) provides a detailed characterization of the "Balanced" cycles, for which analytical expressions can be derived for properties in the time and frequency domains. The spectrum of this class of processes has the form

$$f_{\psi_n}(\lambda) = \sigma_\kappa^2 V(n, \rho) h_\psi(\lambda) \tag{14}$$

where $V(n, \rho)$ is the cycle variance for $\sigma_\kappa^2 = 1$, so that $\sigma_\kappa^2 V(n, \rho)$ is the overall cycle variance for any combination of $n, \rho$, and $\sigma_\kappa^2 = 1$. The formulas for $V(n, \rho)$ and $h_\psi(\lambda)$ are given in Trimbur (2006). The examples of possible cyclical spectra in figure 10 are based on this class with particular parameter values. For reference the frequency range corresponding to six to thirty-two quarters is indicated by the vertical dashed segments. These three cases of $h_\psi(\lambda)$ give representations of different kinds of cyclical behavior. The dotted curve represents a dispersed density that was generated with $n = 6$, $\rho = 0.65$, and $\lambda_c = 2\pi/12$. This illustrates how a targeted spectral band or shape may be represented by densities that are flexibly constructed. The solid-line spectral density was designed by setting the central frequency equal to $2\pi/18$, giving a major period of 4 1/2 years, and setting $n = 2$ with $\rho = 0.72$. There are three primary differences in the two spectra. First, the solid density peaks at a lower frequency; second, it has a greater spread and includes some very low frequencies connected with more persistence, and third, the solid density includes less power at higher frequencies linked to periods of three years or less. Lastly, a more subtle difference is that the $n = 2$ spectrum contains slightly more of the very high frequency components than does the sixth order case. The dashed curve indicates a more sharply designated cycle.

For macroeconomic data, we might have some vague information about the spectrum of $\psi_t$ related to business cycle notions; so our broad expectation is that the cyclical spectrum resemble one of those in figure 10. However, given that the precise shape of $h_\psi(\lambda)$ is unclear, it becomes important to use the available data for a given series to determine its precise cycle properties. One can choose among different forms of $\psi_{n,t}$, each of which may be estimated to tailor its underlying parameters. For the "Butterworth" model, the dynamics are similar to those of the Balanced form. The power spectrum has a compact form; however, as there is no known analytical expression for the variance, the spectral density cannot be formulated in general.

For the third component in equation (1), the irregular $\varepsilon_t$ is specified as white noise, a mean zero, serially uncorrelated sequence of random variables, denoted by $WN(0, \sigma_\varepsilon^2)$ where $\sigma_\varepsilon^2$ is the variance. If the additional noise in the series stems from unusual events and perhaps also measurement error specific to each period, then this assumption gives a reasonable starting point.

Throughout, we assume that uncorrelated disturbances drive the different components; this assumption seems natural for components that have rather distinct properties, and it gives symmetric filters near the middle of the series. This assumption can be generalized; however, other restrictions may become necessary to make sure the model stays identified, and the close link with the band pass literature in statistics would be broken.

## 5.2 Filter Design

Now assume that the decomposition in (1) holds with trend given by (8) and cycle following the n-th order process $\psi_{n,t}$ of Butterworth form. Then, using (5) and (6), forming the ratios gives the general class of filters in (2) and (3). Expressions (2) and (3) remain relatively compact and with various choices of parameter values, a diverse array of gain contours is achieved. We assume that trend follows (7) so the order $m$ is set to two. We focus attention on the order $n$ of the cycle, which directly affects the sharpness of the filters. The quantities $q_\zeta = \sigma_\zeta^2/\sigma_\varepsilon^2$ and $q_\kappa = \sigma_\kappa^2/\sigma_\varepsilon^2$ are "signal-noise" ratios for the trend and cycles, as they indicate the variability of each regular component relative to irregular or noisy fluctuations. The cyclical parameter $\lambda_c$ giving the main frequency of oscillation of the cycle is intuitively related to the region of highest gain in the band-pass filter.

Our preferred approach is to adapt the filter across various cases by choosing parameters fitted to each input series. Here, we pursue a classical framework; note that in some cases, the finite sample likelihood may not be sufficiently well identified to produce suitable estimates. One option is to use a Bayesian approach as in Harvey, Trimbur, and van Dijk (2007); an alternative is to impose sufficient constraints on parameter values. Here, the filtering strategy is made operational by preserving loose conditions on the model parameters, which reflect reasonable expectations about economic cycle periodicity and trend behavior and preserve the low-pass and band-pass filter's usual shapes. In applications in macroeconomics, these relaxed bounds tend to have little effect on performance diagnostics (indeed, the model may actually perform better under the constraints).

In our more general framework, the defining characteristic of the band-pass is that they tend to concentrate on, or select out, mid-range frequencies, in the same way that low-pass filters are characterized by a tendency to cut out higher frequencies. These definitions are rough and allow for a great deal of flexibility in the particulars of the respective gain function; there is no assumption that a "pure" band-pass filter merely allows frequency parts between two edges through without affecting their strengths or amplitudes. Indeed, in the context of stochastic data in statistical applications, perfectly sharp gain functions are the exception rather than the rule. Features such as time-varying cyclical amplitudes may be studied directly. In economic applications, properties of the business cycle such as asymmetries in expansions and recessions may be captured – even though the base model utilizes symmetric disturbances – because the components condition on the data and reveal the data's underlying properties.

The use of orders with $n > 1$ is crucial for providing a link with band-pass filters and for improving models' fit and performance. In much previous work, it has been assumed that $n = 1$, which has a number of limitations, such as keeping noise in the estimated cycle and making it difficult to see cyclical transitions.

# 6 Applications to US macroeconomic series

Quarterly data on U.S. real Gross Domestic Product and other national accounts indicators such as Investment are taken from the Bureau of Economic Analysis for the period 1947Q1 to 2017Q4. These series are chosen because they represent the primary indicator of economic activity for the U.S.; the components of GDP behave differently from a dynamic perspective, and it is important to understand the major components to fully understand the dynamics of overall GDP. In total 12 indicators are considered.

Starting with the unrestricted case, model (1) is estimated with (13) of either the Butterworth or

Balanced form for different orders $n$ ranging up to eight; all parameters are free except for minimal constraints, which represents the fully adaptive case. We compute the parameter estimates for each model and series by Maximum Likelihood [4]. Given a feasible parameter vector, the likelihood function is evaluated from the prediction error decomposition from the Kalman filter; see Harvey (1989). The values of parameters are found by optimizing over the likelihood surface in each case. To do the calculations for the results given below, programs were written in the Ox language (Doornik 2006).

## 6.1 Parameter Constraints

The period corresponding to the central frequency, that is $2\pi/\lambda_c$, is constrained to lie on an interval from three and a half to eight years, ensuring that the cyclical component is reflective of business cycle movements rather than a different type of dynamic such as seasonal cycles or persistent longer run patterns. Correspondingly, the periodicity conditions help guarantee a band-pass profile for the resulting cycle filter, meaning that the gain cuts out low frequencies sufficiently well (that is, falls to one-half by some positive cutoff frequency as the origin is approached from the right – the mid-frequency region) and likewise effectively removes the noisy part of a series' fluctuations (that is, the gain dips to one-half as the frequency increases from its intermediate to its maximum range).

There are two autoregressive-type parameters, that is $\phi$ for the trend-slope and $\rho$ for the cycle. The interval [0.95,1) is used for $\phi$ to ensure sufficient persistence in the slope, so as to give a regular growth rate of the trend and a corresponding focus on low-frequency regions. Additionally, the interval (0,1) is used for $\rho$ to ascertain a well-formed cycle (in practice, the actual estimates of $\rho$ seem to always lie well within these bounds, and the occurrence of $\rho$ very near either boundary indicates a problem with the numerical optimization.)

Also, very loose constraints are placed on the trend disturbance variance, in order to ensure at least slightly stochastic behavior in the trend and correspondingly minimal cutting out of very low-frequencies by the band pass filters. The specific periodicity conditions and other constraints used are detailed in Appendix A.

## 6.2 Parameter Estimates

Complete results on parameter estimates and diagnostics and fit measures are reported in Appendix D. We start by discussing the results for real GDP that are given in the first table of the set of the 12 tables

[4] The difficulties associated with estimating the cyclical period for real GDP are discussed in Harvey and Trimbur (2003). For a Bayesian solution to this problem, see Harvey, Trimbur, and van Dijk (2007).

of D.1. The estimates show that the variance of the slope disturbance is relatively small but nonzero, indicating clear stochastic variation in trend growth rates. The value of $\widehat{\sigma}_\zeta^2$ (which denotes the Maximum Likelihood Estimate of $\sigma_\zeta^2$) for the Balanced model is less than half the value for the Butterworth model when $n = 1$, thus illustrating differences in how the Balanced and Butterworth model accommodate trend and cyclical movements. Also, $\widehat{\sigma}_\zeta^2$ decreases significantly in moving from first to second order. For all $n > 1$ this variance parameter is increasing at a modest rate and remains on the order of $10^{-6}$. This indicates how the trend becomes more stable, with the cycle accounting for more of the overall variation, as the order increases. For real GDP, $\beta_t$ can be interpreted as a growth rate of potential output. The stationary slope's mean $\overline{\beta}$ is estimated to be 0.77, and hence long-run potential growth is about 3.1%. For all orders and models, the coefficient $\phi$ is estimated to lie at the lower bound of 0.95; intuitively, this means that any deviation in trend growth rate per annum from 3.1% is attenuated by nearly 20%, on average, over the course of the following year.

As $n$ increases, the variance of the irregular rises, with the biggest change occurring from $n = 1$ to $n = 2$. In table D.1, the q-ratio is defined as $q_{\zeta *} = \sigma_\zeta^2/(\sigma_\psi^2 + \sigma_\varepsilon^2)$, where $\sigma_\psi^2$ is the unconditional variance of the cycle. This ratio therefore represents the trend's disturbance variance divided by the total stationary components' variance. The estimates of $q_{\zeta *}$ decline considerably as the model changes from first to second order cycles. The estimated value of the variance parameter $\sigma_\kappa^2$ declines with increasing $n$. For $n = 1$ the estimate of the damping parameter $\rho$ of nearly 0.9 indicates a persistent cycle. The value of $\rho$ also declines for larger $n$. Given the resonance property of higher order cycles, where shocks are increasingly reinforced within the system, a fixed value of $\rho$ would lead to a more pronounced cycle as the order increases; however, there is an off-set to this effect via smaller damping parameters for larger $n$. For the frequency MLE for real GDP, the corresponding period is around four-and-a-half years for $n = 1$ with the Balanced model and is about one year shorter for the Butterworth form.

For Investment, central periods from 20.1 to 31.8 are estimated for orders up to six. The values of $\widehat{\rho}$ are close to those for GDP for each $n$. There is also a similar pattern in the declines of the cycle and irregular disturbance variances, though both $\widehat{\sigma}_\kappa^2$ and $\widehat{\sigma}_\varepsilon^2$ are larger in magnitude for Investment, reflecting the stronger cyclical properties and greater noise content of this primary sub-sector of GDP. The same jump in $\sigma_\varepsilon^2$ from $n = 1$ to 2 holds for Investment, with further increases that taper off as $n$ approaches 8 (this pattern continues for even higher $n$ than are shown in the tables). The long-run trend growth rate of Investment is somewhat larger than that of GDP. The values of $q_{\zeta *}$ are considerably lower than for GDP, conveying statistically the stronger cyclical content of Investment. The slope's damping coefficient is estimated again as $\widehat{\phi} = 0.95$ for all orders.

Various aspects of the pattern of parameter estimates for real GDP and Investment also hold for other series. Perhaps most notably, the irregular variance jumps in value from first to second order and then continues to steadily increase for $n > 2$, implying that ever larger amounts of noise are removed from the cycle, which as a result, becomes smoother as $n$ rises. For all series except Inventory Investment, the ratio $q_{\zeta *}$ is above $10^{-5}$ and supports the use of a stochastic model for the trend. The patterns of $\rho$ and $\sigma^2_\kappa$ with respect to different $n$ are rather similar across the different data series. Indeed, based on our overall experience with quarterly macroeconomic series, having the values of $\rho$ and $\sigma^2_\kappa$ fall with $n$ in this manner is a regular feature shared by most series related to economic cycles. The decline in these parameters at higher orders serves to mitigate some of the cycle's rise in intensity with $n$ that is related to the resonance feature of the models.

In terms of periodicity, $\widehat{\lambda}_c$ is within the constraining interval in several cases and appears reflective of business cycles for first and higher orders; for instance, for Exports, most of the central periods are between 4 and 7 years. In other cases, the period estimate lies at the upper bound of 8 years. There is sizeable variation across series in the estimated periods over the range of permissible orders. For the general univariate filters and methodology developed here, the use of an upper bound on the period is a valuable device to ensure the cycle remains separated from trend or other component types.

There are three points that may be noted in passing. First, the models and parameter values pertain to the historical sample – they do not pertain to a specific episode such as the recent Great Recession, which some have argued represent a different kind of cycle than the traditional business cycle, but include many years of economic fluctuations. Second, the estimates are empirical and statistical and let each individual series have its own particular kind of cycle that may be related to concepts like business, housing, and financial cycles; the estimates rely on past behavior and are free of strong assumptions such as those incorporated in debatable economic theories. Third, a central period of eight years carries with it the possibility of actual cyclical episodes that last 12 or so years or even longer; the cycle spectrum obtained for the MLEs of the parameters has power at lower frequencies indicating frequency parts whose periodicity is longer than the central one. The power declines away from the cyclical mid-range but still remains positive over a nontrivial range of frequencies.

## 6.3 Diagnostics and Fit

Several diagnostics are reported in table D.2. $R^2_D$ is the coefficient of determination with respect to first differences, and $\widehat{\sigma}$ is the equation standard error. $Q(P)$ is the Box-Ljung statistic based on the first P residual autocorrelations. Three different values of $P$ are used in reporting these statistics, and in each case $Q(P)$ should be compared with a $\chi^2$ distribution with $P - 4$ degrees of freedom. The

Akaike Information Criterion (AIC) is defined by $AIC = -2\log\hat{L} + 2k$, where $\log\hat{L}$ is the maximized log-likelihood for each model/series combination, and $k$ is the number of model parameters. Similarly, the Schwarz Information Criterion (SIC) is computed as $SIC = -2\log\hat{L} + (\log T)k$, where $T$ is the sample size. The $AIC$ and $SIC$ are comparable across different cyclical orders because the process is stationary for all $n$.

In the first table of the set in D.2, for real GDP, there is a large improvement in diagnostics and fit measures in moving from first to second order. For the Balanced form, the Q(24) statistic decreases by about 65%, while the $R_D^2$ measure increases around 28%; the AIC drops by more than 10. The overall trough of the AIC occurs for the third order Butterworth model, which also marks the optimum order and form for Q(24) and $R_D^2$. The AIC increases by around 0.4 for the second best model, which is the fourth order Butterworth, whereas the Q(24) and $R_D^2$ statistics are little changed. Starting from $n = 3$, further increases in cycle index lead to a very gradual worsening of diagnostics so that, for $n = 8$, the AIC is a few units higher and the Q(24) and $R_D^2$ are slightly less favorable than for third order. All Q(24) and Q(32) statistics are insignificant at the 5% level for orders above one.

For Investment, the large advance in model performance for $n = 2$ from first order is also present. Further, the diagnostics continue to improve as $n$ increases so that in the Butterworth case, the $R_D^2$ is about 20% higher for the 8th order compared to the 2nd order. Likewise, Q(24) is approximately 25% lower, while the AIC decreases by over 7.0 in going from $n = 2$ to $n = 8$. The overall optimal performance occurs for 7-th or 8-th order (the diagnostics give mixed implications), with order 6 performing nearly as well. Hence, there are clear merits, in terms of model fit, in treating relatively high order specifications. In contrasting the two model forms, the Balanced cycles exhibit superior fit compared to Butterworth type for $n = 1$ and 2, whereas for higher orders, the two forms show an increasing tendency to converge in their diagnostics. All Q-statistics lie below the 5% critical value for $n \geq 3$.

Nearly all series show improvements in fit measures in going from $n = 1$ to 2, which are often sizable. The only exception appears to be Consumption of Services, which displays a very slight worsening in AIC and SIC along with $R_D^2$ and which in any event has a rather weak cycle. For seven out of twelve series the selected order is 2 or 3; for four series the preferred index is 7 or 8. Two of the high-order selections are Investment series known to have very pronounced cycles. The other two consist of Consumption, which is a highly aggregated series, as well as Exports, which depend on a large mix of foreign economies.

Overall, the smallest Q(24) statistics are obtained for Inventory Changes (in particular the 8th order Balanced model), which is almost entirely dominated by cyclical movements. Similarly, small values of Q(24) occur for real GDP, Investment, and Exports, with the corresponding p-values exceeding 0.1. Of these four series with favorable Q-statistics, three have intermediate central periods that are estimated

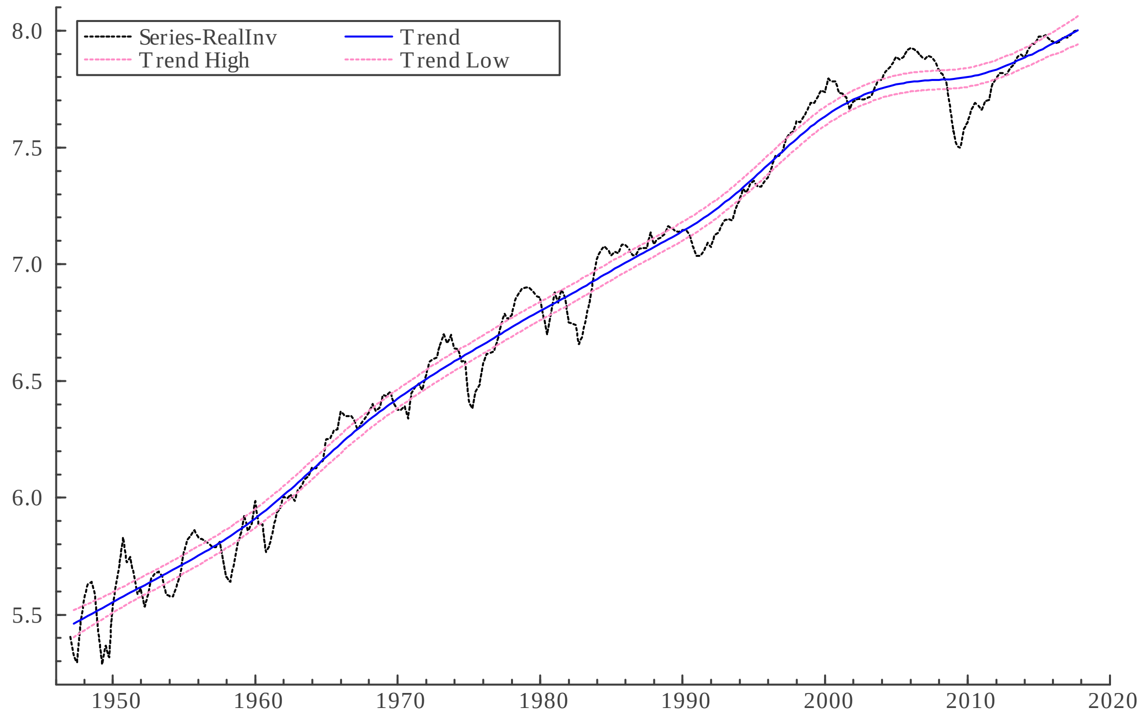


Figure 13: Investment with estimated trend for n = 6 with Balanced form.

within the restriction interval and plausible for business cycle fluctuations. Also for three out of these four series with low serial correlation in residuals, whether the focus lies on Q(24), $R_D^2$, or AIC doesn't affect conclusions about relative performance – all three point to the same form-order combination as the best. Overall for all 12 series the model selection results appear very similar for SIC vs. AIC, and the implications of $R_D^2$ comparisons matched nearly all of selections made based on AICs, whereas the Q-statistics sometimes leads to different conclusions.

## 6.4 Smoothed Components and Filters

Figure 13 shows the (logged) series Real Investment along with the estimated trend (for the Balanced model with $n = 6$, corresponding to the cycle given by the solid line in figure 16). The trend varies smoothly throughout the sample period, undergoing modest changes in growth rate up to the Great Recession. In the 2000's the trend decelerates somewhat rapidly and reaches a nearly flat trajectory; in the recovery period after the GR, the pace of growth again starts to increase so that, at the end of the sample, the slope has almost reverted to its long-run value.

Figure 14 displays the estimated cycles for $n = 1$ and 2 with the balanced form. The use of the higher order model is crucial to describe smoothly varying cyclical dynamics, which make the turning points

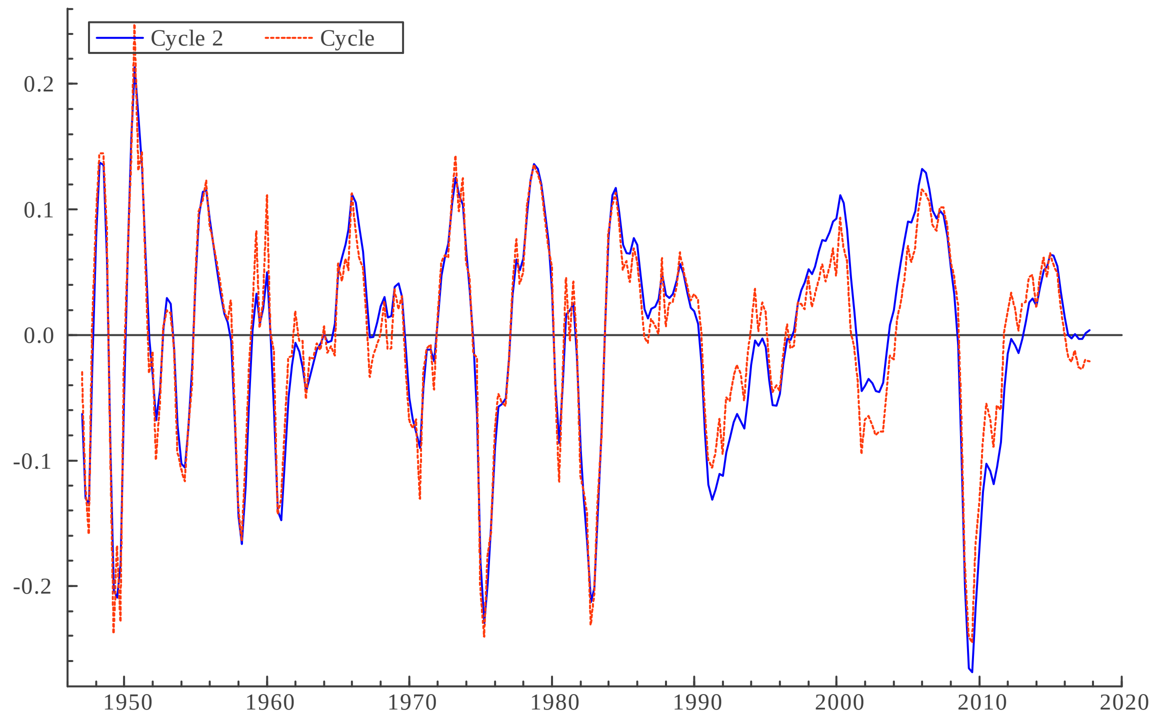


Figure 14: Estimated cycle in Investment for n = 1 and 2 with Balanced form.

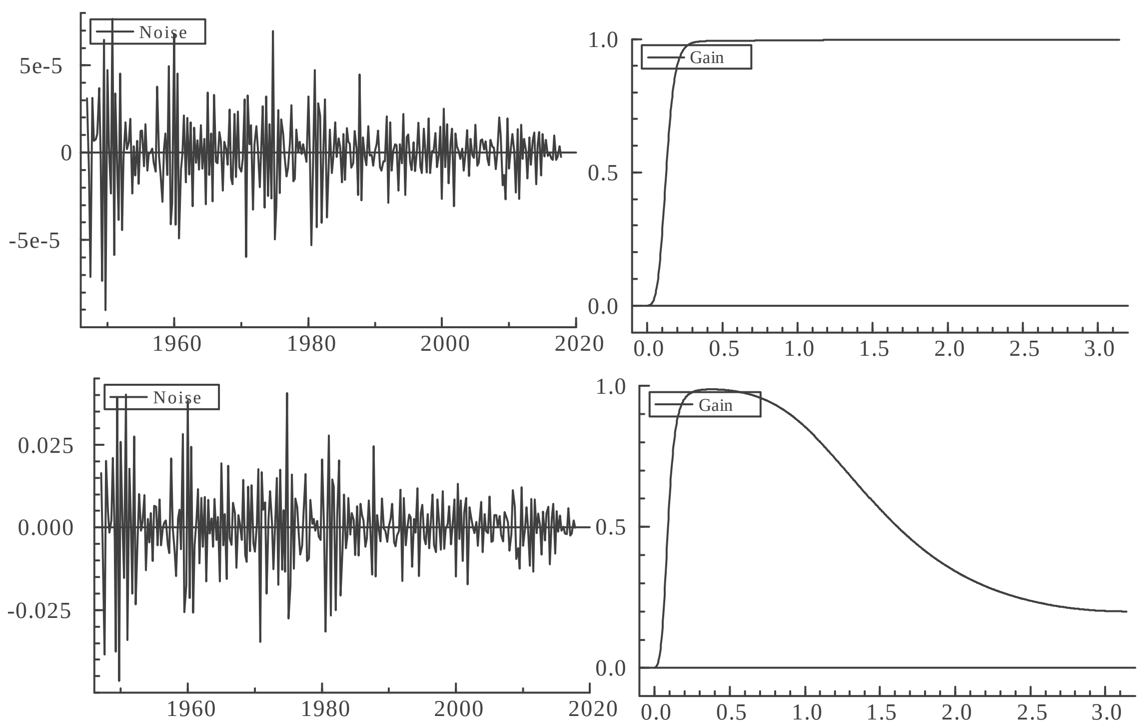


Figure 15: Noise removed from Investment in estimating the cycle, shown along with the corresponding Band Pass filter for n = 1 and 2.

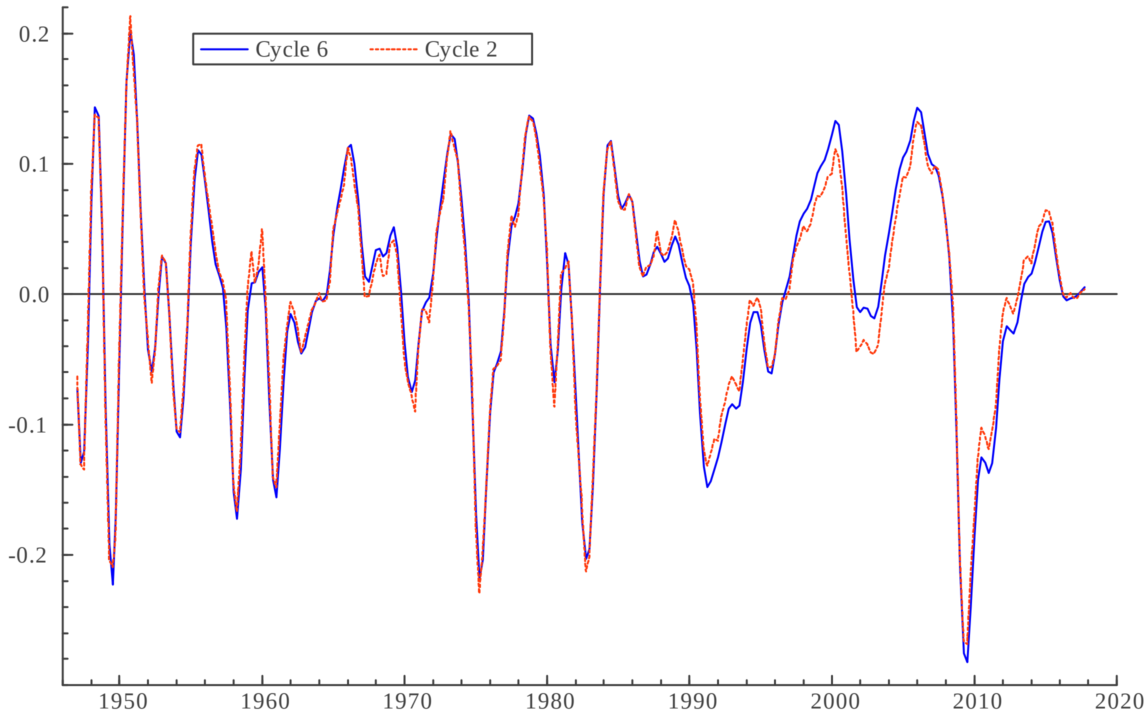


Figure 16: Estimated cycle in Investment for n = 2 and 6 with Balanced form.

transparent, and also to yield sizable advances in terms of diagnostics. In contrast, the first order model gives a noisy estimated cycle that makes it difficult to see the main transitions and general evolution. There are also differences in the message given by the cycle at times, such as the 2001-2 recession being more shallow for the second order case. The GR stands out as being both very intense and long-lasting. There are somewhat comparable cyclical episodes, in terms of peak-to-trough magnitude, earlier in the sample period, but the combination of intensity and duration cause the GR to be partially absorbed as a prolonged slowdown in trend. The functioning of the higher order processes becomes clear by looking at the extracted noise and implied filters directly, as in figure 15. The first order pertains to $n = 1$ and shows a small amount of noise being relegated to the irregular; the associated filter is a high-pass filter – a detrending filter that does not descend at higher frequencies and that does not have the characteristic shape of a band-pass filter. The second row shows the noise component and the implied filter for $n = 2$; the gain function declines as the frequency increases beyond the mid-range and so has a malleable band-pass shape.

There are also further enhancements to the smooth contour of the cycle for higher orders - figure 16 displays the extracted cyclical signals for $n = 2$ and 6. The peaks are more clearly defined for sixth order, for instance in the early 60's; there are also some discrepancies in what the cycle is signalling at times, as in the 2001-2 recession, which is less intense for the higher order case. The phases of increase

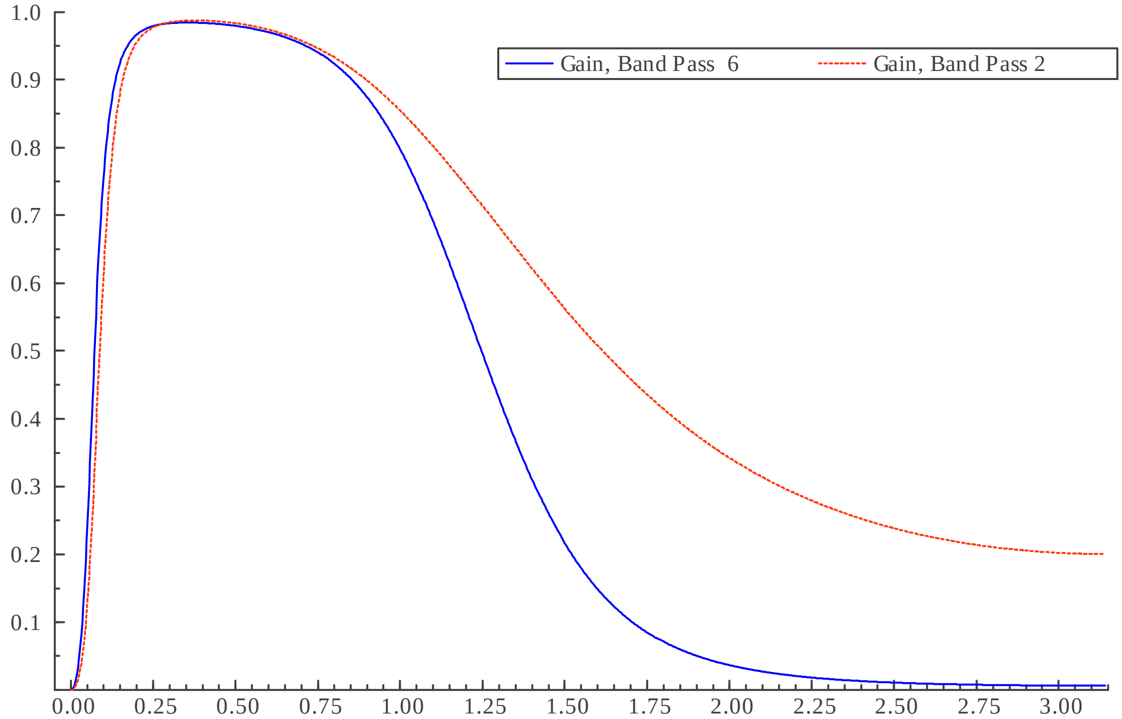


Figure 17: Estimated Band Pass filter for extracting cycle in Investment for n = 2 and 6 with Balanced form.

appear smoother, for example the rise in the cycle into positive territory just before the decline at the end of the sample. The differences in smoothness and overall path in the two cycles along with the improvements in diagnostics make the $n = 6$ model an attractive choice (the number of parameters is the same, as just the state space dimension increases, making estimation take longer – though still entirely feasible given modern computational power in a typical PC). The extracted cycles in Investment for $n$ greater than six appear very similar, though there are still incremental changes, with the turning points and evolution between peaks and troughs being clearer for the higher orders. This greater smoothness is connected with how the implied filter generates a successful rendition of a band-pass that becomes sharper for higher values of $n$; the gain function is shown in figure 17 for 2nd and 6th orders. The higher order band pass effectively eliminates high-frequency movements, while the second order band pass admits some noise and includes it in the estimated cycle. This illustrates an advantage of high order models, whereby the objectives of using a band pass filter – to eliminate both trend and noise and yield smooth signals of cyclical fluctuations – are realized more fully.

For highly cyclical series, there is some differentiation between the alternate forms of the cycle model. Figures F.2 and F.3 in Appendix F show the estimated cycle and gain function for Residential Investment for $n = 8$ for both Balanced and Butterworth models. The Butterworth cycle displays a larger amplitude, for instance, in the 90-91 downturn and the Great Recession, and the evolution around the sample end-point is somewhat different. The gain function for the Balanced cycle cuts out more low-frequency movements than the Butterworth version, which provides an illustration of the flexibility afforded by the different model forms.

## 6.5 Ideal Filter Approximations: Parameters and Diagnostic Measures

Next, the selected 36 possible ideal filter approximations are applied to each of the time series considered. For each of the orders $n = 4, 6$, and 8, there are a dozen representations of the ideal filter, which are fit to each series by maximizing the log-likelihood with respect to $\overline{\beta}$ and $\sigma_\varepsilon^2$ with filter parameters and ratios held fixed at the values determined previously. The results are given in Appendix D entitled "Detailed Tables of Empirical Results". Tables D3-4 pertain to n = 6, which is considered the benchmark representation, while D5-6 reports results for n = 4, and D7-8 focus on n = 8. In each table in D3, the first two columns indicate the q-ratios for each ideal filter representation; recall that $q_\zeta$ is now defined as $\sigma_\zeta^2/\sigma_\varepsilon^2$ in line with the expression for the gain function of the generalized Butterworth filter in (3) that is linked to the model's variances. The last two columns display the MLEs; the results for real GDP are shown in the first table.

The $\widehat{\sigma}_\varepsilon^2$ values show a consistent pattern of increasing as the q-ratios fall. Indeed there is a large difference between the two boundary situations illustrated in figure 2, in that the irregular variance for the low q case is more than double the value for the high $q$ case. This parallels the more effective cutting out of high frequencies displayed in figure 3 by the gain function with low $q$. Two series with a particularly large differential in noise variance are Residential Investment (third table, bottom of p.2 of D3) and Government Expenditures (tenth table, top of p.6 of D3), for which the maximal $\widehat{\sigma}_\varepsilon^2$ obtained is over four times the minimal $\widehat{\sigma}_\varepsilon^2$. Therefore, from the perspective of how much irregular variation is removed, it can make a significant difference how the ideal filter is represented, even after accounting for filter index. To help assess the various possibilities, we now focus on the examination of model estimation results to see which scenarios fit especially well or poorly among those considered.

Table set D4 displays the same diagnostic and fit statistics as before for the adaptive results (that is, based on unrestricted MLEs), but now based on the parameter values in Table D3. For real GDP, the Q-statistics are well beyond the 5% critical value, indicating decisive rejections from these models that underlie ideal filter approximations. Further, the $R_D^2$ is negative, indicating that the models perform worse than a random walk with drift. The results are in stark contrast to those for the adaptive models.

There is a similar message for the other 11 series, with varying degrees of poor fit. Under no conditions would such models be entertained from a statistical perspective. Nevertheless, there are differences in how the 12 representations perform, and it is worthwhile to make comparisons to arrive at the best possible ideal filter models when such a filter is very strongly desired.

For real GDP with $n = 6$, as shown in the first table in D4, the AIC reaches a minimum for the 6th row representation in the table that has $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{$ 0.0661, 0.0496, 0.455$\}$, which also marks the maximum $R^2_D$. For this representation, the Ljung-Box statistic with $P = 24$ takes on a value of about 105 compared to its minimum of just under 90 that is attained for the first row approximation; these statistics rise sharply toward the end of the table for the lowest q-ratio filters. For Investment with $n = 6$, the second table in the set D4 shows that the AIC is minimum for the 6th representation for which $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{$ 0.03178 0.04081 0.4709$\}$. Upon perusal of the set of results, it becomes clear that the AIC minimization corresponds almost always to the maximization of $R^2_D$. However, the minimization of $Q(P)$ frequently gives a different implication. Hence, we concentrate on the AIC and $R^2_D$ while recognizing that $Q(P)$ has some complementary information useful for certain purposes. Focussing on table set D4 for $n = 6$, the 8th row approximation ($\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{$ 0.04946, 0.04589, 0.4611$\}$) appears to work the best, in terms of giving the best overall fit and avoiding extremely poor diagnostics when used across the 12 series. This choice strikes a balance between the case of Residential Investment for which the fourth row case is preferred and that of Inventory Investment where the last row gives the best result. Since the ideal filter tool is designed for widespread use and diverse time series, these considerations are important to bear in mind.

Tables D5 and D6 report results for the fourth order model. Table set D5 again focusses on the unrestricted parameter MLEs while D6 contains diagnostics and fit measures. The major finding is that the results again look extremely negative, with the models explaining less of the variation than a random walk would, and with substantial serial correlation in the residuals. Doing the same exercise as before, and comparing model selections across the 12 series, it appears that the 7th representation with $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{$ 0.05722, 0.174, 0.4146$\}$ works the best in terms of evading a drastically poor fit and achieving a relatively decent fit (compared to other ideal filter models). Similarly, tables D7 and D8 report results for eighth order, for which the representation in the 9th row, which has $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{$0.05188, 0.01226, 0.4815$\}$, is chosen.

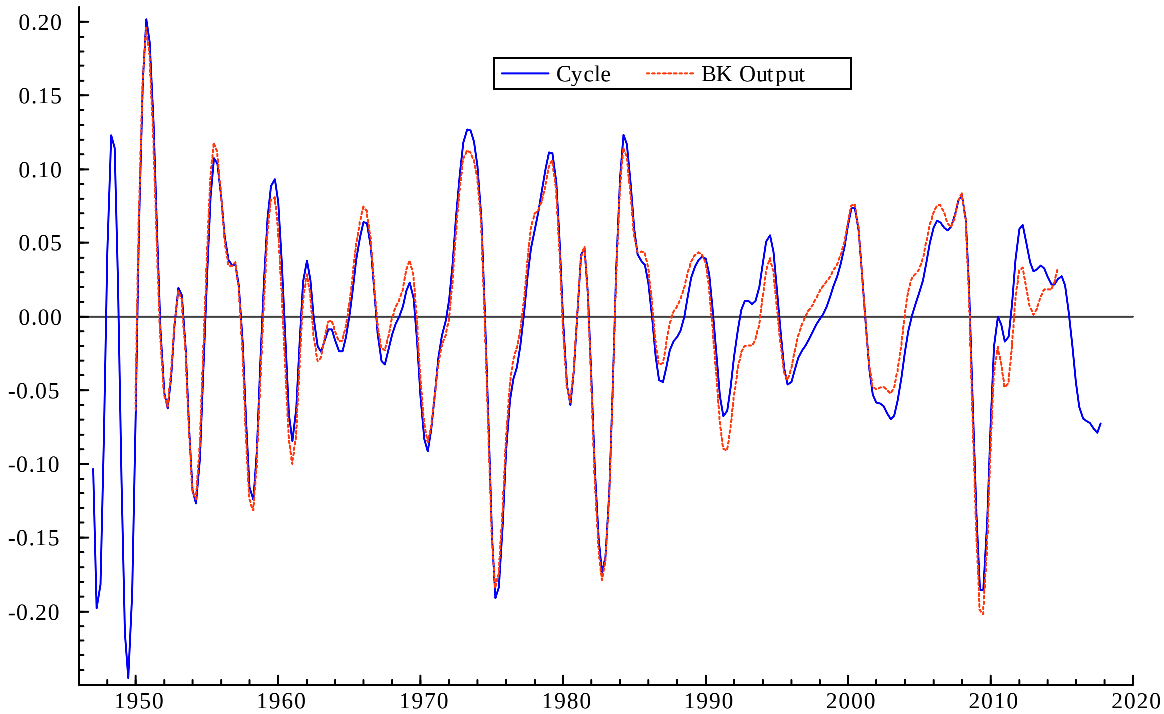


Figure 18: Estimated cycle in Investment (in logs) for the modelled ideal filter with n = 6, compared to the BK filter output.

### 6.6 Comparison of Smoothed Components and Filters

Figure 18 displays the estimated cycle with the modelled ideal filter for $n = 6$ vs. the BK output. The two series generally track each other; over the first half of the sample, there are very minor differences in trajectory. In the second half, there are some discrepancies, which we now describe. First, the troughs in the early 90's and in 2001-2 have a different magnitude for the modelled vs. BK filter. Also, the three different upswings between the mid-80's and the early 2000's have somewhat contrasting contours. For example, the BK output and the modelled cycle appear non-synchronous during part of the upswing starting around the mid-90's. While the BK output flattens out beginning in late 2001, the model-based estimates reach a trough around the end of 2002 (such pinpointing of transitions being useful for timing turning points). Toward the end of the series, when the BK filter fails to show a clear direction, the modelled cycle exhibits a peak followed by a lengthy contraction, with a possible trough in the cycle starting to emerge in the last year of the sample. These differences are modest, but they do demonstrate how the BK gain function – with its unstable ripples and fixed structure over time – gives a somewhat different path for the estimated cycle than the modelled ideal filter.

The divergence between the adaptively measured cycle and the BK output (or ideal filtered cycle

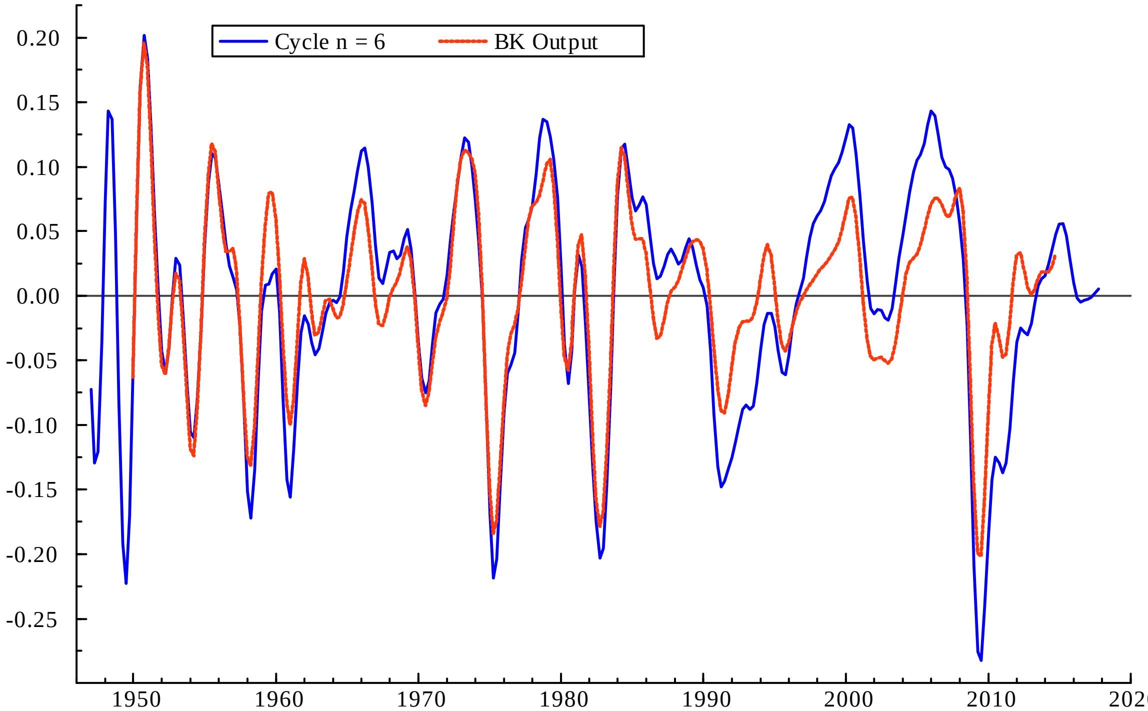


Figure 19: Estimated cycle in Investment (in logs) for the adaptively modelled case with n = 6, compared to the BK filter output.

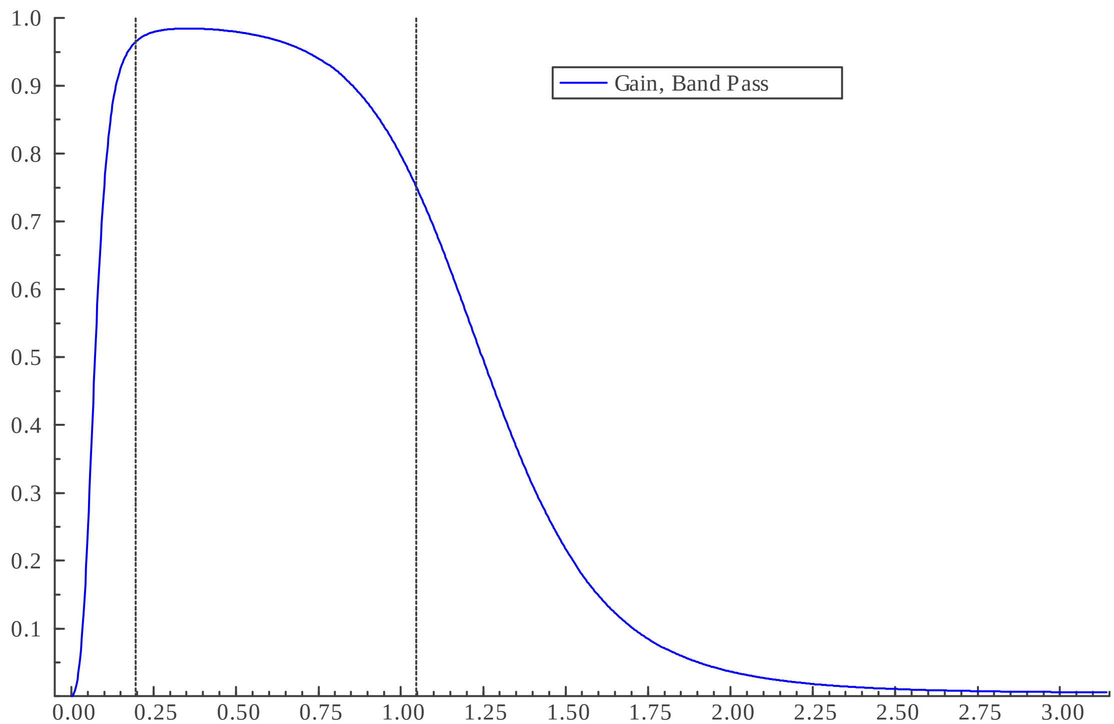


Figure 20: Estimated Band Pass filter for extracting the cycle in Investment for n = 6 with Balanced form, shown with ideal filter boundaries (dotted lines).

from the modelling approach) is substantial for the Investment time series. Figure 19 shows the two extracted cycles. Most notably, the amplitude of fluctuations is significantly greater for the modelled cycle; this is particularly evident over the last half of the sample period. The recession of 1990-1 has a deeper trough by nearly 5 percentage points whereas the peaks during 2001 and prior to the Great Recession's onset are about 6 percentage points higher than for the BK output. The modelled cycle starts to move down during 2008 a few quarters before the BK filtered series, which may indicate more timely turning point detection. All told, the peak-to-bottom distance is almost 15 percentage points wider over the course of the Great Recession. During the last few years of the sample, a shallow trough in the cycle arises (around the 0.0 mark), followed by the beginning of a slight turnaround.

Figure 20 shows the estimated gain function for Investment with $n = 6$ (Balanced model) with the ideal filter boundaries marked by dotted lines. The gain passes through substantially more lower frequency movements below the left edge. The ideal filter remains fixed and fails to adapt to the stronger cycle, which makes the cross-over point (which is where the gain falls to approximately 1/2) between cycle spectrum and trend pseudo-spectrum occur at a lower frequency. The gain also admits somewhat more high frequency parts than the ideal filter.

Appendix F shows results for Consumption of Services, an example for which the estimated cycle is instead magnified by the BK filter,

# 7 Conclusions

The modelled representations of the ideal filter provide a solution to the Baxter-King filter's unstable gain profile, address the sample endpoint problem associated with the Baxter-King filter's truncation, and allow one to evaluate this filter's implicit assumptions about trend-cycle relationships and to quantify problems of mis-measurement that may occur when indiscriminately using such an ideal filter approximation. In certain situations, an ideal filter may be used with only moderate complications; however, to know whether this is the case requires some extra effort or insight beyond the mere construction of a gain function. Usually it will not be the case; for strongly cyclical series like Investment or for series such as Consumption of Services that have a weaker cyclical part, even basic findings about cyclical patterns can be adversely affected.

This paper advocates instead adaptive band pass filtering *through spectral emulation as opposed to gain emulation* and considers methodology based on various unobserved components models. This approach recognizes that frequency domain definitions such as Burns-Mitchell (1946) are intended for the cycle and are suggestive for its spectral shape. The entire premise of band-pass filtering is built on implicit decomposition – hence the need to separate out the cycle from the observed time series, by cutting out the low frequency parts linked to stochastic trend and the higher-frequency noise. It is intuitive that the best filter for the task capitalizes on information about how the cyclical dynamics relate to those of trend and noise. Here, we consider models that make the decomposition explicit and use a modelling framework with several advantages.

As empirical evidence, we have examined twelve series drawn from U.S. national accounts data that have diverse stochastic properties and analyzed them with respect to 36 possible ideal filter representations. The basic conclusion is that the ideal filter involves a pervasive mis-specification problem whose severity varies across series. Even for a series like real GDP where the divergence of the gain function (from the optimal one) happens to be more moderate and the resulting consequences less dire, the models underpinning the ideal filter provide a poor statistical representation of the data. This is problematic for forecasting and for designing policy or making economic decisions, which routinely rely on having accurate portrayals of series' dynamics and their trend path or cyclical position. Using an adaptive band-pass provides the remedy via property-consistent extraction of cycles achieved in an econometrically optimal manner.

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# *Supplement to* Modelled approximations to the ideal filter with application to GDP and its components

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U.S. Census Bureau

December 7, 2018

## Appendix A Adaptive Band-Pass filters: Practical Application

This appendix discusses adaptive band-pass filtering of time series as implemented in the Ox computer language. The motivation for the program is to provide a flexible and consistent function, that estimates the cyclical component of a given sample of observations. The method is based on the broad class of generalized Butterworth band pass filters. A perfectly sharp, or "ideal" gain is one possible shape within this class, and we allow for approximations to this shape, though generally it is inadvisable to use indiscriminately. In each case, the approach that the program follows is to choose the optimal band-pass through model fitting and analysis, along with smoothness considerations. We impose a rough band-pass profile, which in practice is quite flexible, as it simply means that the gain cuts out low frequencies sufficiently well (e.g. falls to one-half by some cutoff frequency on the left as the origin is approached and likewise dips to one-half on the right as the frequency increases to its maximum values associated with the noisy part of a series' fluctuations).

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## A.1 Method Summary

The model-based method helps ensure consistency by tailoring the band-pass to the dynamics of the input series. The generalized Butterworth filters encompass a broad array of gain patterns, and the program has conditions appropriate for the range of dynamics present in macroeconomic series. The stochastic cycle models allow for a variety of time-varying dynamics. Generally, the shape of the model-based band pass is reflective of the dynamic characteristics of the series. A band-pass that tapers off gradually reflects in part the role of stochastic movements in the cycle. The maximum of the cycle order is eight; this allows for very sharp gains. Intermediate values of $n$, for instance four to six, may be appealing in many applications, as this correspond to relatively well-formed band-pass filters and underlying cycles with smooth dynamics.

## A.2 Estimation and restrictions

The adaptive approach is implemented in a way that makes the minimal necessary restrictions, in a classical framework, to generate a representative band-pass for the input series. The parameters are kept within certain bounds to preserve the characteristic shape of the low pass and band pass.

Specifically, there is an interval of permissible values for the frequency to ensure sensible business cycle periods. To be appropriate for economic applications, the program constrains the central business cycle period, $2\pi/\lambda_c$, to lie between two and eight years. This ensures the band pass lies in the appropriate region in the frequency domain. If $\lambda_c$ is estimated without constraint, a plausible cycle or an appropriately shaped band-pass may not be obtained. The program is easily modified to estimate stochastic cycles of different periodicity ranges, suitable for data outside economics, by changing the central frequency interval.

Estimation of parameters is by maximum likelihood. The value of $m$ is two; this value is associated with smooth trend estimates and is appealing for low-pass filtering. Excessive sharpness in the low-pass is not necessarily desirable, and high orders $m$ do not correspond to plausible trend models as they generate unreasonable forecast paths. The cyclical index $n$ is in part selected from one to eight on the basis of a modified information criterion.

Signal-noise ratios, such as $q = \sigma_\zeta^2/(\sigma_\psi^2 + \sigma_\varepsilon^2)$, measure the relative variation of trend and stationary parts. Intervals are used for the signal-noise ratios, which impact the filter shapes, to make sure the estimated trend and cycle series fit with basic intuition. The trend, for instance, should have smoothness properties as well as some degree of adaptivity, and this is reflected in the form of the low-pass, that tapers off in the appropriate way at higher frequencies in the continuum.

In seeking to estimate a workable decomposition, the program attains a value of $q$ such that $10^{-5} < q < 1$. This covers a broad range of relative variation. Though $q$ will naturally vary among series, for ratios above one the trend is absorbing too much of the variation in the series. The trend is approaching the point of being deterministic for $q < 10^{-5}$, and a well-formed low-pass filter is not ensured. In bounding $q$ at $10^{-5}$, the gain function clearly displays the characteristic removal of the lowest frequencies. For all practical purposes, values of $q$ near the lower bound yield a close approximation to a deterministic trend - any lower value of $q$ would produce very similar observed patterns of variation in the trend. Further, deterministic trends are less reasonable for economic data that are subject to extensive sources of stochastic variation; at least some adaptability in the trend is desirable. For small samples especially, obtaining a $q$ of zero is likely due to the finite-series properties of the Maximum Likelihood Estimator of the trend shock variance.

In frequentist treatments, the shape of finite sample likelihood surfaces may sometimes make it difficult to obtain suitable parameter values through unrestricted estimation. This issue can be highly relevant for unobserved components models. As a computationally more costly alternative, a Bayesian approach could provide one route to solving this problem. Here, we retain a classical framework and impose loose and reasonable constraints.

## A.3 Implementation

The program executes model-fitting, filter design. and analysis of filter output. The appropriate finite sample filters are obtained and the end-point adjustments are done optimally through either the Kalman smoothing algorithm or the matrix formula method. One aspect of interest is the analysis of gain functions which the program shows graphically. In the adaptive method, the different shapes of band-pass suitable for diverse time series may be compared in an objective fashion. The procedure leads to the choice of the optimal band-pass through model estimation and an emphasis on a well-formed band-pass and smooth cycle. The "ideal" filter is approximated with cyclical indices $n = 4, 6$, and 8.

Note that if the use of a band-pass filter seems inappropriate for a given series, then the resulting output will reflect this. For instance, if the series appears to follow a trend-only process, then the program would give highly diluted output from a greatly dampened band-pass filter. This contrasts with an automated procedure such as the Hodrick-Prescott filter or Baxter-King band-pass option where a nonsense "cycle" would be computed from white noise or trend-only input series.

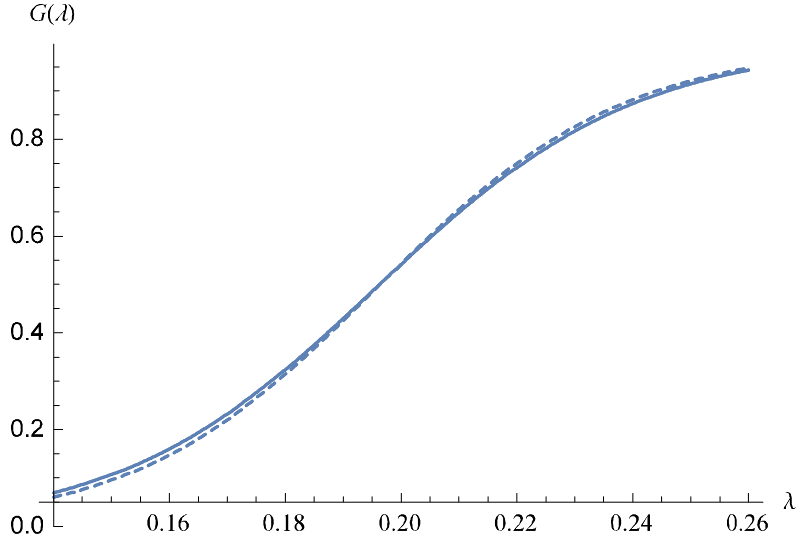


Figure B.1: Close-up showing a low-frequency part – the left-side tail – of gain functions for different ideal filter approximations with $n = 6$,both with $\rho = 0.8, \phi = 0.97$. For the solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.0124,$ $0.0322, 0.4910\}$, while for the dotted curve, $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{2.524, 0.279, 0.398\}$.

# Appendix B The Ideal Filter: Modelled version

This Appendix contains further details regarding the approximations used for an ideal gain. For $n = 6$ figure B.1 displays the gain functions at the low end of the spectrum for $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{2.524, 0.279,$ $0.398\}$ as the dotted line and for $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.0124, 0.0322, 0.4910\}$ as the solid line. The very small deviation differs from the noticeable divergence at the higher frequencies ostensible in figure 3.

Figure B.2 shows the two extreme cases of ideal filter approximations for $n = 4$, that is, for the solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.007957, 0.111, 0.4785\}$, while for the dotted curve, $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.7635, 0.4276,$ $0.3421\}$. Figure B.3 displays the right tail, where a larger difference between the two extremes can be seen compared to the sixth order case. The divergence at the low end of the spectrum is indicated in B.4; now there is a clearer distinction between the two curves relative to sixth order. Overall, the fourth order case provides a looser approximation to the ideal filter. From the standpoint of gain emulation, this represents a minus, but from the viewpoint of modelling and adaptive filtering, this is neutral and other considerations hold more importance. The plots for the eighth order case (not shown for brevity) show a similar pattern, albeit with a reduced discrepancy between the two extreme cases.

Now, take the sixth order case with the preferred parameter settings; these values are selected as indicated in the text, by fitting the ideal filter models to a set of U.S. macroeconomic series and examining model diagnostics. Hence, the parameters are $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.05722, 0.174, 0.4146\}$. For the corresponding filter, figure B.5 shows the observation weights when viewed over the range of lags and

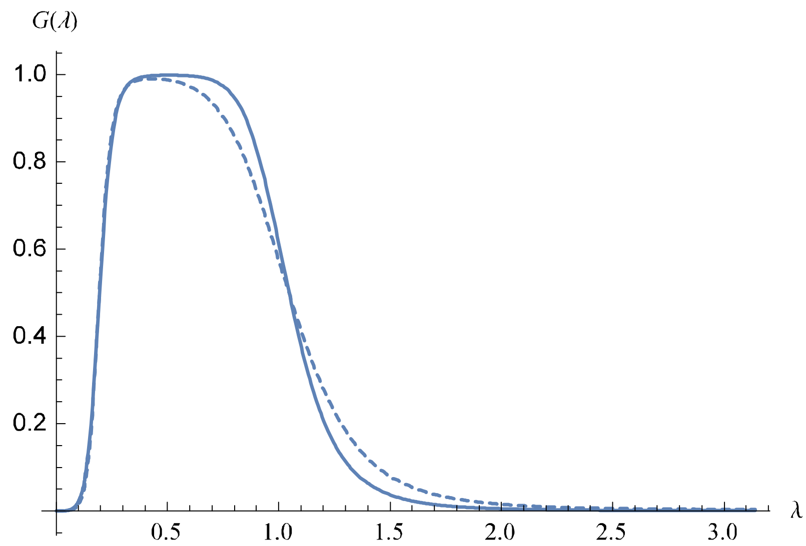


Figure B.2: Gain functions for different ideal filter approximations with $n = 4$,both with $\rho = 0.8, \phi = 0.97$. For the solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.007957,\ 0.111,\ 0.4785\}$, while for the dotted curve, $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.7635,\ 0.4276,\ 0.3421\}$.

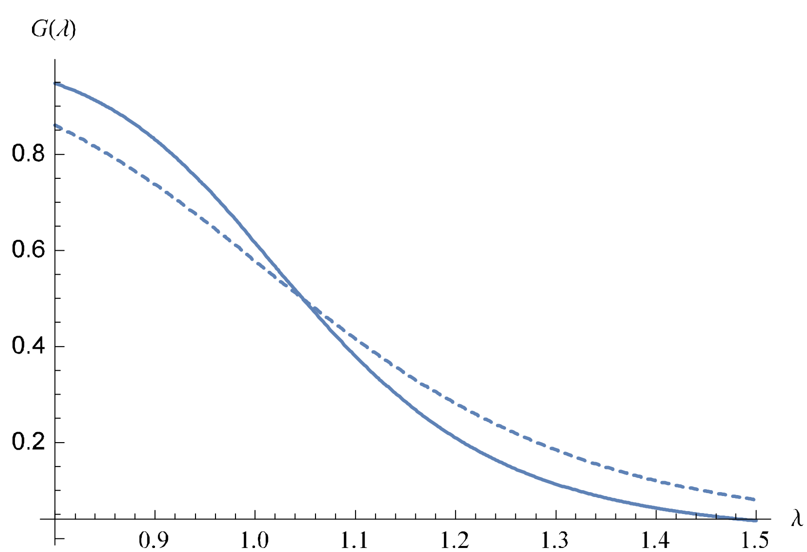


Figure B.3: High-frequency part of gain functions for different ideal filter approximations with $n = 4$,both with $\rho = 0.8, \phi = 0.97$. For the solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.007957,\ 0.111,\ 0.4785\}$, while for the dotted curve, $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.7635,\ 0.4276,\ 0.3421\}$.

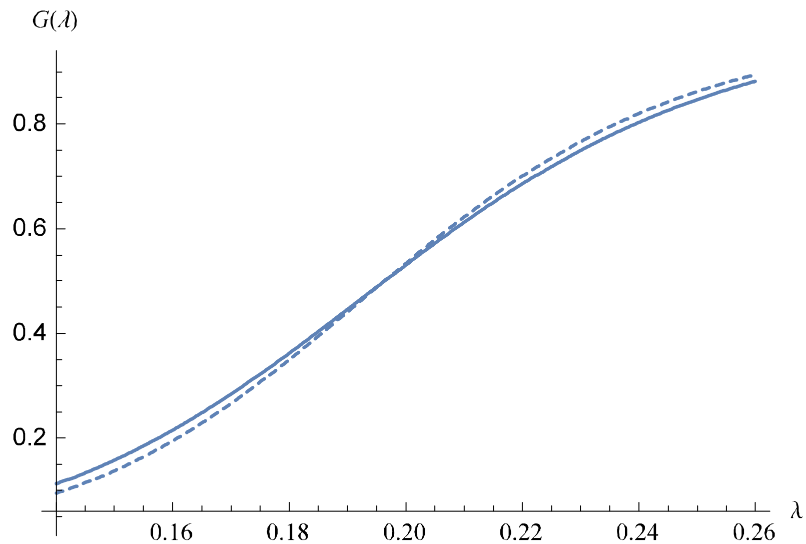


Figure B.4: Low-frequency part of gain functions for different ideal filter approximations with $n = 4$,both with $\rho = 0.8, \phi = 0.97$. For the solid curve $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.007957,\ 0.111,\ 0.4785\}$, while for the dotted curve, $\{\overline{q}_\zeta, \overline{q}_\kappa, \overline{\lambda}_c\} = \{0.7635,\ 0.4276,\ 0.3421\}$.

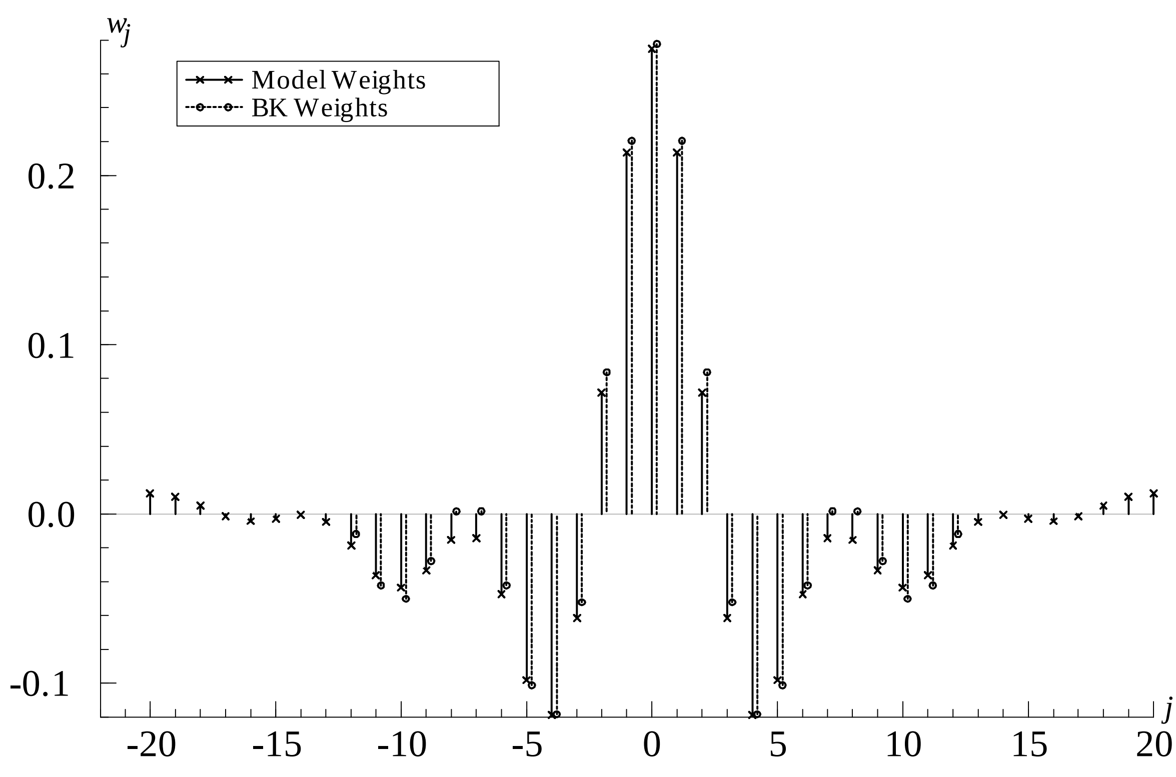


Figure B.5: Weight functions for model representation of ideal filter and for BK filter.

leads from the time point of estimation. Such weighting patterns give a direct picture of what the filters are essentially doing. Figure B.5 shows how the modelled filter places slightly less emphasis on the current and immediately adjacent observations. A bigger difference arises at lags seven and eight, where the model weights are negative, while the BK weights are nearly zero. The modelled filter compensates for this reduced weighting at lower lags by having the weights increase at lag ten and beyond.

# Appendix C The ideal filter and nonstationarity in the frequency domain

Given its convenience and compact expressions, the frequency domain[1] is often used in the analysis of cyclical movements in economic data. The spectrum gives the decomposition of its variability into oscillations with different frequencies. The gain function of a filter gives a visible summary of its effects; it shows the impact on the amplitudes of different frequency parts when applying the filter. As employed here, the defining characteristic of the band-pass filter is a *general* concentration in its gain over mid-range frequencies. (In the same way, low-pass filters are characterized by a tendency to cut out higher frequencies.) The absolute sharpness of the ideal filter is not required and, in fact, will rarely be appropriate for economic time series subject to both stochastic cyclical and trend (and perhaps also noise). Later we will assess the impact of the way, in which the BK filter emulates this filter in a finite sample, on the actual, realized gain. For now, we use the hypothetical limit of the ideal filter to explain the distortions in the simple illustrations above.

Assume that we apply the pure ideal filter to a stationary data process of interest. Then the gain shown in figure C.1 is directly applied to the process. For a white noise process, taking the gain-squared times the original spectrum, gives an adjusted, or output spectrum proportional to the same block shape as in figure C.1. From this illustration we can understand the spurious creation of the cycle in the white noise example above. The absence of a cycle in the process $y_t = \mu_0 + \varepsilon_t$ means that the optimal gain for extracting it is simply zero. Correspondingly, the model estimation gives a conclusion of nearly white noise observations, so that the modelling-estimation-filtering sequence successfully handles this situation. In contrast, applying a fixed band pass filter to white noise generates a process, whose power

[1] Indeed, some of the very early work on the subject by Burns-Mitchell employed a definition of a frequency interval for business cycle fluctuations as discussed below. Here, we use a broader definition of cyclical components and account for their stochastic character and interaction with trend or noise in the series of interest.

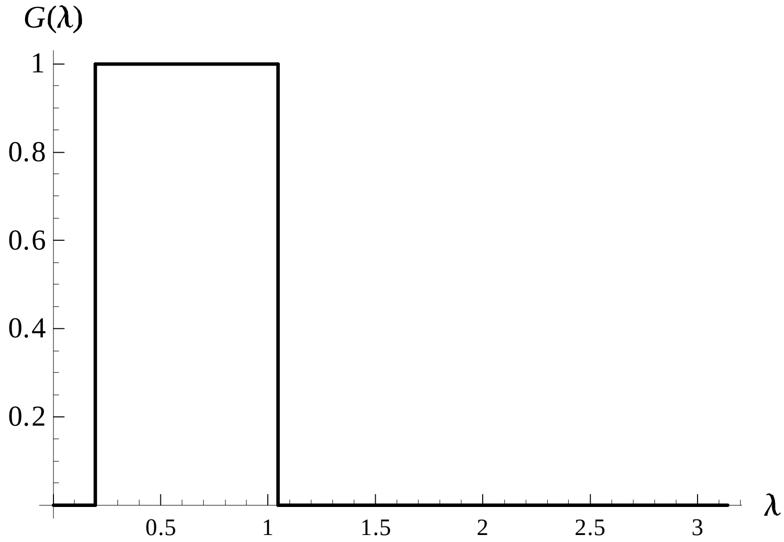


Figure C.1: Gain of ideal filter applicable to a stationary process.

over the restricted band of intermediate frequencies means that its oscillations will mimic those of the business cycle variety.

Most economic data are nonstationary, and several authors have considered potential pitfalls when filtering nonstationary series. This question was originally addressed in Harvey and Jaeger (1993) for the HP filter and Murray (2003) for the BK filter. In the following, I discuss how the source of error in band-pass filtering depends on the presence of additional components, apart from the component of interest, and that such inherent distortions occur regardless of the stationarity each extra component.

It might seem that there are special implications of applying a band-pass filter in the nonstationary case given the dichotomy between stationary and nonstationary dynamics. For the HP filter, Harvey and Jaeger (1993) considered the filtering of nonstationary series and argued that the presence of nonstationarity leads to an artificial spectral maximum, or spurious cyclical behavior. The basis for the argument is that the gain function applied to the (appropriately differenced) stationary series peaks at certain frequency, determined by how the gain is altered to account for the differencing. The same line of reasoning would apply to the ideal filter, and indeed Murray (2003) places special emphasis on the nonstationarity or presence of trend in the input series.

Consider a component given by $\theta_i(L)\xi_{i,t}/(1-L)\varphi_i(L)$. When an ideal gain function is applied, as long as there is a cancellation in the (1-$L$), we can view the equivalent operation of applying a modified gain to the underlying stationary process $\theta_i(L)\xi_{i,t}/\varphi_i(L)$. If we assume that that ideal filter is given by a limiting filter in the Generalized Butterworth class with order $m$ at least one, then such a cancellation is guaranteed to occur. The modified gain, displayed in figure C.2, rises toward lower frequencies and then falls off suddenly. Depending on the spectrum of $\theta_i(L)\xi_{i,t}/\varphi_i(L)$, the operation therefore induces

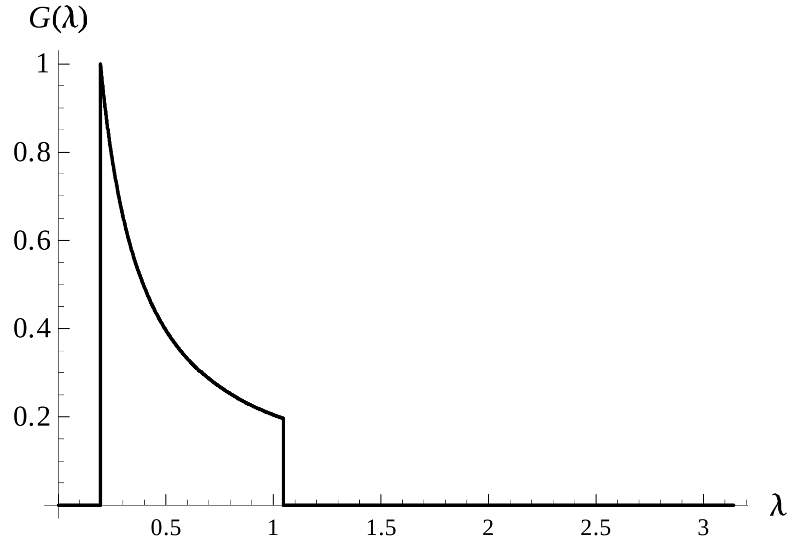


Figure C.2: Gain of ideal filter applicable to the stationary process underlying an integrated of order one process.

some cyclical behavior with frequencies concentrated at the low end. Rather than hinging on the nonstationarity, this effect actually depends on the presence of the extra component and its spectrum or pseudo-spectrum. That is, the pseudospectrum of a stochastic trend has power at mid-range frequencies – this being the key – which the band-pass filter draws.

The simplest stochastic trend is the random walk. It has a pseudospectrum of $f(\lambda) = 2^{-1}(1 - cos\lambda)^{-1}$, and when a band-pass filter is applied, the mid-range of this spectrum is selected out. The square root of the resulting spectrum has the same shape as the altered gain in figure C.2. Therefore, the contribution to the output occurs because a random walk contains some frequency components at intermediate points in the spectrum. As long as the resulting spectrum exists, then the band-pass filter would send through a component whose spectrum resembles figure C.2.

The smooth trend, an I(2) process, often produces an attractive signal that tracks gradual, long-run movements. Figure C.3 shows the gain function applied to the underlying stationary process after differencing twice; it has the shape of the square root of the pseudospectrum of the smooth trend, or integrated random walk, which has a more pronounced rise toward lower frequencies. Here, as before, the key aspect of nonstationarity is that it requires a certain condition, that the filter cancels out the differencing operator of the process; in the frequency domain, this is identical to annihilation of the infinite spectrum at zero. Apart from this needed cancellation, the essence of the error just involves the presence of extra components that have share frequency parts with the cycle.

In the case of a trend, the gain function applied to the stationarized trend process could just as well be viewed as the original band-pass applied to the trend pseudospectrum, which rises toward lower

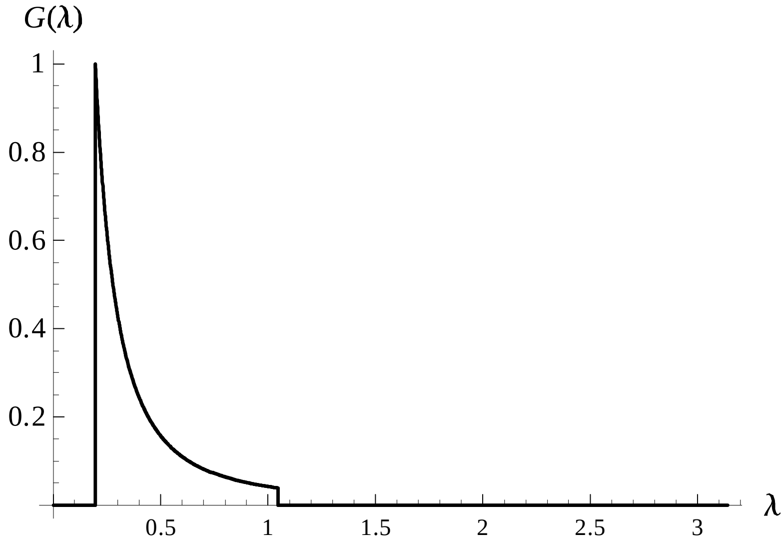


Figure C.3: Gain of ideal filter applicable to the stationary process underlying an integrated of order two process.

frequencies. Then, the additional component contributes to the filtered process. If one replaces the $I(d)$ process by a stationary AR(1) process with coefficient near unity, the spectrum of the output would have a similar contribution. Instead of the order of integration, the key to the problem lies in the spectrum of the extra component, which shares frequencies with the cycle. So the presence of additional components, apart from the component of interest, is paramount regardless of whether each extra component is stationary or nonstationary.

The motivation for using band-pass filters is a frequency-based concept: in the context of economic time series, the presence of a peak in the spectrum at some intermediate periodicity, with most of their spectral power in a surrounding interval, defines cyclical fluctuations. Since the evolution in the overall business cycle tend to dominate these fluctuations, in macroeconomics a more specific definition that focusses on business cycle frequencies has often been used in previous work. Here we work with the more general definition, recognizing that recurring and persistent episodes of advance or decline may, for a given aggregate, arise from factors other than changes in the underlying state of the business cycle. Correspondingly, here we use "band-pass filter" to denote a general and flexible kind of filter compared to the ideal filter, for instance (or any fixed filter, for that matter). Requiring only a *general* concentration over a neighborhood of medium-term periodicity still allows for a gain that falls off continuously at high- and low- frequency extremes; such curvature is crucial for time series with frequency overlap between components, as occurs for the stochastic trend and cyclical parts in economic data.

In contrast to a stationary process, for which the spectrum has a finite value at the zero frequency (so no question arises about the existence of the spectral output), the spectral power of a nonstationary component rises indefinitely as the frequency approaches zero. Nevertheless, when the product of the squared gain function and the pseudospectrum has a finite limit at $\lambda = 0$, we can still work with the pseudospectrum itself. Given the dichotomy between stationary and nonstationary dynamics, it may seem, at first, that there are special implications and very distinct conditions in the presence of nonstationarity; actually, however, the occurrence of distortions when applying a band-pass filter to a nonstationary process arise from the same basic source as those for a stationary input process. The technical requirement for existence of the output is that the filter factors as $F(L) = h(L)(1-L)^d$, so application of the filter gives a cancellation of the $(1-L)^d$ factor in the time domain. With this mathematical condition satisfied, rather than having to consider the adjusted filter applied to a differenced component, we can effectively analyze $F(L)$ applied directly to the observed time series. Working with the process's pseudospectrum and the original gain is conceptually simpler and gives a clearer interpretation of filter formation and effects.

# Appendix D Tables of Empirical Results

The attached Appendix D file contains the full set of empirical results relating to parameter estimates and diagnostics for the adaptive filter, with models fully fitted to series. Additionally, results on the ideal filter fits and diagnostics are presented for the four cycle indices 4, 6, and 8.

# Appendix E Stochastic Cycle Models

The first form is referred to as the "Balanced" model, as there are symmetric assumptions on the shock vectors. In particular, the balanced *nth-order stochastic cycle* $\psi_{n,t}$, for positive integer $n$, is defined by

$$\begin{bmatrix} \psi_{1,t} \\ \psi^*_{1,t} \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{1,t-1} \\ \psi^*_{1,t-1} \end{bmatrix} + \begin{bmatrix} \kappa_t \\ \kappa^*_t \end{bmatrix} \tag{E.1}$$

$$\begin{bmatrix} \psi_{i,t} \\ \psi^*_{i,t} \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{i,t-1} \\ \psi^*_{i,t-1} \end{bmatrix} + \begin{bmatrix} \psi_{i-1,t-1} \\ \psi^*_{i-1,t-1} \end{bmatrix}, \quad i = 2, ..., n$$

where $\kappa_t, \kappa^*_t \sim WN(0, \sigma^2_\kappa)$ are uncorrelated.

The term balanced refers to the symmetry in its equations. For the balanced form, analytical expressions are available for the time and frequency domain properties, as set out in Trimbur (2006). The expression for the power spectrum is

$$f_{\psi_n}(\lambda;\rho,\lambda_c) = \sigma_\kappa^2 \left[ \frac{\sum_{j=0}^{n}\sum_{k=0}^{n}(-1)^{j+k}\begin{pmatrix} n \\ j \end{pmatrix}\begin{pmatrix} n \\ k \end{pmatrix}\rho^{j+k}\cos\lambda_c(j-k)\cos\lambda(j-k)}{(1+4\rho^2\cos^2\lambda_c+\rho^4-4\rho(1+\rho^2)\cos\lambda_c\cos\lambda+2\rho^2\cos 2\lambda)^n} \right] \tag{E.2}$$

and the spectral density is

$$h_{\psi_n}(\lambda;\rho,\lambda_c) = (1/\sigma_\psi^2) f_{\psi_n}(\lambda;\rho,\lambda_c) \tag{E.3}$$

where the cycle variance is

$$\sigma_\psi^2 = \frac{\sigma_\kappa^2 \sum_{i=0}^{n-1}\begin{pmatrix} n-1 \\ i \end{pmatrix}^2 \rho^{2i}}{(1-\rho^2)^{2n-1}} \tag{E.4}$$

$$y_t = \mu_{m,t} + \psi_{n,t} + \varepsilon_t, \quad , \quad \varepsilon_t \sim WN(0,\sigma_\varepsilon^2), \quad t = 1,...,T \tag{E.5}$$

where $WN$ denotes 'white noise', a serially uncorrelated process with mean zero and constant variance.

The Butterworth form of the $n-$th order stochastic cycle $\psi_{n,t}$ is given by

$$\begin{bmatrix} \psi_{1,t} \\ \psi_{1,t}^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{1,t-1} \\ \psi_{1,t-1}^* \end{bmatrix} + \begin{bmatrix} \kappa_t \\ 0 \end{bmatrix} \tag{E.6}$$

$$\begin{bmatrix} \psi_{i,t} \\ \psi_{i,t}^* \end{bmatrix} = \rho \begin{bmatrix} \cos\lambda_c & \sin\lambda_c \\ -\sin\lambda_c & \cos\lambda_c \end{bmatrix} \begin{bmatrix} \psi_{i,t-1} \\ \psi_{i,t-1}^* \end{bmatrix} + \begin{bmatrix} \psi_{i-1,t} \\ 0 \end{bmatrix}, \quad i = 2,...,n$$

where $\kappa_t \sim WN(0,\sigma_\kappa^2)$. For (E.6), the symmetry in the shock vectors no longer holds. The advantage of such a form is that it allows for compact gain functions when optimal filters/estimators are formed from an overall model.

For both (E.1) and (E.6), $2\pi/\lambda_c$ is a central period of oscillation, and $\rho$ is a damping parameter. The parameter $\rho$ satisfies $0 < \rho \leq 1$, while $0 \leq \lambda_c \leq \pi$. As (E.1) and (E.6) have pairs of complex conjugate roots $\rho e^{\pm i\lambda_c}$ with modulus $\rho$, $\psi_{n,t}$ is stationary if $|\rho| < 1$. The models guarantee a peaked spectral shape, with a certain width, centered around $\lambda_c$. For $n = 1$, the shocks to $\psi_{1,t}$ are random, but for $n > 1$, the disturbances to $\psi_{n,t}$ are periodic; this leads to a reinforcement of the cycle at the frequency $\lambda_c$. The interpretation of the damping factor $\rho$ changes with $n$, so that different values of $\rho$ are appropriate for different orders.

# Appendix F Additional Graphs of Empirical Results

In moving from high to very high orders, there are further increases in the sharpness of the estimated band-pass filter's gain function. This finding is illustrated in figure F.1, which shows results for $n = 6$ vs. $n = 12$ for Investment. This results in a slight increase in the amount of noise removed from the cycle.

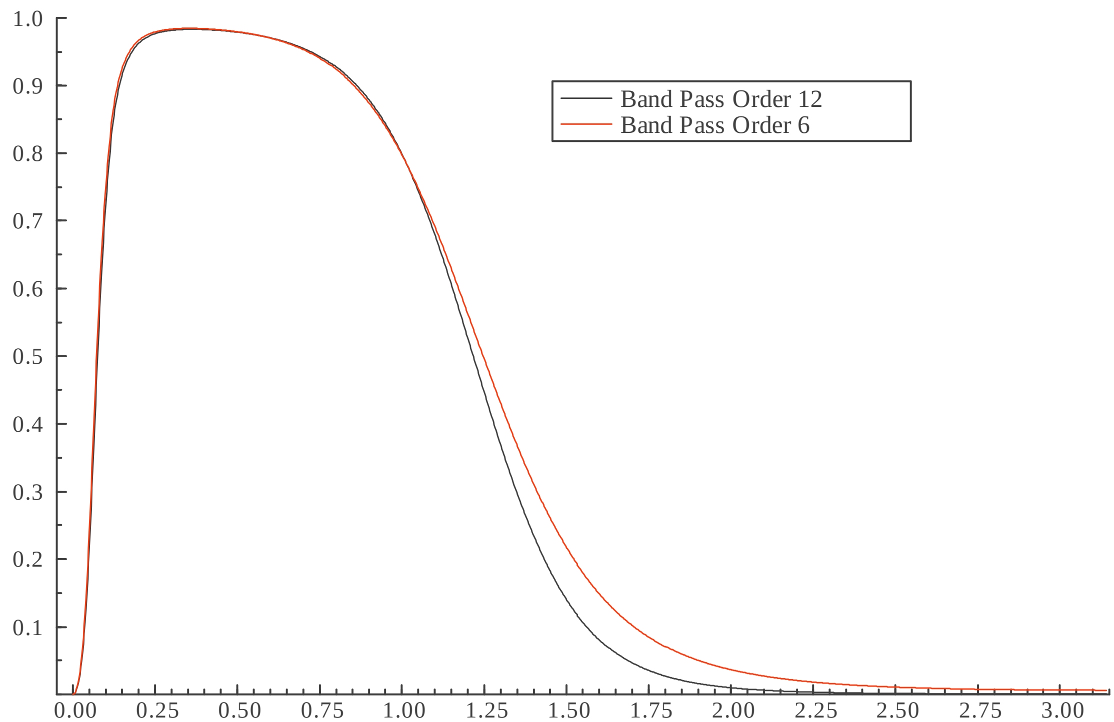


Figure F.1: Estimated Band Pass filter for extracting cycle in Investment (in logs) for n = 6 and $n = 12$ with Balanced form.

The extracted cycles for Residential Investment for $n = 8$ are displayed for both Balanced and Butterworth models in figure F.2. This illustrates how the two different forms can have different implications for the extracted cycle, as the amplitude of fluctuations are larger for the Butterworth forms. The larger amplitude is reflected in figure F.3 by greater inclusion of the lower-frequency portion of the cyclical frequency range. This example involves a highly cyclical series that also has substantial

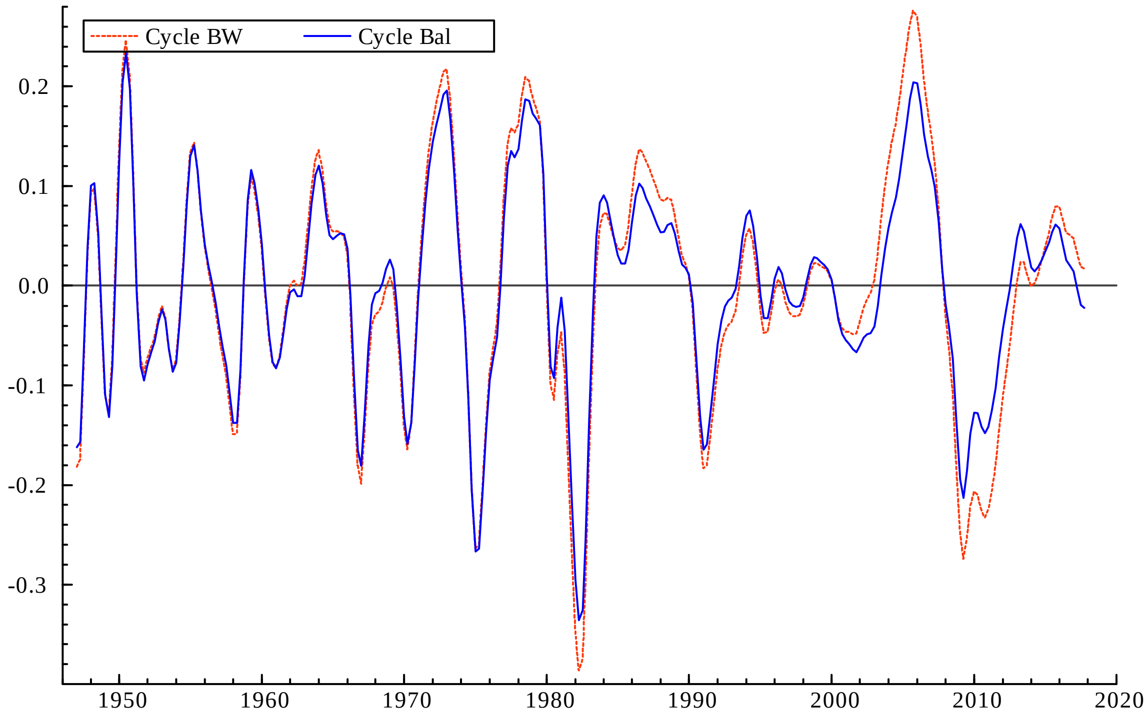


Figure F.2: Estimated cycle in Residential Investment (in logs) for n = 8 with Balanced and Butterworth forms.

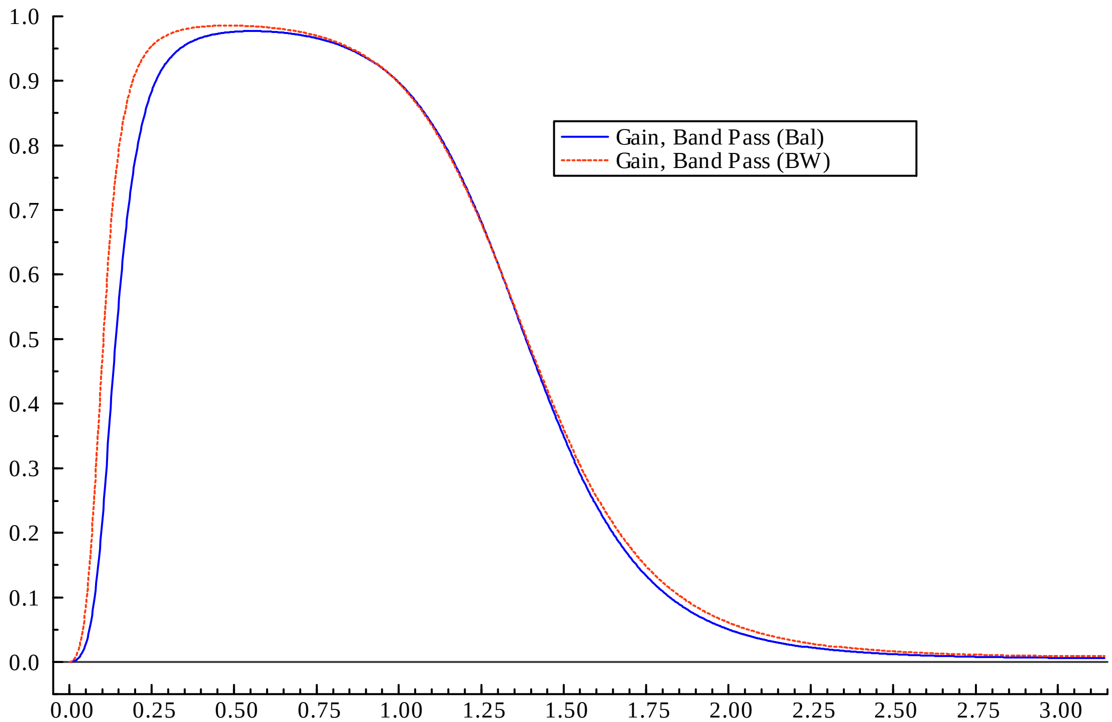


Figure F.3: Estimated Band Pass filter for extracting cycle in Residential Investment (in logs) for n = 8 with Balanced and Butterworth forms.

variability in the direction of its stochastic trend. In many cases, the differences between Balanced and Butterworth forms appear more subtle in practice.

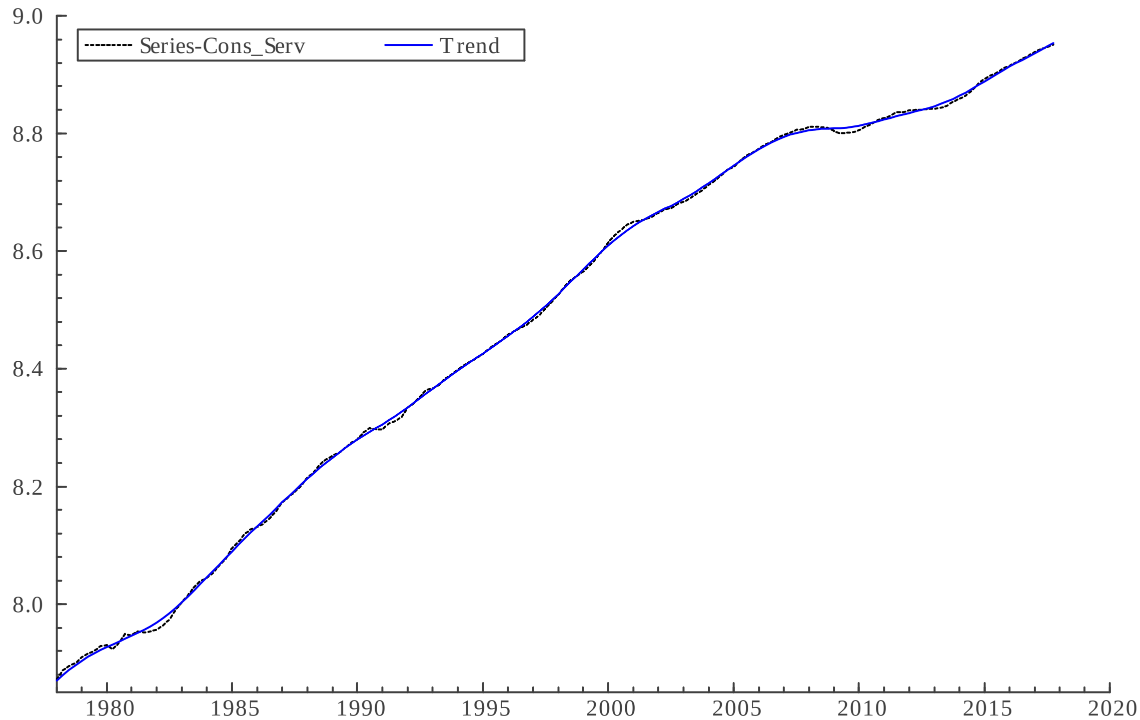


Figure F.4: Estimated Trend in Consumption of Services for $n = 6$ with adaptive modelled filter, compared to the BK filter.

Figures F.4 and F.5 show the resulting trend and cycle for Consumption of Services, an example with a relatively muted cyclical component; the trend is shown over the second half of the sample period to help distinguish it from the observed series. In this example, the trend growth dominates short-term movements, and the estimated cycle is overstated by the BK filter and modelled ideal filter. Hence, the BK filter artificially enhances the cyclical movements and gives a falsely amplified signal. The end-of-series estimates, which are unavailable with the BK procedure, reveal a continued down-turn in the cycle.

Figure F.6 displays the associated gain function for Consumption of Services. The estimated gain cuts out significant frequency parts around the left boundary of BK's ideal filter. The gain does not rise until a higher frequency within the band is reached, and the peak of the band-pass is well below 0.9. In contrast to Baxter and King's targeted filter, there is no requirement on the gain being unity or close to unity at its maximum, as that depends on the power spectrum of the cycle and how it relates to trend and noise (pseudo-)spectra.

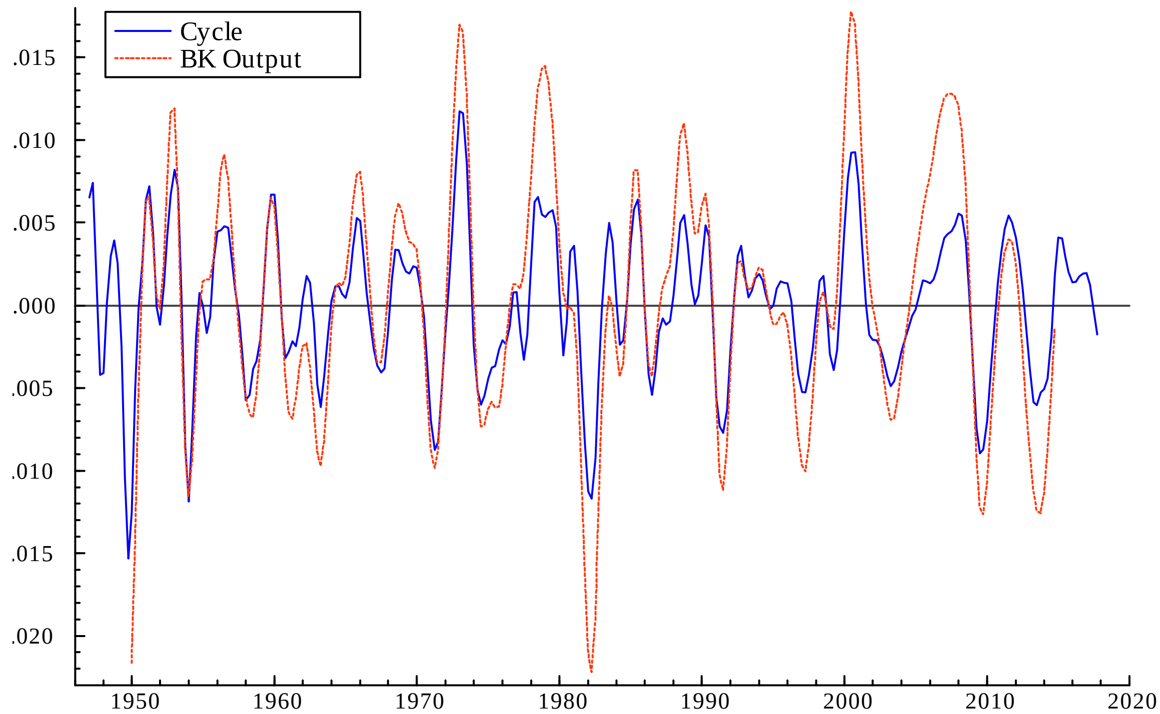


Figure F.5: Estimated Cycle in Consumption of Services for $n = 6$ with adaptive modelled filter, compared to the BK filter.

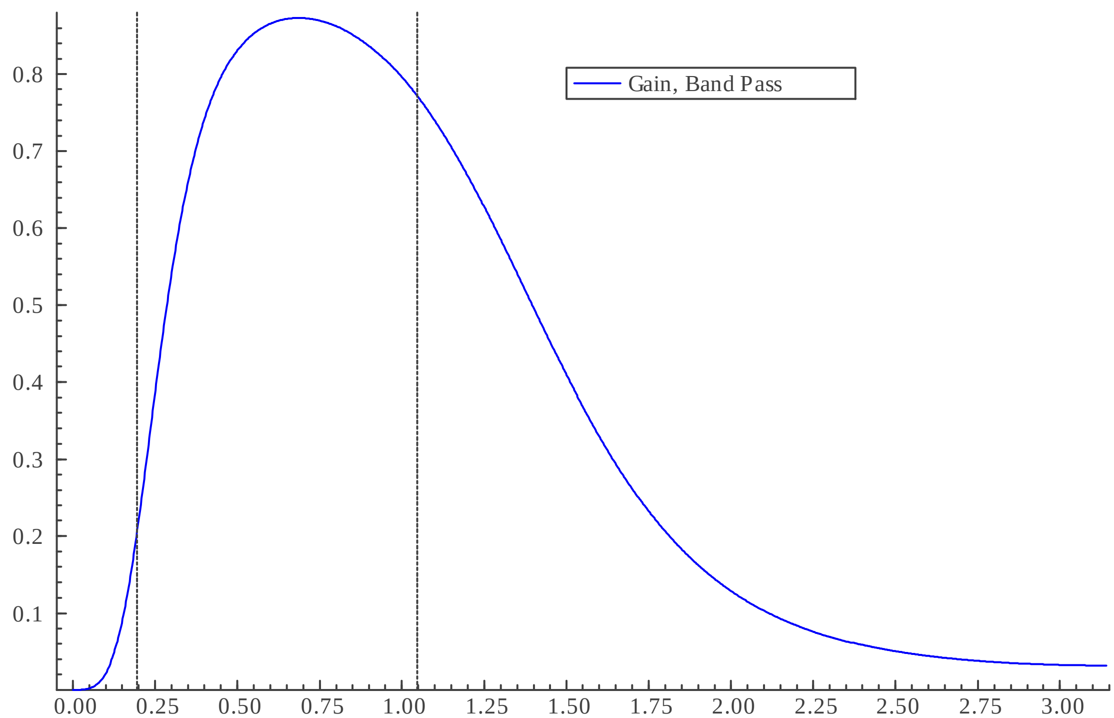


Figure F.6: Estimated Band Pass filter for extracting cycle in Consumption of Services for n = 6 with Balanced form, shown with ideal filter boundaries (dotted lines).

Tables of Parameter Estimates - Set D1

-Results for time series data taken from BEA: RealGDP and components

Univariate Models.
Model Type for Observations: Trend + Cycle + Irreg
Trend Model: Damped (Order 2) in Standard Form
The cycle is an n-th order stochastic cycle, as in Harvey-Trimbur (2003).
The cycle's model form is either Butterworth ('BW') or Balanced ('Bal').

Results for Series: 'Real GDP'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'RealGDP'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 5.426e-006 | 1.219e-008 | 0.0445 | 0.007326 | 0.95 | 4.341e-005 | 0.8903 | 14.09 |
| Balanced (Order 1) | 2.294e-006 | 2.229e-008 | 0.01029 | 0.007623 | 0.95 | 4.557e-005 | 0.892 | 17.51 |
| Butterworth (Order 2) | 9.009e-007 | 7.817e-006 | 0.001992 | 0.007737 | 0.95 | 4.186e-005 | 0.7443 | 32 |
| Balanced (Order 2) | 8.98e-007 | 8.426e-006 | 0.001969 | 0.007735 | 0.95 | 3.477e-005 | 0.7145 | 32 |
| Butterworth (Order 3) | 9.043e-007 | 1.182e-005 | 0.001977 | 0.007724 | 0.95 | 2.6e-005 | 0.6009 | 32 |
| Balanced (Order 3) | 1.016e-006 | 1.223e-005 | 0.002448 | 0.007723 | 0.95 | 2.034e-005 | 0.5832 | 32 |
| Butterworth (Order 4) | 9.548e-007 | 1.317e-005 | 0.002185 | 0.00772 | 0.95 | 1.894e-005 | 0.5019 | 32 |
| Balanced (Order 4) | 1.109e-006 | 1.351e-005 | 0.002864 | 0.007715 | 0.95 | 1.407e-005 | 0.494 | 32 |
| Butterworth (Order 5) | 1.004e-006 | 1.38e-005 | 0.002396 | 0.007718 | 0.95 | 1.508e-005 | 0.4313 | 32 |
| Balanced (Order 5) | 1.179e-006 | 1.409e-005 | 0.003191 | 0.007708 | 0.95 | 1.069e-005 | 0.4298 | 32 |
| Butterworth (Order 6) | 1.045e-006 | 1.415e-005 | 0.002575 | 0.007716 | 0.95 | 1.266e-005 | 0.3788 | 32 |
| Balanced (Order 6) | 1.232e-006 | 1.442e-005 | 0.003444 | 0.007703 | 0.95 | 8.612e-006 | 0.3814 | 32 |
| Butterworth (Order 7) | 1.078e-006 | 1.437e-005 | 0.002725 | 0.007709 | 0.95 | 1.102e-005 | 0.3381 | 32 |
| Balanced (Order 7) | 1.271e-006 | 1.461e-005 | 0.003637 | 0.007698 | 0.95 | 7.223e-006 | 0.3434 | 32 |
| Butterworth (Order 8) | 1.106e-006 | 1.452e-005 | 0.00285 | 0.007709 | 0.95 | 9.836e-006 | 0.3055 | 32 |
| Balanced (Order 8) | 1.302e-006 | 1.475e-005 | 0.003794 | 0.007695 | 0.95 | 6.23e-006 | 0.3127 | 32 |

Results for Series: 'Real Investment'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'RealInv'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 6.937e-006 | 6.529e-007 | 0.001062 | 0.009161 | 0.95 | 0.001835 | 0.9124 | 24.78 |
| Balanced (Order 1) | 6.256e-006 | 7.065e-007 | 0.0008853 | 0.009267 | 0.95 | 0.001572 | 0.8818 | 20.14 |
| Butterworth (Order 2) | 2.32e-006 | 0.0003089 | 0.0002309 | 0.009161 | 0.95 | 0.001111 | 0.7225 | 30.11 |
| Balanced (Order 2) | 3.363e-006 | 0.0003591 | 0.0003709 | 0.008971 | 0.95 | 0.000737 | 0.7055 | 23.59 |
| Butterworth (Order 3) | 2.419e-006 | 0.0004379 | 0.0002415 | 0.009161 | 0.95 | 0.0005897 | 0.6187 | 24.39 |
| Balanced (Order 3) | 2.632e-006 | 0.0004488 | 0.0002685 | 0.008918 | 0.95 | 0.0004269 | 0.5923 | 26.64 |
| Butterworth (Order 4) | 2.404e-006 | 0.0004765 | 0.0002399 | 0.009161 | 0.95 | 0.0003753 | 0.5424 | 21.78 |
| Balanced (Order 4) | 2.395e-006 | 0.000478 | 0.000238 | 0.008909 | 0.95 | 0.0002918 | 0.5104 | 29.05 |
| Butterworth (Order 5) | 2.384e-006 | 0.0004923 | 0.000238 | 0.009161 | 0.95 | 0.0002682 | 0.4831 | 20.18 |
| Balanced (Order 5) | 2.295e-006 | 0.0004908 | 0.0002257 | 0.008908 | 0.95 | 0.0002211 | 0.4484 | 31.19 |
| Butterworth (Order 6) | 2.369e-006 | 0.0005001 | 0.0002366 | 0.009161 | 0.95 | 0.0002071 | 0.4356 | 19.05 |
| Balanced (Order 6) | 2.312e-006 | 0.0004984 | 0.0002287 | 0.008905 | 0.95 | 0.0001707 | 0.4021 | 31.83 |
| Butterworth (Order 7) | 2.356e-006 | 0.0005045 | 0.0002356 | 0.009161 | 0.95 | 0.0001689 | 0.3966 | 18.21 |
| Balanced (Order 7) | 2.35e-006 | 0.0005032 | 0.0002342 | 0.008905 | 0.95 | 0.0001361 | 0.3655 | 32 |
| Butterworth (Order 8) | 2.347e-006 | 0.0005071 | 0.0002347 | 0.009161 | 0.95 | 0.0001433 | 0.364 | 17.56 |
| Balanced (Order 8) | 2.393e-006 | 0.0005062 | 0.0002404 | 0.008902 | 0.95 | 0.000112 | 0.3355 | 32 |

Results for Series: 'Real Residential Investment'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'ResInv'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 0.0002916 | 2.534e-007 | 0.115 | 0.005758 | 0.95 | 0.0008608 | 0.8969 | 14 |
| Balanced (Order 1) | 0.0002261 | 3.556e-007 | 0.06359 | 0.006705 | 0.95 | 0.0007869 | 0.8824 | 14 |
| Butterworth (Order 2) | 1.142e-005 | 7.849e-005 | 0.0005913 | 0.005758 | 0.95 | 0.001275 | 0.7903 | 32 |
| Balanced (Order 2) | 7.981e-006 | 0.0001026 | 0.0003678 | 0.005247 | 0.95 | 0.001017 | 0.7612 | 32 |
| Butterworth (Order 3) | 7.195e-006 | 0.0002016 | 0.0003329 | 0.005758 | 0.95 | 0.0007501 | 0.6539 | 32 |
| Balanced (Order 3) | 1.22e-005 | 0.0002124 | 0.000645 | 0.005537 | 0.95 | 0.000561 | 0.6275 | 32 |
| Butterworth (Order 4) | 7.767e-006 | 0.0002418 | 0.0003702 | 0.005758 | 0.95 | 0.0005134 | 0.5527 | 32 |
| Balanced (Order 4) | 1.784e-005 | 0.0002462 | 0.001072 | 0.005813 | 0.95 | 0.0003726 | 0.5319 | 32 |
| Butterworth (Order 5) | 9.546e-006 | 0.0002588 | 0.0004827 | 0.005758 | 0.95 | 0.0003914 | 0.4771 | 32 |
| Balanced (Order 5) | 2.263e-005 | 0.0002606 | 0.001486 | 0.005972 | 0.95 | 0.0002758 | 0.4624 | 32 |
| Butterworth (Order 6) | 1.187e-005 | 0.0002673 | 0.0006378 | 0.005758 | 0.95 | 0.0003202 | 0.419 | 32 |
| Balanced (Order 6) | 2.676e-005 | 0.0002677 | 0.001876 | 0.006068 | 0.95 | 0.0002188 | 0.4095 | 32 |
| Butterworth (Order 7) | 1.413e-005 | 0.0002721 | 0.0008002 | 0.005758 | 0.95 | 0.0002739 | 0.3735 | 32 |
| Balanced (Order 7) | 3.041e-005 | 0.0002717 | 0.002242 | 0.006131 | 0.95 | 0.000182 | 0.3677 | 32 |
| Butterworth (Order 8) | 1.617e-005 | 0.000275 | 0.0009554 | 0.005758 | 0.95 | 0.000242 | 0.3369 | 32 |
| Balanced (Order 8) | 3.365e-005 | 0.0002741 | 0.002584 | 0.006174 | 0.95 | 0.0001564 | 0.3339 | 32 |

Results for Series: 'Real Non-Residential Investment'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'NonRes_Inv'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 4.918e-005 | 8.841e-008 | 0.05562 | 0.01015 | 0.95 | 0.000249 | 0.9178 | 15.19 |
| Balanced (Order 1) | 1.213e-005 | 2.209e-007 | 0.005489 | 0.009647 | 0.95 | 0.0002934 | 0.9312 | 20.89 |
| Butterworth (Order 2) | 3.556e-006 | 3.683e-005 | 0.0008979 | 0.01015 | 0.95 | 0.0002803 | 0.781 | 32 |
| Balanced (Order 2) | 3.287e-006 | 4.201e-005 | 0.0007934 | 0.01001 | 0.95 | 0.000226 | 0.7475 | 32 |
| Butterworth (Order 3) | 3.319e-006 | 6.676e-005 | 0.0008165 | 0.01015 | 0.95 | 0.0001651 | 0.6373 | 32 |
| Balanced (Order 3) | 4.112e-006 | 7.007e-005 | 0.001118 | 0.009925 | 0.95 | 0.0001241 | 0.6164 | 32 |
| Butterworth (Order 4) | 3.619e-006 | 7.695e-005 | 0.0009319 | 0.01015 | 0.95 | 0.0001146 | 0.5363 | 32 |
| Balanced (Order 4) | 4.86e-006 | 7.939e-005 | 0.001438 | 0.009864 | 0.95 | 8.17e-005 | 0.5257 | 32 |
| Butterworth (Order 5) | 3.947e-006 | 8.171e-005 | 0.001061 | 0.01015 | 0.95 | 8.759e-005 | 0.4639 | 32 |
| Balanced (Order 5) | 5.463e-006 | 8.359e-005 | 0.001711 | 0.00982 | 0.95 | 5.955e-005 | 0.4599 | 32 |
| Butterworth (Order 6) | 4.232e-006 | 8.435e-005 | 0.001178 | 0.01015 | 0.95 | 7.108e-005 | 0.4096 | 32 |
| Balanced (Order 6) | 5.946e-006 | 8.585e-005 | 0.001937 | 0.009786 | 0.95 | 4.633e-005 | 0.4099 | 32 |
| Butterworth (Order 7) | 4.474e-006 | 8.6e-005 | 0.001278 | 0.01015 | 0.95 | 6.01e-005 | 0.3671 | 32 |
| Balanced (Order 7) | 6.336e-006 | 8.72e-005 | 0.002126 | 0.00976 | 0.95 | 3.769e-005 | 0.3703 | 32 |
| Butterworth (Order 8) | 4.677e-006 | 8.71e-005 | 0.001365 | 0.01015 | 0.95 | 5.234e-005 | 0.333 | 32 |
| Balanced (Order 8) | 6.659e-006 | 8.809e-005 | 0.002285 | 0.00974 | 0.95 | 3.169e-005 | 0.3382 | 32 |

Results for Series: 'Change in Inventories (As Percentage of Total Investment) '.     Sample period:  1947-1 to 2017-4.
Data Transformation: No transform.  Original series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'DPct_Inven'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 1.283e-008 | 0.000159 | 1e-005 | -7.618e-005 | 0.95 | 0.0006088 | 0.7678 | 18.42 |
| Balanced (Order 1) | 1.296e-008 | 0.0002229 | 1e-005 | -5.808e-005 | 0.95 | 0.0004249 | 0.7772 | 14.73 |
| Butterworth (Order 2) | 1.311e-008 | 0.0003474 | 1e-005 | -7.618e-005 | 0.95 | 0.0002442 | 0.6628 | 15.46 |
| Balanced (Order 2) | 1.317e-008 | 0.000364 | 1e-005 | -6.124e-005 | 0.95 | 0.0001405 | 0.6387 | 14.49 |
| Butterworth (Order 3) | 1.323e-008 | 0.000383 | 1e-005 | -7.618e-005 | 0.95 | 0.0001318 | 0.5893 | 14 |
| Balanced (Order 3) | 1.321e-008 | 0.0003871 | 1e-005 | -6.247e-005 | 0.95 | 6.914e-005 | 0.5468 | 14.4 |
| Butterworth (Order 4) | 1.367e-008 | 0.0003981 | 1e-005 | -7.618e-005 | 0.95 | 8.248e-005 | 0.523 | 14 |
| Balanced (Order 4) | 1.322e-008 | 0.000394 | 1e-005 | -6.306e-005 | 0.95 | 4.089e-005 | 0.4814 | 14.36 |
| Butterworth (Order 5) | 1.4e-008 | 0.000399 | 1e-005 | -7.618e-005 | 0.95 | 7.545e-005 | 0.4378 | 14 |
| Balanced (Order 5) | 1.322e-008 | 0.0003968 | 1e-005 | -6.341e-005 | 0.95 | 2.7e-005 | 0.4319 | 14.33 |
| Butterworth (Order 6) | 1.406e-008 | 0.0003947 | 1e-005 | -7.618e-005 | 0.95 | 8.441e-005 | 0.3579 | 14 |
| Balanced (Order 6) | 1.322e-008 | 0.0003982 | 1e-005 | -6.363e-005 | 0.95 | 1.922e-005 | 0.3927 | 14.31 |
| Butterworth (Order 7) | 1.405e-008 | 0.0003926 | 1e-005 | -7.618e-005 | 0.95 | 8.895e-005 | 0.3042 | 14 |
| Balanced (Order 7) | 1.322e-008 | 0.000399 | 1e-005 | -6.381e-005 | 0.95 | 1.442e-005 | 0.3608 | 14.3 |
| Butterworth (Order 8) | 1.403e-008 | 0.0003916 | 1e-005 | -7.618e-005 | 0.95 | 9.041e-005 | 0.2662 | 14 |
| Balanced (Order 8) | 1.322e-008 | 0.0003994 | 1e-005 | -6.392e-005 | 0.95 | 1.128e-005 | 0.3341 | 14.3 |

Results for Series: 'Real Consumption'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'Cons'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 1.451e-006 | 6.734e-006 | 0.01096 | 0.008118 | 0.95 | 3.147e-005 | 0.9262 | 21.88 |
| Balanced (Order 1) | 1.391e-006 | 8.554e-006 | 0.009708 | 0.008142 | 0.95 | 2.418e-005 | 0.9058 | 19.66 |
| Butterworth (Order 2) | 9.33e-007 | 1.464e-005 | 0.00431 | 0.008118 | 0.95 | 1.418e-005 | 0.7963 | 28.29 |
| Balanced (Order 2) | 1.01e-006 | 1.534e-005 | 0.005059 | 0.008141 | 0.95 | 8.454e-006 | 0.7634 | 23.5 |
| Butterworth (Order 3) | 7.776e-007 | 1.614e-005 | 0.002768 | 0.008118 | 0.95 | 7.601e-006 | 0.6733 | 32 |
| Balanced (Order 3) | 8.368e-007 | 1.633e-005 | 0.003281 | 0.008131 | 0.95 | 5.3e-006 | 0.6506 | 32 |
| Butterworth (Order 4) | 7.716e-007 | 1.664e-005 | 0.002701 | 0.008118 | 0.95 | 4.601e-006 | 0.5825 | 32 |
| Balanced (Order 4) | 8.684e-007 | 1.679e-005 | 0.003544 | 0.008133 | 0.95 | 2.857e-006 | 0.5726 | 32 |
| Butterworth (Order 5) | 7.807e-007 | 1.686e-005 | 0.002769 | 0.008118 | 0.95 | 3.133e-006 | 0.5139 | 32 |
| Balanced (Order 5) | 8.9e-007 | 1.702e-005 | 0.003732 | 0.008133 | 0.95 | 1.721e-006 | 0.5147 | 32 |
| Butterworth (Order 6) | 7.919e-007 | 1.698e-005 | 0.002859 | 0.008118 | 0.95 | 2.302e-006 | 0.4606 | 32 |
| Balanced (Order 6) | 9.08e-007 | 1.716e-005 | 0.003888 | 0.008134 | 0.95 | 1.125e-006 | 0.4692 | 32 |
| Butterworth (Order 7) | 8.023e-007 | 1.707e-005 | 0.002944 | 0.008118 | 0.95 | 1.784e-006 | 0.418 | 32 |
| Balanced (Order 7) | 9.205e-007 | 1.725e-005 | 0.004001 | 0.008132 | 0.95 | 7.782e-007 | 0.4325 | 32 |
| Butterworth (Order 8) | 8.115e-007 | 1.712e-005 | 0.003021 | 0.008118 | 0.95 | 1.438e-006 | 0.3832 | 32 |
| Balanced (Order 8) | 9.317e-007 | 1.732e-005 | 0.004094 | 0.008133 | 0.95 | 5.628e-007 | 0.4021 | 32 |

Results for Series: 'Real Consumption of Durables'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'ConsDur'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 7.678e-006 | 0.0001485 | 0.002469 | 0.01281 | 0.95 | 0.000805 | 0.9105 | 32 |
| Balanced (Order 1) | 5.073e-006 | 0.0001629 | 0.001274 | 0.01298 | 0.95 | 0.0007434 | 0.8974 | 32 |
| Butterworth (Order 2) | 5.341e-006 | 0.0003653 | 0.001348 | 0.01281 | 0.95 | 0.0003204 | 0.752 | 32 |
| Balanced (Order 2) | 4.971e-006 | 0.0003746 | 0.001206 | 0.013 | 0.95 | 0.0002549 | 0.7278 | 32 |
| Butterworth (Order 3) | 4.179e-006 | 0.0004089 | 0.0009523 | 0.01281 | 0.95 | 0.0001693 | 0.6343 | 32 |
| Balanced (Order 3) | 5.856e-006 | 0.0004124 | 0.001531 | 0.01308 | 0.95 | 0.0001268 | 0.61 | 32 |
| Butterworth (Order 4) | 4.069e-006 | 0.0004251 | 0.0009202 | 0.01281 | 0.95 | 0.0001056 | 0.5473 | 32 |
| Balanced (Order 4) | 6.451e-006 | 0.000426 | 0.001772 | 0.01312 | 0.95 | 7.572e-005 | 0.5293 | 32 |
| Butterworth (Order 5) | 4.359e-006 | 0.0004323 | 0.001012 | 0.01281 | 0.95 | 7.477e-005 | 0.4799 | 32 |
| Balanced (Order 5) | 6.848e-006 | 0.0004328 | 0.001943 | 0.01314 | 0.95 | 5.034e-005 | 0.4703 | 32 |
| Butterworth (Order 6) | 4.685e-006 | 0.0004362 | 0.001119 | 0.01281 | 0.95 | 5.74e-005 | 0.4273 | 32 |
| Balanced (Order 6) | 7.131e-006 | 0.0004369 | 0.002069 | 0.01316 | 0.95 | 3.597e-005 | 0.4248 | 32 |
| Butterworth (Order 7) | 4.959e-006 | 0.0004387 | 0.001213 | 0.01281 | 0.95 | 4.632e-005 | 0.3856 | 32 |
| Balanced (Order 7) | 7.325e-006 | 0.0004397 | 0.002158 | 0.01317 | 0.95 | 2.702e-005 | 0.3884 | 32 |
| Butterworth (Order 8) | 5.181e-006 | 0.0004405 | 0.001291 | 0.01281 | 0.95 | 3.878e-005 | 0.3517 | 32 |
| Balanced (Order 8) | 7.472e-006 | 0.0004418 | 0.002226 | 0.01318 | 0.95 | 2.104e-005 | 0.3587 | 32 |

Results for Series: 'Real Consumption of Non-Durables'.     Sample period:  1947-1 to 2017-4.
Data Transformation: Logarithmic transform.  Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'Cons_NonDur'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 7.05e-007 | 1.81e-006 | 0.004201 | 0.006105 | 0.95 | 4.435e-005 | 0.9129 | 32 |
| Balanced (Order 1) | 7.053e-007 | 4.03e-006 | 0.003991 | 0.006138 | 0.95 | 3.632e-005 | 0.8886 | 27.04 |
| Butterworth (Order 2) | 6.341e-007 | 1.458e-005 | 0.003142 | 0.006105 | 0.95 | 1.69e-005 | 0.7502 | 32 |
| Balanced (Order 2) | 6.32e-007 | 1.498e-005 | 0.00311 | 0.00614 | 0.95 | 1.374e-005 | 0.7208 | 32 |
| Butterworth (Order 3) | 6.463e-007 | 1.668e-005 | 0.0032 | 0.006105 | 0.95 | 9.809e-006 | 0.6114 | 32 |
| Balanced (Order 3) | 7.153e-007 | 1.691e-005 | 0.003915 | 0.006137 | 0.95 | 7.482e-006 | 0.5933 | 32 |
| Butterworth (Order 4) | 6.797e-007 | 1.729e-005 | 0.00353 | 0.006105 | 0.95 | 6.983e-006 | 0.5121 | 32 |
| Balanced (Order 4) | 7.772e-007 | 1.744e-005 | 0.004578 | 0.006136 | 0.95 | 5.101e-006 | 0.5031 | 32 |
| Butterworth (Order 5) | 7.131e-007 | 1.754e-005 | 0.003863 | 0.006105 | 0.95 | 5.495e-006 | 0.4407 | 32 |
| Balanced (Order 5) | 8.194e-007 | 1.762e-005 | 0.005052 | 0.006135 | 0.95 | 3.88e-006 | 0.4376 | 32 |
| Butterworth (Order 6) | 7.393e-007 | 1.768e-005 | 0.004137 | 0.006105 | 0.95 | 4.588e-006 | 0.3873 | 32 |
| Balanced (Order 6) | 8.491e-007 | 1.772e-005 | 0.005396 | 0.006136 | 0.95 | 3.134e-006 | 0.3882 | 32 |
| Butterworth (Order 7) | 7.562e-007 | 1.774e-005 | 0.004326 | 0.006105 | 0.95 | 3.975e-006 | 0.346 | 32 |
| Balanced (Order 7) | 8.724e-007 | 1.776e-005 | 0.005671 | 0.00613 | 0.95 | 2.637e-006 | 0.3492 | 32 |
| Butterworth (Order 8) | 7.725e-007 | 1.78e-005 | 0.0045 | 0.006105 | 0.95 | 3.513e-006 | 0.3132 | 32 |
| Balanced (Order 8) | 8.896e-007 | 1.779e-005 | 0.005884 | 0.006137 | 0.95 | 2.277e-006 | 0.3179 | 32 |

Results for Series: 'Real Consumption of Services'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'Cons_Serv'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 8.237e-007 | 5.028e-009 | 0.01638 | 0.008333 | 0.95 | 1.382e-005 | 0.9156 | 23.96 |
| Balanced (Order 1) | 7.074e-007 | 6.062e-009 | 0.01167 | 0.008176 | 0.9557 | 1.29e-005 | 0.8872 | 23.57 |
| Butterworth (Order 2) | 7.766e-007 | 2.279e-006 | 0.01214 | 0.008333 | 0.9524 | 9.689e-006 | 0.6665 | 32 |
| Balanced (Order 2) | 8.207e-007 | 2.393e-006 | 0.01358 | 0.008181 | 0.9507 | 8.429e-006 | 0.6407 | 32 |
| Butterworth (Order 3) | 8.916e-007 | 3.221e-006 | 0.01594 | 0.008333 | 0.95 | 6.619e-006 | 0.5109 | 32 |
| Balanced (Order 3) | 9.388e-007 | 3.311e-006 | 0.01834 | 0.008164 | 0.95 | 5.578e-006 | 0.4986 | 32 |
| Butterworth (Order 4) | 9.435e-007 | 3.587e-006 | 0.01825 | 0.008333 | 0.95 | 5.224e-006 | 0.4161 | 32 |
| Balanced (Order 4) | 1.004e-006 | 3.685e-006 | 0.02133 | 0.008153 | 0.95 | 4.243e-006 | 0.4116 | 32 |
| Butterworth (Order 5) | 9.848e-007 | 3.771e-006 | 0.02009 | 0.008333 | 0.95 | 4.441e-006 | 0.3515 | 32 |
| Balanced (Order 5) | 1.054e-006 | 3.857e-006 | 0.02367 | 0.008147 | 0.95 | 3.524e-006 | 0.3507 | 32 |
| Butterworth (Order 6) | 1.008e-006 | 3.888e-006 | 0.02127 | 0.008333 | 0.95 | 3.929e-006 | 0.3052 | 32 |
| Balanced (Order 6) | 1.092e-006 | 3.952e-006 | 0.02562 | 0.008133 | 0.95 | 3.095e-006 | 0.3049 | 32 |
| Butterworth (Order 7) | 1.03e-006 | 3.961e-006 | 0.02241 | 0.008333 | 0.95 | 3.574e-006 | 0.2695 | 32 |
| Balanced (Order 7) | 1.109e-006 | 4.045e-006 | 0.02657 | 0.008139 | 0.95 | 2.707e-006 | 0.2728 | 32 |
| Butterworth (Order 8) | 1.044e-006 | 4.015e-006 | 0.02319 | 0.008333 | 0.95 | 3.312e-006 | 0.2417 | 32 |
| Balanced (Order 8) | 1.129e-006 | 4.089e-006 | 0.02769 | 0.008136 | 0.95 | 2.491e-006 | 0.2451 | 32 |

Results for Series: 'Real Government Expenditures'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'Gov'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 3.672e-005 | 2.175e-008 | 0.1688 | 0.006368 | 0.95 | 8.043e-005 | 0.8848 | 14 |
| Balanced (Order 1) | 3.215e-005 | 2.871e-008 | 0.112 | 0.0064 | 0.95 | 7.363e-005 | 0.8623 | 14 |
| Butterworth (Order 2) | 8.971e-006 | 1.682e-005 | 0.005852 | 0.006368 | 0.95 | 9.134e-005 | 0.8016 | 32 |
| Balanced (Order 2) | 8.859e-006 | 1.846e-005 | 0.005506 | 0.006902 | 0.95 | 7.275e-005 | 0.7637 | 32 |
| Butterworth (Order 3) | 1.003e-005 | 2.69e-005 | 0.006706 | 0.006368 | 0.95 | 4.946e-005 | 0.6579 | 32 |
| Balanced (Order 3) | 1.123e-005 | 2.801e-005 | 0.008615 | 0.006844 | 0.95 | 3.541e-005 | 0.6336 | 32 |
| Butterworth (Order 4) | 1.076e-005 | 3.074e-005 | 0.007466 | 0.006368 | 0.95 | 3.056e-005 | 0.5636 | 32 |
| Balanced (Order 4) | 1.296e-005 | 3.145e-005 | 0.01116 | 0.006794 | 0.95 | 1.987e-005 | 0.5499 | 32 |
| Butterworth (Order 5) | 1.124e-005 | 3.266e-005 | 0.007995 | 0.006368 | 0.95 | 2.067e-005 | 0.4973 | 32 |
| Balanced (Order 5) | 1.401e-005 | 3.314e-005 | 0.01275 | 0.006766 | 0.95 | 1.217e-005 | 0.4921 | 32 |
| Butterworth (Order 6) | 1.157e-005 | 3.378e-005 | 0.008363 | 0.006368 | 0.95 | 1.495e-005 | 0.4471 | 32 |
| Balanced (Order 6) | 1.463e-005 | 3.411e-005 | 0.01368 | 0.006751 | 0.95 | 7.962e-006 | 0.4488 | 32 |
| Butterworth (Order 7) | 1.182e-005 | 3.446e-005 | 0.008651 | 0.006368 | 0.95 | 1.14e-005 | 0.4072 | 32 |
| Balanced (Order 7) | 1.508e-005 | 3.469e-005 | 0.01434 | 0.006741 | 0.95 | 5.518e-006 | 0.414 | 32 |
| Butterworth (Order 8) | 1.204e-005 | 3.493e-005 | 0.008901 | 0.006368 | 0.95 | 9.046e-006 | 0.3744 | 32 |
| Balanced (Order 8) | 1.54e-005 | 3.508e-005 | 0.01482 | 0.006734 | 0.95 | 4.008e-006 | 0.385 | 32 |

Results for Series: 'Real Exports'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'Exports'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 1.861e-005 | 0.0002625 | 0.006549 | 0.009363 | 0.95 | 0.0008012 | 0.9032 | 20.51 |
| Balanced (Order 1) | 1.484e-005 | 0.0002909 | 0.004406 | 0.009638 | 0.95 | 0.0006816 | 0.8823 | 20.08 |
| Butterworth (Order 2) | 1.053e-005 | 0.0004704 | 0.002368 | 0.01012 | 0.95 | 0.0003931 | 0.7539 | 27.45 |
| Balanced (Order 2) | 1.086e-005 | 0.0004854 | 0.002515 | 0.01008 | 0.95 | 0.0002756 | 0.7223 | 24.89 |
| Butterworth (Order 3) | 9.728e-006 | 0.0005159 | 0.00199 | 0.01025 | 0.95 | 0.0002186 | 0.6306 | 27.37 |
| Balanced (Order 3) | 9.958e-006 | 0.0005198 | 0.002096 | 0.01021 | 0.95 | 0.0001653 | 0.6058 | 30.29 |
| Butterworth (Order 4) | 9.563e-006 | 0.0005304 | 0.001921 | 0.01028 | 0.95 | 0.0001436 | 0.5415 | 26.84 |
| Balanced (Order 4) | 9.979e-006 | 0.0005331 | 0.002103 | 0.01021 | 0.95 | 0.0001048 | 0.5254 | 31.96 |
| Butterworth (Order 5) | 9.536e-006 | 0.0005366 | 0.001911 | 0.01028 | 0.95 | 0.0001051 | 0.4745 | 26.61 |
| Balanced (Order 5) | 1.015e-005 | 0.0005396 | 0.002176 | 0.01019 | 0.95 | 7.023e-005 | 0.4675 | 32 |
| Butterworth (Order 6) | 9.548e-006 | 0.0005399 | 0.001918 | 0.01028 | 0.95 | 8.283e-005 | 0.422 | 26.74 |
| Balanced (Order 6) | 1.029e-005 | 0.0005431 | 0.002235 | 0.01017 | 0.95 | 5.048e-005 | 0.4225 | 32 |
| Butterworth (Order 7) | 9.573e-006 | 0.0005419 | 0.001929 | 0.01028 | 0.95 | 6.868e-005 | 0.3796 | 27.23 |
| Balanced (Order 7) | 1.039e-005 | 0.0005453 | 0.002279 | 0.01016 | 0.95 | 3.821e-005 | 0.3864 | 32 |
| Butterworth (Order 8) | 9.599e-006 | 0.0005431 | 0.00194 | 0.01126 | 0.95 | 5.917e-005 | 0.3443 | 28.22 |
| Balanced (Order 8) | 1.046e-005 | 0.0005468 | 0.002312 | 0.01015 | 0.95 | 3.005e-005 | 0.3566 | 32 |

Results for Series: 'Real Imports'.     Sample period:  1947-1 to 2017-4.
Data Transformation: Logarithmic transform.  Logged series used in estimation.

Table of Maximum likelihood estimates of parameters for Series: 'Imports'

| Cycle Model | ZetaVar | EpsVar | Q_Zeta | BetaMean | Phi | KappaVar | Rho | Period |
|---|---|---|---|---|---|---|---|---|
| Butterworth (Order 1) | 5.136e-006 | 3.728e-007 | 0.001378 | 0.01396 | 0.95 | 0.001225 | 0.879 | 32 |
| Balanced (Order 1) | 4.387e-006 | 4.232e-007 | 0.001037 | 0.01392 | 0.95 | 0.001166 | 0.8512 | 32 |
| Butterworth (Order 2) | 4.648e-006 | 0.0002796 | 0.001115 | 0.01396 | 0.95 | 0.0006417 | 0.6584 | 32 |
| Balanced (Order 2) | 4.767e-006 | 0.0002909 | 0.00116 | 0.01389 | 0.95 | 0.0005541 | 0.6408 | 32 |
| Butterworth (Order 3) | 4.617e-006 | 0.0003585 | 0.001097 | 0.01396 | 0.95 | 0.0004037 | 0.5334 | 32 |
| Balanced (Order 3) | 4.974e-006 | 0.0003682 | 0.00124 | 0.01388 | 0.95 | 0.0003265 | 0.5273 | 32 |
| Butterworth (Order 4) | 4.686e-006 | 0.0003877 | 0.001124 | 0.01396 | 0.95 | 0.0002949 | 0.4496 | 32 |
| Balanced (Order 4) | 5.11e-006 | 0.000397 | 0.001294 | 0.01388 | 0.95 | 0.0002229 | 0.4519 | 32 |
| Butterworth (Order 5) | 4.76e-006 | 0.0004022 | 0.001153 | 0.01396 | 0.95 | 0.0002339 | 0.3896 | 32 |
| Balanced (Order 5) | 5.201e-006 | 0.0004111 | 0.001331 | 0.01387 | 0.95 | 0.000166 | 0.3973 | 32 |
| Butterworth (Order 6) | 4.821e-006 | 0.0004106 | 0.001178 | 0.01396 | 0.95 | 0.0001954 | 0.3445 | 32 |
| Balanced (Order 6) | 5.27e-006 | 0.0004192 | 0.001359 | 0.01387 | 0.95 | 0.0001311 | 0.3555 | 32 |
| Butterworth (Order 7) | 4.873e-006 | 0.0004161 | 0.001199 | 0.01396 | 0.95 | 0.000169 | 0.3091 | 32 |
| Balanced (Order 7) | 5.319e-006 | 0.0004244 | 0.00138 | 0.01387 | 0.95 | 0.0001079 | 0.3223 | 32 |
| Butterworth (Order 8) | 4.917e-006 | 0.0004199 | 0.001216 | 0.01396 | 0.95 | 0.00015 | 0.2806 | 32 |
| Balanced (Order 8) | 5.358e-006 | 0.0004279 | 0.001396 | 0.01387 | 0.95 | 9.149e-005 | 0.2952 | 32 |

Note:  The Model Type for Observations 'Standard (Order m) + Irregular' indicates an Unobserved Components model with Standard m-th order Stochastic Trend plus a White Noise Irregular.
'Canonical' indicates a Canonical Stochastic Trend.
'ZetaVar' is the variance of the core Trend disturbance. 'EpsVar' is the variance of the Irregular.  'Q_Zeta' equals ZetaVar/EpsVar, the Signal-Noise ratio.
'BetaMean' is the Mean of the slope [AR(1)] at the  core of the Trend and is only applicable to models with |Phi| < 1, where Phi is damping coefficient of the Slope.

Program used for computations : 'UnivDST+Cycle--Vers4--RunAH.ox'

Tables of Diagnostics - Set D2

Results for time series data taken from BEA: RealGDP and components

Univariate Models.
Model Type for Observations: Trend + Cycle + Irreg
Trend Model: Damped (Order 2) in Standard Form
The cycle is an n-th order stochastic cycle, as in Harvey-Trimbur (2003).
The cycle's model form is either Butterworth ('BW') or Balanced ('Bal').

Results for Series: 'Real GDP'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'RealGDP'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 7.271 | 27.68 | 35.68 | 0.00888 | -1853.79 | -1779.40 | 0.106 |
| Balanced (n = 1) | 6.396 | 31.36 | 38.32 | 0.00881 | -1858.43 | -1784.05 | 0.12 |
| Butterworth (n = 2) | 2.373 | 16.98 | 25.09 | 0.00865 | -1869.31 | -1794.93 | 0.152 |
| Balanced (n = 2) | 2.259 | 16.66 | 24.53 | 0.00864 | -1869.53 | -1795.14 | 0.153 |
| Butterworth (n = 3) | 2.593 | 15.88 | 22.75 | 0.00863 | -1870.13 | -1795.75 | 0.155 |
| Balanced (n = 3) | 2.956 | 16.3 | 23.02 | 0.00864 | -1869.53 | -1795.15 | 0.153 |
| Butterworth (n = 4) | 3.307 | 16.28 | 22.78 | 0.00864 | -1869.74 | -1795.35 | 0.154 |
| Balanced (n = 4) | 3.96 | 17.11 | 23.41 | 0.00865 | -1868.76 | -1794.37 | 0.151 |
| Butterworth (n = 5) | 3.858 | 16.76 | 23.11 | 0.00865 | -1869.24 | -1794.86 | 0.152 |
| Balanced (n = 5) | 4.676 | 17.82 | 23.92 | 0.00866 | -1868.06 | -1793.68 | 0.149 |
| Butterworth (n = 6) | 4.266 | 17.17 | 23.43 | 0.00865 | -1868.82 | -1794.44 | 0.151 |
| Balanced (n = 6) | 5.176 | 18.37 | 24.35 | 0.00867 | -1867.52 | -1793.13 | 0.147 |
| Butterworth (n = 7) | 4.574 | 17.51 | 23.7 | 0.00866 | -1868.48 | -1794.09 | 0.15 |
| Balanced (n = 7) | 5.538 | 18.77 | 24.69 | 0.00868 | -1867.10 | -1792.71 | 0.146 |
| Butterworth (n = 8) | 4.814 | 17.78 | 23.92 | 0.00866 | -1868.19 | -1793.81 | 0.149 |
| Balanced (n = 8) | 5.809 | 19.09 | 24.95 | 0.00868 | -1866.77 | -1792.38 | 0.145 |

Results for Series: 'Real Investment'.     Sample period:  1947-1 to 2017-4.
Data Transformation: Logarithmic transform.  Logged series used in estimation.

Table of Diagnostics for Series: 'RealInv'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 29.3 | 56.78 | 69.91 | 0.0469 | -911.91 | -837.53 | 0.0484 |
| Balanced (n = 1) | 16.74 | 41.31 | 51.68 | 0.0464 | -917.83 | -843.45 | 0.0681 |
| Butterworth (n = 2) | 8.906 | 22.57 | 31.51 | 0.0454 | -931.08 | -856.69 | 0.111 |
| Balanced (n = 2) | 6.902 | 20.18 | 28.67 | 0.0453 | -932.15 | -857.76 | 0.114 |
| Butterworth (n = 3) | 5.621 | 17.57 | 25.7 | 0.045 | -935.36 | -860.98 | 0.124 |
| Balanced (n = 3) | 5.333 | 17.24 | 25.3 | 0.045 | -935.59 | -861.21 | 0.125 |
| Butterworth (n = 4) | 4.789 | 16.25 | 24.12 | 0.0449 | -936.97 | -862.59 | 0.129 |
| Balanced (n = 4) | 4.801 | 16.36 | 24.27 | 0.0449 | -936.92 | -862.53 | 0.129 |
| Butterworth (n = 5) | 4.515 | 15.89 | 23.66 | 0.0448 | -937.66 | -863.28 | 0.131 |
| Balanced (n = 5) | 4.568 | 16.04 | 23.89 | 0.0448 | -937.54 | -863.15 | 0.131 |
| Butterworth (n = 6) | 4.393 | 15.79 | 23.52 | 0.0448 | -938.00 | -863.61 | 0.132 |
| Balanced (n = 6) | 4.428 | 15.89 | 23.68 | 0.0448 | -937.87 | -863.49 | 0.132 |
| Butterworth (n = 7) | 4.325 | 15.77 | 23.49 | 0.0448 | -938.18 | -863.79 | 0.133 |
| Balanced (n = 7) | 4.35 | 15.84 | 23.58 | 0.0448 | -938.06 | -863.68 | 0.133 |
| Butterworth (n = 8) | 4.281 | 15.78 | 23.49 | 0.0448 | -938.28 | -863.90 | 0.133 |
| Balanced (n = 8) | 4.308 | 15.83 | 23.54 | 0.0448 | -938.17 | -863.79 | 0.133 |

Results for Series: 'Real Residential Investment'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'ResInv'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 17.05 | 44.17 | 51.53 | 0.0434 | -954.93 | -880.54 | 0.2 |
| Balanced (n = 1) | 11.6 | 39.41 | 45.62 | 0.0431 | -959.82 | -885.43 | 0.214 |
| Butterworth (n = 2) | 5.633 | 26.17 | 35.53 | 0.0415 | -981.93 | -907.55 | 0.272 |
| Balanced (n = 2) | 5.728 | 25.79 | 34.54 | 0.0414 | -982.62 | -908.24 | 0.274 |
| Butterworth (n = 3) | 7.708 | 26.13 | 34.6 | 0.0414 | -982.24 | -907.86 | 0.273 |
| Balanced (n = 3) | 8.23 | 26.97 | 36.3 | 0.0416 | -980.15 | -905.76 | 0.267 |
| Butterworth (n = 4) | 9.642 | 27.72 | 36.18 | 0.0415 | -980.83 | -906.45 | 0.269 |
| Balanced (n = 4) | 10.19 | 29.5 | 38.92 | 0.0418 | -977.78 | -903.39 | 0.261 |
| Butterworth (n = 5) | 10.84 | 29.11 | 37.78 | 0.0416 | -979.41 | -905.03 | 0.266 |
| Balanced (n = 5) | 11.37 | 31.53 | 40.78 | 0.0419 | -976.08 | -901.69 | 0.257 |
| Butterworth (n = 6) | 11.51 | 30.16 | 39.01 | 0.0417 | -978.26 | -903.88 | 0.262 |
| Balanced (n = 6) | 12.08 | 32.99 | 42.02 | 0.042 | -974.86 | -900.48 | 0.254 |
| Butterworth (n = 7) | 11.93 | 30.99 | 39.91 | 0.0418 | -977.37 | -902.98 | 0.26 |
| Balanced (n = 7) | 12.52 | 34.04 | 42.83 | 0.042 | -973.98 | -899.60 | 0.251 |
| Butterworth (n = 8) | 12.23 | 31.66 | 40.6 | 0.0418 | -976.66 | -902.28 | 0.258 |
| Balanced (n = 8) | 12.79 | 34.79 | 43.37 | 0.0421 | -973.31 | -898.93 | 0.25 |

Results for Series: 'Real Non-Residential Investment'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'NonRes_Inv'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 17.13 | 34.86 | 46.8 | 0.0219 | -1342.34 | -1267.96 | 0.154 |
| Balanced (n = 1) | 16.86 | 35.22 | 45.66 | 0.0217 | -1347.93 | -1273.55 | 0.17 |
| Butterworth (n = 2) | 14.99 | 26.39 | 40.68 | 0.0212 | -1362.53 | -1288.15 | 0.211 |
| Balanced (n = 2) | 15.87 | 27.3 | 41.49 | 0.0212 | -1362.20 | -1287.82 | 0.21 |
| Butterworth (n = 3) | 20.73 | 32.52 | 46.48 | 0.0213 | -1358.63 | -1284.25 | 0.2 |
| Balanced (n = 3) | 22.12 | 34.61 | 48.65 | 0.0214 | -1356.61 | -1282.23 | 0.195 |
| Butterworth (n = 4) | 23.54 | 35.77 | 49.53 | 0.0214 | -1355.81 | -1281.42 | 0.192 |
| Balanced (n = 4) | 25.28 | 38.67 | 52.52 | 0.0215 | -1352.96 | -1278.58 | 0.184 |
| Butterworth (n = 5) | 25.11 | 37.74 | 51.38 | 0.0215 | -1353.89 | -1279.51 | 0.187 |
| Balanced (n = 5) | 27.02 | 41.07 | 54.78 | 0.0216 | -1350.64 | -1276.26 | 0.178 |
| Butterworth (n = 6) | 26.11 | 39.08 | 52.63 | 0.0215 | -1352.53 | -1278.15 | 0.183 |
| Balanced (n = 6) | 28.08 | 42.61 | 56.22 | 0.0217 | -1349.08 | -1274.70 | 0.173 |
| Butterworth (n = 7) | 26.8 | 40.04 | 53.54 | 0.0216 | -1351.53 | -1277.14 | 0.18 |
| Balanced (n = 7) | 28.79 | 43.68 | 57.19 | 0.0217 | -1347.98 | -1273.60 | 0.17 |
| Butterworth (n = 8) | 27.31 | 40.77 | 54.23 | 0.0216 | -1350.76 | -1276.37 | 0.178 |
| Balanced (n = 8) | 29.28 | 44.44 | 57.89 | 0.0217 | -1347.17 | -1272.78 | 0.168 |

Results for Series: 'Change in Inventories (As Percentage of Total Investment) '. Sample period: 1947-1 to 2017-4.
Data Transformation: No transform. Original series used in estimation.

Table of Diagnostics for Series: 'DPct_Inven'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 11.81 | 18.41 | 29.38 | 0.0294 | -1175.24 | -1100.86 | 0.2 |
| Balanced (n = 1) | 10.54 | 16.3 | 26.96 | 0.0293 | -1176.92 | -1102.53 | 0.204 |
| Butterworth (n = 2) | 9.378 | 14.47 | 24.91 | 0.0292 | -1179.37 | -1104.98 | 0.211 |
| Balanced (n = 2) | 9.012 | 13.91 | 24.21 | 0.0291 | -1179.94 | -1105.55 | 0.213 |
| Butterworth (n = 3) | 8.687 | 13.34 | 23.53 | 0.0291 | -1180.85 | -1106.46 | 0.215 |
| Balanced (n = 3) | 8.496 | 13.22 | 23.34 | 0.0291 | -1180.91 | -1106.53 | 0.215 |
| Butterworth (n = 4) | 9.488 | 14.27 | 24.78 | 0.0291 | -1180.63 | -1106.25 | 0.214 |
| Balanced (n = 4) | 8.242 | 12.91 | 22.93 | 0.0291 | -1181.37 | -1106.98 | 0.217 |
| Butterworth (n = 5) | 10.63 | 15.72 | 26.54 | 0.0292 | -1179.56 | -1105.17 | 0.211 |
| Balanced (n = 5) | 8.093 | 12.74 | 22.69 | 0.0291 | -1181.62 | -1107.24 | 0.217 |
| Butterworth (n = 6) | 11.34 | 16.69 | 27.64 | 0.0292 | -1178.78 | -1104.39 | 0.209 |
| Balanced (n = 6) | 7.997 | 12.63 | 22.54 | 0.0291 | -1181.79 | -1107.41 | 0.218 |
| Butterworth (n = 7) | 11.72 | 17.19 | 28.2 | 0.0292 | -1178.37 | -1103.99 | 0.207 |
| Balanced (n = 7) | 7.93 | 12.56 | 22.44 | 0.0291 | -1181.90 | -1107.52 | 0.218 |
| Butterworth (n = 8) | 11.94 | 17.46 | 28.5 | 0.0293 | -1178.15 | -1103.76 | 0.207 |
| Balanced (n = 8) | 7.882 | 12.51 | 22.37 | 0.029 | -1181.99 | -1107.60 | 0.218 |

Results for Series: 'Real Consumption'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'Cons'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 27.57 | 42.57 | 47.13 | 0.00785 | -1923.81 | -1849.43 | 0.0557 |
| Balanced (n = 1) | 24.2 | 40.79 | 44.4 | 0.00781 | -1926.48 | -1852.10 | 0.0644 |
| Butterworth (n = 2) | 18.56 | 32.18 | 36.33 | 0.00769 | -1935.32 | -1860.93 | 0.0929 |
| Balanced (n = 2) | 17.36 | 31.03 | 35.07 | 0.00768 | -1936.54 | -1862.15 | 0.0969 |
| Butterworth (n = 3) | 16.44 | 28.8 | 32.78 | 0.00765 | -1938.80 | -1864.42 | 0.104 |
| Balanced (n = 3) | 16.29 | 28.53 | 32.57 | 0.00765 | -1939.00 | -1864.62 | 0.104 |
| Butterworth (n = 4) | 15.99 | 27.69 | 31.74 | 0.00763 | -1939.86 | -1865.47 | 0.107 |
| Balanced (n = 4) | 16.17 | 27.73 | 31.85 | 0.00763 | -1939.75 | -1865.37 | 0.107 |
| Butterworth (n = 5) | 15.94 | 27.3 | 31.41 | 0.00763 | -1940.22 | -1865.83 | 0.108 |
| Balanced (n = 5) | 16.34 | 27.58 | 31.78 | 0.00763 | -1939.93 | -1865.54 | 0.108 |
| Butterworth (n = 6) | 15.99 | 27.17 | 31.32 | 0.00763 | -1940.34 | -1865.96 | 0.109 |
| Balanced (n = 6) | 16.57 | 27.62 | 31.89 | 0.00763 | -1939.93 | -1865.54 | 0.108 |
| Butterworth (n = 7) | 16.06 | 27.12 | 31.32 | 0.00763 | -1940.38 | -1866.00 | 0.109 |
| Balanced (n = 7) | 16.77 | 27.72 | 32.05 | 0.00763 | -1939.87 | -1865.49 | 0.108 |
| Butterworth (n = 8) | 16.14 | 27.12 | 31.35 | 0.00763 | -1940.38 | -1866.00 | 0.109 |
| Balanced (n = 8) | 16.95 | 27.84 | 32.22 | 0.00763 | -1939.80 | -1865.42 | 0.107 |

Results for Series: 'Real Consumption of Durables'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'ConsDur'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 12.34 | 28.63 | 33.86 | 0.0358 | -1065.41 | -991.03 | -0.00145 |
| Balanced (n = 1) | 11.13 | 28.15 | 33.27 | 0.0357 | -1067.06 | -992.68 | 0.0042 |
| Butterworth (n = 2) | 8.012 | 24.53 | 29.13 | 0.0354 | -1071.05 | -996.67 | 0.0183 |
| Balanced (n = 2) | 7.987 | 24.64 | 29.16 | 0.0354 | -1071.24 | -996.86 | 0.0189 |
| Butterworth (n = 3) | 8.972 | 24.77 | 28.94 | 0.0354 | -1071.95 | -997.56 | 0.0214 |
| Balanced (n = 3) | 9.579 | 25.05 | 29.17 | 0.0354 | -1071.10 | -996.71 | 0.0185 |
| Butterworth (n = 4) | 10.03 | 25.52 | 29.48 | 0.0354 | -1071.79 | -997.40 | 0.0209 |
| Balanced (n = 4) | 10.88 | 25.87 | 29.78 | 0.0355 | -1070.55 | -996.17 | 0.0167 |
| Butterworth (n = 5) | 10.75 | 26.03 | 29.89 | 0.0354 | -1071.45 | -997.07 | 0.0198 |
| Balanced (n = 5) | 11.79 | 26.56 | 30.35 | 0.0355 | -1070.07 | -995.68 | 0.0151 |
| Butterworth (n = 6) | 11.25 | 26.39 | 30.19 | 0.0354 | -1071.13 | -996.75 | 0.0187 |
| Balanced (n = 6) | 12.44 | 27.1 | 30.81 | 0.0355 | -1069.68 | -995.29 | 0.0138 |
| Butterworth (n = 7) | 11.63 | 26.67 | 30.43 | 0.0354 | -1070.87 | -996.48 | 0.0178 |
| Balanced (n = 7) | 12.92 | 27.53 | 31.19 | 0.0355 | -1069.37 | -994.99 | 0.0127 |
| Butterworth (n = 8) | 11.92 | 26.9 | 30.63 | 0.0355 | -1070.65 | -996.26 | 0.017 |
| Balanced (n = 8) | 13.3 | 27.88 | 31.49 | 0.0356 | -1069.12 | -994.74 | 0.0119 |

Results for Series: 'Real Consumption of Non-Durables'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'Cons_NonDur'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 15.7 | 21.93 | 33.95 | 0.00772 | -1933.67 | -1859.29 | 0.00908 |
| Balanced (n = 1) | 14.46 | 21.33 | 33.46 | 0.00771 | -1934.69 | -1860.31 | 0.0127 |
| Butterworth (n = 2) | 16.2 | 22.61 | 34.75 | 0.0077 | -1935.44 | -1861.05 | 0.0154 |
| Balanced (n = 2) | 16.62 | 23.05 | 35.29 | 0.0077 | -1935.25 | -1860.87 | 0.0147 |
| Butterworth (n = 3) | 18.49 | 24.5 | 37.02 | 0.00771 | -1934.07 | -1859.69 | 0.0103 |
| Balanced (n = 3) | 19.25 | 25.37 | 37.97 | 0.00772 | -1933.27 | -1858.88 | 0.00803 |
| Butterworth (n = 4) | 19.49 | 25.32 | 37.97 | 0.00773 | -1933.07 | -1858.69 | 0.00724 |
| Balanced (n = 4) | 20.39 | 26.31 | 38.95 | 0.00774 | -1931.93 | -1857.55 | 0.00313 |
| Butterworth (n = 5) | 20.03 | 25.74 | 38.42 | 0.00774 | -1932.40 | -1858.02 | 0.00492 |
| Balanced (n = 5) | 20.95 | 26.71 | 39.3 | 0.00775 | -1931.10 | -1856.71 | 0.000456 |
| Butterworth (n = 6) | 20.35 | 25.99 | 38.66 | 0.00774 | -1931.93 | -1857.55 | 0.00306 |
| Balanced (n = 6) | 21.27 | 26.91 | 39.45 | 0.00776 | -1930.54 | -1856.15 | -0.00189 |
| Butterworth (n = 7) | 20.58 | 26.17 | 38.83 | 0.00775 | -1931.59 | -1857.20 | 0.00197 |
| Balanced (n = 7) | 21.47 | 27.02 | 39.5 | 0.00777 | -1930.15 | -1855.76 | -0.00298 |
| Butterworth (n = 8) | 20.76 | 26.32 | 38.98 | 0.00775 | -1931.32 | -1856.94 | 0.00114 |
| Balanced (n = 8) | 21.61 | 27.09 | 39.52 | 0.00777 | -1929.86 | -1855.47 | -0.00389 |

Results for Series: 'Real Consumption of Services'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'Cons_Serv'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 12.57 | 27.49 | 29.93 | 0.0045 | -2239.10 | -2164.72 | 0.218 |
| Balanced (n = 1) | 11.45 | 26.28 | 28.45 | 0.00449 | -2239.78 | -2165.40 | 0.219 |
| Butterworth (n = 2) | 9.934 | 24.44 | 25.84 | 0.00449 | -2239.70 | -2165.32 | 0.219 |
| Balanced (n = 2) | 10.06 | 24.67 | 26.11 | 0.0045 | -2239.46 | -2165.08 | 0.218 |
| Butterworth (n = 3) | 11.34 | 26.66 | 28.2 | 0.00451 | -2238.10 | -2163.72 | 0.215 |
| Balanced (n = 3) | 11.64 | 27.33 | 28.9 | 0.00451 | -2237.58 | -2163.20 | 0.214 |
| Butterworth (n = 4) | 12.26 | 28.12 | 29.75 | 0.00451 | -2237.07 | -2162.68 | 0.212 |
| Balanced (n = 4) | 12.69 | 28.97 | 30.67 | 0.00452 | -2236.40 | -2162.02 | 0.21 |
| Butterworth (n = 5) | 12.86 | 29.08 | 30.78 | 0.00452 | -2236.36 | -2161.98 | 0.21 |
| Balanced (n = 5) | 13.39 | 30.02 | 31.8 | 0.00453 | -2235.62 | -2161.24 | 0.208 |
| Butterworth (n = 6) | 13.3 | 29.75 | 31.49 | 0.00452 | -2235.86 | -2161.47 | 0.208 |
| Balanced (n = 6) | 13.84 | 30.71 | 32.55 | 0.00453 | -2235.07 | -2160.68 | 0.206 |
| Butterworth (n = 7) | 13.59 | 30.23 | 32.01 | 0.00453 | -2235.48 | -2161.09 | 0.208 |
| Balanced (n = 7) | 14.2 | 31.23 | 33.13 | 0.00453 | -2234.66 | -2160.27 | 0.206 |
| Butterworth (n = 8) | 13.83 | 30.6 | 32.41 | 0.00453 | -2235.18 | -2160.80 | 0.207 |
| Balanced (n = 8) | 14.44 | 31.6 | 33.53 | 0.00453 | -2234.35 | -2159.96 | 0.205 |

Results for Series: 'Real Government Expenditures'.     Sample period:  1947-1 to 2017-4.
Data Transformation: Logarithmic transform.  Logged series used in estimation.

Table of Diagnostics for Series: 'Gov'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 13.75 | 26.82 | 32.6 | 0.0138 | -1605.29 | -1530.91 | 0.306 |
| Balanced (n = 1) | 12.01 | 25.44 | 31.43 | 0.0137 | -1606.85 | -1532.47 | 0.31 |
| Butterworth (n = 2) | 14.14 | 20.43 | 26.65 | 0.0136 | -1614.74 | -1540.35 | 0.327 |
| Balanced (n = 2) | 14.15 | 20.66 | 26.85 | 0.0136 | -1614.66 | -1540.27 | 0.327 |
| Butterworth (n = 3) | 15.3 | 23.11 | 29.29 | 0.0136 | -1612.81 | -1538.42 | 0.323 |
| Balanced (n = 3) | 15.43 | 24.13 | 30.39 | 0.0136 | -1611.88 | -1537.50 | 0.321 |
| Butterworth (n = 4) | 16.73 | 25.18 | 31.27 | 0.0136 | -1611.41 | -1537.03 | 0.32 |
| Balanced (n = 4) | 16.64 | 26.43 | 32.54 | 0.0137 | -1610.15 | -1535.77 | 0.317 |
| Butterworth (n = 5) | 17.67 | 26.42 | 32.47 | 0.0136 | -1610.50 | -1536.12 | 0.318 |
| Balanced (n = 5) | 17.41 | 27.64 | 33.66 | 0.0137 | -1609.17 | -1534.79 | 0.315 |
| Butterworth (n = 6) | 18.24 | 27.17 | 33.2 | 0.0137 | -1609.90 | -1535.51 | 0.316 |
| Balanced (n = 6) | 17.87 | 28.28 | 34.25 | 0.0137 | -1608.59 | -1534.21 | 0.313 |
| Butterworth (n = 7) | 18.59 | 27.63 | 33.65 | 0.0137 | -1609.48 | -1535.10 | 0.315 |
| Balanced (n = 7) | 18.13 | 28.62 | 34.57 | 0.0137 | -1608.23 | -1533.85 | 0.312 |
| Butterworth (n = 8) | 18.81 | 27.93 | 33.94 | 0.0137 | -1609.19 | -1534.81 | 0.315 |
| Balanced (n = 8) | 18.28 | 28.81 | 34.73 | 0.0137 | -1608.00 | -1533.61 | 0.312 |

Results for Series: 'Real Exports'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'Exports'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 10.61 | 28.95 | 36.01 | 0.0408 | -990.73 | -916.34 | 0.0201 |
| Balanced (n = 1) | 9.747 | 27.59 | 33.99 | 0.0407 | -992.27 | -917.89 | 0.0247 |
| Butterworth (n = 2) | 6.839 | 23.4 | 27.73 | 0.0404 | -996.26 | -921.88 | 0.0382 |
| Balanced (n = 2) | 6.339 | 22.96 | 27.24 | 0.0404 | -996.75 | -922.37 | 0.0396 |
| Butterworth (n = 3) | 5.583 | 21.87 | 25.59 | 0.0403 | -997.71 | -923.33 | 0.0431 |
| Balanced (n = 3) | 5.475 | 21.77 | 25.5 | 0.0403 | -997.88 | -923.50 | 0.0433 |
| Butterworth (n = 4) | 5.133 | 21.29 | 24.81 | 0.0403 | -998.23 | -923.84 | 0.0448 |
| Balanced (n = 4) | 5.052 | 21.29 | 24.86 | 0.0403 | -998.30 | -923.92 | 0.0448 |
| Butterworth (n = 5) | 4.928 | 21.03 | 24.46 | 0.0403 | -998.45 | -924.07 | 0.0455 |
| Balanced (n = 5) | 4.863 | 21.1 | 24.61 | 0.0403 | -998.46 | -924.08 | 0.0455 |
| Butterworth (n = 6) | 4.819 | 20.89 | 24.28 | 0.0403 | -998.56 | -924.18 | 0.046 |
| Balanced (n = 6) | 4.775 | 21.03 | 24.49 | 0.0403 | -998.53 | -924.14 | 0.0456 |
| Butterworth (n = 7) | 4.754 | 20.81 | 24.17 | 0.0403 | -998.62 | -924.24 | 0.0462 |
| Balanced (n = 7) | 4.732 | 21.01 | 24.45 | 0.0403 | -998.55 | -924.17 | 0.0458 |
| Butterworth (n = 8) | 4.712 | 20.76 | 24.11 | 0.0403 | -998.66 | -924.28 | 0.0463 |
| Balanced (n = 8) | 4.708 | 21.01 | 24.43 | 0.0403 | -998.56 | -924.18 | 0.0458 |

Results for Series: 'Real Imports'. Sample period: 1947-1 to 2017-4.
Data Transformation: Logarithmic transform. Logged series used in estimation.

Table of Diagnostics for Series: 'Imports'

| Cycle Model | Q(8) | Q(16) | Q(24) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|
| Butterworth (n = 1) | 7.553 | 34.82 | 43.31 | 0.0378 | -1034.45 | -960.07 | 0.00236 |
| Balanced (n = 1) | 7.097 | 34.28 | 42.37 | 0.0378 | -1035.17 | -960.79 | 0.00479 |
| Butterworth (n = 2) | 5.047 | 29.56 | 38.17 | 0.0377 | -1036.49 | -962.10 | 0.00957 |
| Balanced (n = 2) | 5.031 | 29.33 | 38.08 | 0.0377 | -1036.36 | -961.98 | 0.00916 |
| Butterworth (n = 3) | 5.238 | 27.95 | 36.89 | 0.0377 | -1036.60 | -962.21 | 0.01 |
| Balanced (n = 3) | 5.298 | 27.93 | 37.19 | 0.0377 | -1036.18 | -961.80 | 0.00865 |
| Butterworth (n = 4) | 5.439 | 27.39 | 36.6 | 0.0377 | -1036.42 | -962.03 | 0.00941 |
| Balanced (n = 4) | 5.57 | 27.47 | 37.08 | 0.0377 | -1035.91 | -961.52 | 0.00768 |
| Butterworth (n = 5) | 5.591 | 27.15 | 36.57 | 0.0377 | -1036.23 | -961.84 | 0.00879 |
| Balanced (n = 5) | 5.771 | 27.29 | 37.15 | 0.0377 | -1035.68 | -961.30 | 0.00699 |
| Butterworth (n = 6) | 5.704 | 27.03 | 36.61 | 0.0377 | -1036.07 | -961.69 | 0.0083 |
| Balanced (n = 6) | 5.918 | 27.22 | 37.25 | 0.0377 | -1035.50 | -961.12 | 0.00638 |
| Butterworth (n = 7) | 5.791 | 26.96 | 36.66 | 0.0377 | -1035.94 | -961.56 | 0.00786 |
| Balanced (n = 7) | 6.028 | 27.19 | 37.36 | 0.0377 | -1035.37 | -960.98 | 0.00591 |
| Butterworth (n = 8) | 5.858 | 26.93 | 36.72 | 0.0377 | -1035.84 | -961.46 | 0.00754 |
| Balanced (n = 8) | 6.113 | 27.18 | 37.45 | 0.0377 | -1035.26 | -960.88 | 0.00554 |

Note: AIC = -2*LogL_Max +2*k, where k is number of parameters. SIC = -2*LogL_Max +2*log(T)*k for series length T.
R^2_x is the coeff. of determination relative to simple benchmark, a RW (with fixed seasonal dummies for seasonal data.
Specifically, R^2_x = 1-PEV(model)/PEV(benchmark), where PEV is the Prediction Error Variance (in KF steady state).

Program used for computations : 'UnivDST+Cycle--Vers4--RunAH.ox'

Table Set D3- Maximum Likelihood Estimates for BEA data

This table contains parameter estimates for various representations of the ideal filter all of order 6
Results are given for 12 different parameter combinations.
Filter class: Generalized Butterworth Band Pass (index n set to 6)

Underlying Model Type for Observations:
Underlying Trend Model: Damped (Order 2) in Standard Form
The underlying cycle model is an n-th order stochastic cycle of the Butterworth ('BW') form.

Results for time series taken from Bureau of Economic Analysis (GDP data)
All series logged, except 'Inventory Change' which is percent of Investment.

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Gross Domestic Product'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 8.919e-006 | 0.007918 |
| 0.7504 | 0.1244 | 0.4197 | 1.215e-005 | 0.007933 |
| 0.3793 | 0.08929 | 0.4197 | 1.361e-005 | 0.008005 |
| 0.1213 | 0.05927 | 0.4416 | 1.549e-005 | 0.008075 |
| 0.08806 | 0.05388 | 0.4484 | 1.606e-005 | 0.008108 |
| 0.06612 | 0.0496 | 0.4546 | 1.668e-005 | 0.008111 |
| 0.05506 | 0.04725 | 0.4587 | 1.712e-005 | 0.008117 |
| 0.04946 | 0.04589 | 0.4611 | 1.742e-005 | 0.008126 |
| 0.04376 | 0.04441 | 0.4638 | 1.783e-005 | 0.008133 |
| 0.03178 | 0.04081 | 0.4709 | 1.897e-005 | 0.00815 |
| 0.01551 | 0.03396 | 0.487 | 2.319e-005 | 0.008173 |
| 0.01242 | 0.0322 | 0.4911 | 2.506e-005 | 0.008182 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 0.0002664 | 0.006834 |
| 0.7504 | 0.1244 | 0.4197 | 0.0003471 | 0.007252 |
| 0.3793 | 0.08929 | 0.4197 | 0.0003814 | 0.007733 |
| 0.1213 | 0.05927 | 0.4416 | 0.0004189 | 0.008415 |
| 0.08806 | 0.05388 | 0.4484 | 0.0004285 | 0.008583 |
| 0.06612 | 0.0496 | 0.4546 | 0.0004381 | 0.008722 |
| 0.05506 | 0.04725 | 0.4587 | 0.0004447 | 0.008805 |
| 0.04946 | 0.04589 | 0.4611 | 0.000449 | 0.008854 |
| 0.04376 | 0.04441 | 0.4638 | 0.0004544 | 0.008907 |
| 0.03178 | 0.04081 | 0.4709 | 0.0004707 | 0.009035 |
| 0.01551 | 0.03396 | 0.487 | 0.0005272 | 0.009283 |
| 0.01242 | 0.0322 | 0.4911 | 0.0005515 | 0.009356 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 0.0001766 | 0.002671 |
| 0.7504 | 0.1244 | 0.4197 | 0.0002497 | 0.004141 |
| 0.3793 | 0.08929 | 0.4197 | 0.0002853 | 0.005513 |
| 0.1213 | 0.05927 | 0.4416 | 0.0003457 | 0.005965 |
| 0.08806 | 0.05388 | 0.4484 | 0.0003692 | 0.006106 |
| 0.06612 | 0.0496 | 0.4546 | 0.0003953 | 0.006318 |
| 0.05506 | 0.04725 | 0.4587 | 0.0004154 | 0.006406 |
| 0.04946 | 0.04589 | 0.4611 | 0.0004288 | 0.006466 |
| 0.04376 | 0.04441 | 0.4638 | 0.0004457 | 0.006511 |
| 0.03178 | 0.04081 | 0.4709 | 0.0004997 | 0.006626 |
| 0.01551 | 0.03396 | 0.487 | 0.0006961 | 0.006858 |
| 0.01242 | 0.0322 | 0.4911 | 0.0007823 | 0.006922 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Non-Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 5.302e-005 | 0.008887 |
| 0.7504 | 0.1244 | 0.4197 | 7.386e-005 | 0.008882 |
| 0.3793 | 0.08929 | 0.4197 | 8.368e-005 | 0.009233 |
| 0.1213 | 0.05927 | 0.4416 | 9.775e-005 | 0.009579 |
| 0.08806 | 0.05388 | 0.4484 | 0.0001025 | 0.009669 |
| 0.06612 | 0.0496 | 0.4546 | 0.0001077 | 0.009746 |
| 0.05506 | 0.04725 | 0.4587 | 0.0001116 | 0.009795 |
| 0.04946 | 0.04589 | 0.4611 | 0.0001142 | 0.009822 |
| 0.04376 | 0.04441 | 0.4638 | 0.0001174 | 0.009852 |
| 0.03178 | 0.04081 | 0.4709 | 0.0001276 | 0.009931 |
| 0.01551 | 0.03396 | 0.487 | 0.0001641 | 0.01009 |
| 0.01242 | 0.0322 | 0.4911 | 0.00018 | 0.01015 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Inventory Change'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 0.0001996 | -7.619e-005 |
| 0.7504 | 0.1244 | 0.4197 | 0.0002515 | -7.685e-005 |
| 0.3793 | 0.08929 | 0.4197 | 0.0002715 | -7.621e-005 |
| 0.1213 | 0.05927 | 0.4416 | 0.0002887 | -8.162e-005 |
| 0.08806 | 0.05388 | 0.4484 | 0.0002915 | -0.0002211 |
| 0.06612 | 0.0496 | 0.4546 | 0.0002935 | -0.0001912 |
| 0.05506 | 0.04725 | 0.4587 | 0.0002946 | -0.0001697 |
| 0.04946 | 0.04589 | 0.4611 | 0.0002952 | -0.0001613 |
| 0.04376 | 0.04441 | 0.4638 | 0.0002958 | -0.0001492 |
| 0.03178 | 0.04081 | 0.4709 | 0.0002975 | -0.000119 |
| 0.01551 | 0.03396 | 0.487 | 0.0003014 | -5.85e-005 |
| 0.01242 | 0.0322 | 0.4911 | 0.000303 | -4.225e-005 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 8.441e-006 | 0.008694 |
| 0.7504 | 0.1244 | 0.4197 | 1.067e-005 | 0.008512 |
| 0.3793 | 0.08929 | 0.4197 | 1.158e-005 | 0.008422 |
| 0.1213 | 0.05927 | 0.4416 | 1.281e-005 | 0.00831 |
| 0.08806 | 0.05388 | 0.4484 | 1.321e-005 | 0.008287 |
| 0.06612 | 0.0496 | 0.4546 | 1.364e-005 | 0.008269 |
| 0.05506 | 0.04725 | 0.4587 | 1.397e-005 | 0.008258 |
| 0.04946 | 0.04589 | 0.4611 | 1.419e-005 | 0.008253 |
| 0.04376 | 0.04441 | 0.4638 | 1.444e-005 | 0.008247 |
| 0.03178 | 0.04081 | 0.4709 | 1.529e-005 | 0.008231 |
| 0.01551 | 0.03396 | 0.487 | 1.835e-005 | 0.008204 |
| 0.01242 | 0.0322 | 0.4911 | 1.973e-005 | 0.008197 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Durables'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 0.0002123 | 0.01313 |
| 0.7504 | 0.1244 | 0.4197 | 0.0002725 | 0.01311 |
| 0.3793 | 0.08929 | 0.4197 | 0.0002966 | 0.01312 |
| 0.1213 | 0.05927 | 0.4416 | 0.0003224 | 0.01311 |
| 0.08806 | 0.05388 | 0.4484 | 0.0003287 | 0.0131 |
| 0.06612 | 0.0496 | 0.4546 | 0.0003349 | 0.01309 |
| 0.05506 | 0.04725 | 0.4587 | 0.0003393 | 0.01308 |
| 0.04946 | 0.04589 | 0.4611 | 0.0003421 | 0.01308 |
| 0.04376 | 0.04441 | 0.4638 | 0.0003454 | 0.01307 |
| 0.03178 | 0.04081 | 0.4709 | 0.0003559 | 0.01306 |
| 0.01551 | 0.03396 | 0.487 | 0.0003922 | 0.01303 |
| 0.01242 | 0.0322 | 0.4911 | 0.0004082 | 0.01303 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Non-Durables'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 8.98e-006 | 0.006856 |
| 0.7504 | 0.1244 | 0.4197 | 1.18e-005 | 0.006643 |
| 0.3793 | 0.08929 | 0.4197 | 1.301e-005 | 0.006558 |
| 0.1213 | 0.05927 | 0.4416 | 1.441e-005 | 0.006406 |
| 0.08806 | 0.05388 | 0.4484 | 1.479e-005 | 0.006373 |
| 0.06612 | 0.0496 | 0.4546 | 1.521e-005 | 0.006347 |
| 0.05506 | 0.04725 | 0.4587 | 1.546e-005 | 0.00633 |
| 0.04946 | 0.04589 | 0.4611 | 1.564e-005 | 0.006325 |
| 0.04376 | 0.04441 | 0.4638 | 1.586e-005 | 0.006316 |
| 0.03178 | 0.04081 | 0.4709 | 1.656e-005 | 0.006295 |
| 0.01551 | 0.03396 | 0.487 | 1.901e-005 | 0.006259 |
| 0.01242 | 0.0322 | 0.4911 | 2.009e-005 | 0.00625 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Services'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 2.479e-006 | 0.009002 |
| 0.7504 | 0.1244 | 0.4197 | 3.308e-006 | 0.00874 |
| 0.3793 | 0.08929 | 0.4197 | 3.674e-006 | 0.008662 |
| 0.1213 | 0.05927 | 0.4416 | 4.183e-006 | 0.008555 |
| 0.08806 | 0.05388 | 0.4484 | 4.345e-006 | 0.008526 |
| 0.06612 | 0.0496 | 0.4546 | 4.566e-006 | 0.008523 |
| 0.05506 | 0.04725 | 0.4587 | 4.678e-006 | 0.008496 |
| 0.04946 | 0.04589 | 0.4611 | 4.774e-006 | 0.008494 |
| 0.04376 | 0.04441 | 0.4638 | 4.892e-006 | 0.008492 |
| 0.03178 | 0.04081 | 0.4709 | 5.39e-006 | 0.008474 |
| 0.01551 | 0.03396 | 0.487 | 6.602e-006 | 0.008462 |
| 0.01242 | 0.0322 | 0.4911 | 7.284e-006 | 0.008455 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Government Expenditures'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 1.901e-005 | 0.008715 |
| 0.7504 | 0.1244 | 0.4197 | 2.581e-005 | 0.008622 |
| 0.3793 | 0.08929 | 0.4197 | 2.911e-005 | 0.008595 |
| 0.1213 | 0.05927 | 0.4416 | 3.55e-005 | 0.008587 |
| 0.08806 | 0.05388 | 0.4484 | 3.825e-005 | 0.008586 |
| 0.06612 | 0.0496 | 0.4546 | 4.141e-005 | 0.008591 |
| 0.05506 | 0.04725 | 0.4587 | 4.387e-005 | 0.008588 |
| 0.04946 | 0.04589 | 0.4611 | 4.554e-005 | 0.008588 |
| 0.04376 | 0.04441 | 0.4638 | 4.764e-005 | 0.008586 |
| 0.03178 | 0.04081 | 0.4709 | 5.439e-005 | 0.008581 |
| 0.01551 | 0.03396 | 0.487 | 7.952e-005 | 0.008553 |
| 0.01242 | 0.0322 | 0.4911 | 9.083e-005 | 0.008529 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Exports'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 0.0002678 | 0.009679 |
| 0.7504 | 0.1244 | 0.4197 | 0.0003368 | 0.009507 |
| 0.3793 | 0.08929 | 0.4197 | 0.0003634 | 0.009747 |
| 0.1213 | 0.05927 | 0.4416 | 0.0003906 | 0.009874 |
| 0.08806 | 0.05388 | 0.4484 | 0.000397 | 0.009918 |
| 0.06612 | 0.0496 | 0.4546 | 0.0004031 | 0.009962 |
| 0.05506 | 0.04725 | 0.4587 | 0.0004073 | 0.009992 |
| 0.04946 | 0.04589 | 0.4611 | 0.00041 | 0.01001 |
| 0.04376 | 0.04441 | 0.4638 | 0.0004133 | 0.01003 |
| 0.03178 | 0.04081 | 0.4709 | 0.0004234 | 0.01009 |
| 0.01551 | 0.03396 | 0.487 | 0.0004576 | 0.01023 |
| 0.01242 | 0.0322 | 0.4911 | 0.0004724 | 0.01029 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Imports'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 0.0002227 | 0.01819 |
| 0.7504 | 0.1244 | 0.4197 | 0.0002896 | 0.01593 |
| 0.3793 | 0.08929 | 0.4197 | 0.0003163 | 0.01521 |
| 0.1213 | 0.05927 | 0.4416 | 0.0003436 | 0.0146 |
| 0.08806 | 0.05388 | 0.4484 | 0.0003502 | 0.01448 |
| 0.06612 | 0.0496 | 0.4546 | 0.0003564 | 0.01442 |
| 0.05506 | 0.04725 | 0.4587 | 0.0003607 | 0.01439 |
| 0.04946 | 0.04589 | 0.4611 | 0.0003634 | 0.01437 |
| 0.04376 | 0.04441 | 0.4638 | 0.0003668 | 0.01435 |
| 0.03178 | 0.04081 | 0.4709 | 0.0003771 | 0.01431 |
| 0.01551 | 0.03396 | 0.487 | 0.0004125 | 0.01425 |
| 0.01242 | 0.0322 | 0.4911 | 0.0004276 | 0.01424 |

Note: 'ZetaVar' is the variance of the core Trend disturbance. 'EpsVar' is the variance of the Irregular. 'Q_Zeta' equals ZeetaVar/EpsVar, the Trend's Signal-Noise ratio.

'BetaMean' is the Mean of the slope [AR(1)] at the core of the Trend and is only applicable to models with |Phi| < 1, where Phi is the damping coefficient of the Slope.

'KappaVar' is the variance of the core Cycle disturbance. 'Q_Kappa' equals KappaVar/EpsVar, the Cycle's Signal-Noise ratio..

'Rho' is the damping rate of shocks to the Cycle. Lambda_C is the cycle's central frequency.

Table Set D4 - These tables contain fit statistics and diagnostics are reported for various representations of the ideal filter given by 12 different parameter combinations.

Filter class: Generalized Butterworth Band Pass (index n set to 6)
Extension to allow for Damped Trend is incorporated.

Results for time series taken from Bureau of Economic Analysis (GDP data)
All series logged, except 'Inventory Change' which is percent of Investment.

Table of Ideal-Filter Diagnostics for Series: 'Gross Domestic Product'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 81.84 | 88.98 | 109.1 | 0.0138 | -1597.76 | -1523.38 | -1.14 |
| 0.7504 | 0.1244 | 0.4197 | 94.53 | 100.7 | 121.4 | 0.0128 | -1637.59 | -1563.21 | -0.865 |
| 0.3793 | 0.08929 | 0.4197 | 100.4 | 106.2 | 126.9 | 0.0124 | -1656.00 | -1581.62 | -0.748 |
| 0.1213 | 0.05927 | 0.4416 | 96.33 | 104 | 123.5 | 0.0119 | -1678.61 | -1604.22 | -0.613 |
| 0.08806 | 0.05388 | 0.4484 | 92.96 | 102.6 | 121.4 | 0.0119 | -1681.97 | -1607.59 | -0.592 |
| 0.06612 | 0.0496 | 0.4546 | 90.57 | 103.1 | 121.3 | 0.0118 | -1683.27 | -1608.89 | -0.585 |
| 0.05506 | 0.04725 | 0.4587 | 90.09 | 105.4 | 123.1 | 0.0118 | -1682.88 | -1608.50 | -0.586 |
| 0.04946 | 0.04589 | 0.4611 | 90.46 | 107.7 | 125.2 | 0.0118 | -1682.24 | -1607.86 | -0.59 |
| 0.04376 | 0.04441 | 0.4638 | 91.68 | 111.6 | 128.9 | 0.0119 | -1681.05 | -1606.67 | -0.599 |
| 0.03178 | 0.04081 | 0.4709 | 100.6 | 130.1 | 146.8 | 0.012 | -1675.38 | -1600.99 | -0.628 |
| 0.01551 | 0.03396 | 0.487 | 172.6 | 243.1 | 260.1 | 0.0126 | -1645.76 | -1571.38 | -0.806 |
| 0.01242 | 0.0322 | 0.4911 | 213.4 | 303.1 | 321.6 | 0.0129 | -1631.76 | -1557.38 | -0.896 |

Table of Ideal-Filter Diagnostics for Series: 'Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 83.73 | 92.32 | 103.7 | 0.0752 | -636.57 | -562.18 | -1.44 |
| 0.7504 | 0.1244 | 0.4197 | 78.93 | 87.84 | 98.66 | 0.0686 | -688.83 | -614.45 | -1.03 |
| 0.3793 | 0.08929 | 0.4197 | 78.27 | 87.39 | 97.69 | 0.0657 | -712.83 | -638.45 | -0.867 |
| 0.1213 | 0.05927 | 0.4416 | 73.82 | 83.22 | 93.01 | 0.062 | -745.35 | -670.97 | -0.663 |
| 0.08806 | 0.05388 | 0.4484 | 71.06 | 80.69 | 90.4 | 0.0612 | -752.68 | -678.30 | -0.62 |
| 0.06612 | 0.0496 | 0.4546 | 68.19 | 78.2 | 87.89 | 0.0606 | -758.35 | -683.97 | -0.588 |
| 0.05506 | 0.04725 | 0.4587 | 66.35 | 76.76 | 86.48 | 0.0603 | -761.24 | -686.86 | -0.571 |
| 0.04946 | 0.04589 | 0.4611 | 65.33 | 76.04 | 85.81 | 0.0601 | -762.75 | -688.36 | -0.562 |
| 0.04376 | 0.04441 | 0.4638 | 64.31 | 75.44 | 85.28 | 0.06 | -764.20 | -689.82 | -0.554 |
| 0.03178 | 0.04081 | 0.4709 | 62.89 | 75.7 | 85.87 | 0.0597 | -766.55 | -692.17 | -0.54 |
| 0.01551 | 0.03396 | 0.487 | 77.13 | 98.85 | 111.2 | 0.0602 | -761.56 | -687.18 | -0.566 |
| 0.01242 | 0.0322 | 0.4911 | 89.53 | 116 | 129.3 | 0.0607 | -756.94 | -682.56 | -0.591 |

Table of Ideal-Filter Diagnostics for Series: 'Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 96.27 | 105.7 | 110.2 | 0.0612 | -753.11 | -678.72 | -0.587 |
| 0.7504 | 0.1244 | 0.4197 | 105.9 | 115.5 | 119.5 | 0.0581 | -782.08 | -707.69 | -0.433 |
| 0.3793 | 0.08929 | 0.4197 | 106 | 117.1 | 121.1 | 0.0568 | -794.68 | -720.29 | -0.37 |
| 0.1213 | 0.05927 | 0.4416 | 100.2 | 122.4 | 127.8 | 0.0563 | -799.69 | -725.31 | -0.346 |
| 0.08806 | 0.05388 | 0.4484 | 105.7 | 136.2 | 142.7 | 0.0568 | -795.01 | -720.63 | -0.368 |
| 0.06612 | 0.0496 | 0.4546 | 119 | 160.7 | 169 | 0.0576 | -787.46 | -713.08 | -0.404 |
| 0.05506 | 0.04725 | 0.4587 | 133.6 | 184.7 | 194.5 | 0.0583 | -780.60 | -706.21 | -0.438 |
| 0.04946 | 0.04589 | 0.4611 | 144.8 | 202.6 | 213.4 | 0.0587 | -775.84 | -701.46 | -0.462 |
| 0.04376 | 0.04441 | 0.4638 | 160.5 | 226.7 | 238.9 | 0.0594 | -769.66 | -695.27 | -0.494 |
| 0.03178 | 0.04081 | 0.4709 | 217 | 310.8 | 327.3 | 0.0615 | -749.66 | -675.27 | -0.603 |
| 0.01551 | 0.03396 | 0.487 | 433.8 | 616.9 | 646.9 | 0.0692 | -682.92 | -608.54 | -1.03 |
| 0.01242 | 0.0322 | 0.4911 | 515.3 | 730.5 | 763.9 | 0.0722 | -658.02 | -583.63 | -1.21 |

Table of Ideal-Filter Diagnostics for Series: 'Non-Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 113.3 | 139 | 143.4 | 0.0335 | -1093.41 | -1019.02 | -0.98 |
| 0.7504 | 0.1244 | 0.4197 | 125.9 | 147.1 | 151.8 | 0.0316 | -1126.73 | -1052.35 | -0.761 |
| 0.3793 | 0.08929 | 0.4197 | 132.7 | 152.3 | 157.5 | 0.0308 | -1142.10 | -1067.72 | -0.669 |
| 0.1213 | 0.05927 | 0.4416 | 127.2 | 147 | 152.5 | 0.03 | -1157.19 | -1082.80 | -0.581 |
| 0.08806 | 0.05388 | 0.4484 | 125 | 146.4 | 152 | 0.0299 | -1157.43 | -1083.05 | -0.579 |
| 0.06612 | 0.0496 | 0.4546 | 125.5 | 149.8 | 155.5 | 0.03 | -1155.34 | -1080.96 | -0.59 |
| 0.05506 | 0.04725 | 0.4587 | 128.2 | 155.5 | 161.2 | 0.0302 | -1152.49 | -1078.11 | -0.606 |
| 0.04946 | 0.04589 | 0.4611 | 131.2 | 160.5 | 166.3 | 0.0303 | -1150.28 | -1075.90 | -0.618 |
| 0.04376 | 0.04441 | 0.4638 | 136 | 168.3 | 174.2 | 0.0305 | -1147.18 | -1072.80 | -0.636 |
| 0.03178 | 0.04081 | 0.4709 | 157.8 | 200.9 | 207.2 | 0.0311 | -1136.02 | -1061.64 | -0.7 |
| 0.01551 | 0.03396 | 0.487 | 276.9 | 364.1 | 372.5 | 0.0336 | -1091.85 | -1017.47 | -0.985 |
| 0.01242 | 0.0322 | 0.4911 | 332.2 | 438.9 | 447.9 | 0.0347 | -1073.73 | -999.35 | -1.12 |

Table of Ideal-Filter Diagnostics for Series: 'Inventory Change'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 98.17 | 112.8 | 121.4 | 0.0651 | -718.27 | -643.88 | -2.92 |
| 0.7504 | 0.1244 | 0.4197 | 84.35 | 97.25 | 106.3 | 0.0584 | -780.03 | -705.65 | -2.15 |
| 0.3793 | 0.08929 | 0.4197 | 80.24 | 92.49 | 101.8 | 0.0555 | -808.98 | -734.59 | -1.85 |
| 0.1213 | 0.05927 | 0.4416 | 78.42 | 90.03 | 99.46 | 0.0515 | -850.67 | -776.29 | -1.46 |
| 0.08806 | 0.05388 | 0.4484 | 78.04 | 89.53 | 98.96 | 0.0505 | -861.77 | -787.38 | -1.36 |
| 0.06612 | 0.0496 | 0.4546 | 77.63 | 89 | 98.43 | 0.0496 | -871.69 | -797.30 | -1.28 |
| 0.05506 | 0.04725 | 0.4587 | 77.32 | 88.62 | 98.04 | 0.0491 | -877.82 | -803.44 | -1.23 |
| 0.04946 | 0.04589 | 0.4611 | 77.1 | 88.36 | 97.78 | 0.0487 | -881.47 | -807.09 | -1.2 |
| 0.04376 | 0.04441 | 0.4638 | 76.83 | 88.05 | 97.45 | 0.0484 | -885.62 | -811.23 | -1.17 |
| 0.03178 | 0.04081 | 0.4709 | 75.95 | 87.05 | 96.42 | 0.0475 | -896.37 | -821.99 | -1.09 |
| 0.01551 | 0.03396 | 0.487 | 72.82 | 83.66 | 92.93 | 0.0455 | -919.78 | -845.40 | -0.918 |
| 0.01242 | 0.0322 | 0.4911 | 71.52 | 82.3 | 91.55 | 0.045 | -926.44 | -852.05 | -0.873 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 132 | 136.6 | 145.9 | 0.0134 | -1613.43 | -1539.04 | -1.74 |
| 0.7504 | 0.1244 | 0.4197 | 108.3 | 113.5 | 124.1 | 0.012 | -1674.55 | -1600.17 | -1.21 |
| 0.3793 | 0.08929 | 0.4197 | 100.1 | 106 | 118 | 0.0115 | -1701.74 | -1627.35 | -1.01 |
| 0.1213 | 0.05927 | 0.4416 | 85.78 | 93.34 | 106.2 | 0.0108 | -1732.30 | -1657.92 | -0.802 |
| 0.08806 | 0.05388 | 0.4484 | 81.04 | 89.65 | 102.5 | 0.0107 | -1737.34 | -1662.96 | -0.77 |
| 0.06612 | 0.0496 | 0.4546 | 77.35 | 87.53 | 100.3 | 0.0107 | -1740.13 | -1665.75 | -0.752 |
| 0.05506 | 0.04725 | 0.4587 | 75.81 | 87.48 | 100.2 | 0.0107 | -1740.69 | -1666.30 | -0.749 |
| 0.04946 | 0.04589 | 0.4611 | 75.4 | 88.18 | 100.9 | 0.0107 | -1740.64 | -1666.26 | -0.749 |
| 0.04376 | 0.04441 | 0.4638 | 75.55 | 89.85 | 102.5 | 0.0107 | -1740.13 | -1665.75 | -0.751 |
| 0.03178 | 0.04081 | 0.4709 | 80.43 | 100.4 | 113 | 0.0108 | -1736.38 | -1662.00 | -0.773 |
| 0.01551 | 0.03396 | 0.487 | 134.7 | 180.8 | 194.5 | 0.0112 | -1711.85 | -1637.46 | -0.931 |
| 0.01242 | 0.0322 | 0.4911 | 168.3 | 227.4 | 242.5 | 0.0115 | -1699.51 | -1625.12 | -1.02 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption of Durables'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 103.5 | 113.2 | 131.1 | 0.0671 | -700.78 | -626.40 | -2.52 |
| 0.7504 | 0.1244 | 0.4197 | 96.26 | 103.4 | 119 | 0.0607 | -757.30 | -682.92 | -1.88 |
| 0.3793 | 0.08929 | 0.4197 | 96.52 | 102.9 | 117.9 | 0.058 | -784.02 | -709.63 | -1.63 |
| 0.1213 | 0.05927 | 0.4416 | 93.27 | 99.49 | 113.9 | 0.0544 | -819.50 | -745.12 | -1.31 |
| 0.08806 | 0.05388 | 0.4484 | 90.49 | 97 | 111.3 | 0.0536 | -827.72 | -753.34 | -1.25 |
| 0.06612 | 0.0496 | 0.4546 | 87.25 | 94.28 | 108.4 | 0.053 | -834.32 | -759.93 | -1.19 |
| 0.05506 | 0.04725 | 0.4587 | 84.88 | 92.45 | 106.5 | 0.0527 | -837.85 | -763.47 | -1.17 |
| 0.04946 | 0.04589 | 0.4611 | 83.41 | 91.4 | 105.5 | 0.0525 | -839.78 | -765.40 | -1.15 |
| 0.04376 | 0.04441 | 0.4638 | 81.72 | 90.3 | 104.3 | 0.0523 | -841.75 | -767.37 | -1.14 |
| 0.03178 | 0.04081 | 0.4709 | 77.56 | 88.49 | 102.5 | 0.0519 | -845.65 | -771.27 | -1.11 |
| 0.01551 | 0.03396 | 0.487 | 77.52 | 101.3 | 116.1 | 0.0519 | -845.31 | -770.93 | -1.11 |
| 0.01242 | 0.0322 | 0.4911 | 83.57 | 114.9 | 130.5 | 0.0522 | -842.05 | -767.67 | -1.13 |

## Table of Ideal-Filter Diagnostics for Series: 'Consumption of Non-Durables'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 92.42 | 110.4 | 125.9 | 0.0138 | -1595.92 | -1521.54 | -2.17 |
| 0.7504 | 0.1244 | 0.4197 | 87.04 | 107.3 | 124.5 | 0.0126 | -1645.85 | -1571.46 | -1.66 |
| 0.3793 | 0.08929 | 0.4197 | 88.45 | 110.7 | 129.7 | 0.0121 | -1668.90 | -1594.51 | -1.45 |
| 0.1213 | 0.05927 | 0.4416 | 85.66 | 108.6 | 128.9 | 0.0115 | -1699.02 | -1624.64 | -1.2 |
| 0.08806 | 0.05388 | 0.4484 | 83.05 | 105.8 | 126 | 0.0114 | -1705.34 | -1630.95 | -1.15 |
| 0.06612 | 0.0496 | 0.4546 | 80.37 | 103 | 123.1 | 0.0113 | -1709.89 | -1635.51 | -1.12 |
| 0.05506 | 0.04725 | 0.4587 | 78.71 | 101.3 | 121.3 | 0.0112 | -1711.93 | -1637.55 | -1.1 |
| 0.04946 | 0.04589 | 0.4611 | 77.89 | 100.5 | 120.4 | 0.0112 | -1712.88 | -1638.50 | -1.09 |
| 0.04376 | 0.04441 | 0.4638 | 77.16 | 99.97 | 119.7 | 0.0112 | -1713.64 | -1639.26 | -1.09 |
| 0.03178 | 0.04081 | 0.4709 | 77.22 | 101.3 | 120.8 | 0.0112 | -1713.87 | -1639.48 | -1.08 |
| 0.01551 | 0.03396 | 0.487 | 101.4 | 137 | 156.1 | 0.0114 | -1701.98 | -1627.59 | -1.17 |
| 0.01242 | 0.0322 | 0.4911 | 120.2 | 163.4 | 183.1 | 0.0116 | -1694.28 | -1619.90 | -1.23 |

## Table of Ideal-Filter Diagnostics for Series: 'Consumption of Services'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 105.5 | 110.8 | 119.4 | 0.00725 | -1960.21 | -1885.82 | -1.03 |
| 0.7504 | 0.1244 | 0.4197 | 101.3 | 107.8 | 117.3 | 0.00669 | -2005.51 | -1931.13 | -0.732 |
| 0.3793 | 0.08929 | 0.4197 | 95.11 | 101.6 | 111.7 | 0.00645 | -2026.40 | -1952.02 | -0.609 |
| 0.1213 | 0.05927 | 0.4416 | 81.2 | 88.35 | 98.06 | 0.0062 | -2049.28 | -1974.90 | -0.486 |
| 0.08806 | 0.05388 | 0.4484 | 77.5 | 85.2 | 94.44 | 0.00616 | -2051.68 | -1977.29 | -0.469 |
| 0.06612 | 0.0496 | 0.4546 | 76.06 | 84.58 | 93.25 | 0.00619 | -2051.63 | -1977.24 | -0.48 |
| 0.05506 | 0.04725 | 0.4587 | 76.98 | 86.26 | 94.49 | 0.00618 | -2050.12 | -1975.73 | -0.478 |
| 0.04946 | 0.04589 | 0.4611 | 78.48 | 88.31 | 96.26 | 0.0062 | -2048.73 | -1974.35 | -0.486 |
| 0.04376 | 0.04441 | 0.4638 | 81.31 | 91.89 | 99.5 | 0.00622 | -2046.59 | -1972.21 | -0.497 |
| 0.03178 | 0.04081 | 0.4709 | 96.18 | 109.4 | 116 | 0.00639 | -2037.88 | -1963.50 | -0.578 |
| 0.01551 | 0.03396 | 0.487 | 191.1 | 215.2 | 219.6 | 0.00673 | -1999.26 | -1924.87 | -0.754 |
| 0.01242 | 0.0322 | 0.4911 | 241.9 | 270.6 | 274.5 | 0.00697 | -1981.77 | -1907.39 | -0.88 |

Table of Ideal-Filter Diagnostics for Series: 'Government Expenditures'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 85.26 | 95.83 | 107.7 | 0.0201 | -1383.64 | -1309.25 | -0.477 |
| 0.7504 | 0.1244 | 0.4197 | 75.69 | 86.07 | 95.57 | 0.0187 | -1424.34 | -1349.96 | -0.28 |
| 0.3793 | 0.08929 | 0.4197 | 67.02 | 78.45 | 87.45 | 0.0182 | -1440.97 | -1366.58 | -0.207 |
| 0.1213 | 0.05927 | 0.4416 | 69.08 | 97.31 | 107.1 | 0.0181 | -1443.80 | -1369.42 | -0.194 |
| 0.08806 | 0.05388 | 0.4484 | 85.75 | 127.1 | 137.7 | 0.0183 | -1436.52 | -1362.13 | -0.225 |
| 0.06612 | 0.0496 | 0.4546 | 113.8 | 172.6 | 184.2 | 0.0186 | -1425.89 | -1351.51 | -0.272 |
| 0.05506 | 0.04725 | 0.4587 | 140.1 | 213.4 | 225.9 | 0.0189 | -1416.68 | -1342.30 | -0.313 |
| 0.04946 | 0.04589 | 0.4611 | 159 | 242.2 | 255.3 | 0.0191 | -1410.43 | -1336.05 | -0.343 |
| 0.04376 | 0.04441 | 0.4638 | 184 | 279.7 | 293.6 | 0.0194 | -1402.44 | -1328.05 | -0.381 |
| 0.03178 | 0.04081 | 0.4709 | 266.8 | 401.5 | 417.7 | 0.0203 | -1377.21 | -1302.83 | -0.508 |
| 0.01551 | 0.03396 | 0.487 | 533.8 | 782.8 | 806.2 | 0.0234 | -1296.96 | -1222.58 | -1 |
| 0.01242 | 0.0322 | 0.4911 | 625.4 | 912.3 | 938.9 | 0.0246 | -1267.21 | -1192.82 | -1.22 |

Table of Ideal-Filter Diagnostics for Series: 'Exports'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 125.5 | 135.3 | 153.9 | 0.0754 | -635.07 | -560.69 | -2.34 |
| 0.7504 | 0.1244 | 0.4197 | 104.8 | 111.9 | 128.6 | 0.0675 | -697.33 | -622.95 | -1.68 |
| 0.3793 | 0.08929 | 0.4197 | 97.51 | 103.8 | 119.4 | 0.0642 | -726.49 | -652.11 | -1.42 |
| 0.1213 | 0.05927 | 0.4416 | 88.11 | 94.45 | 108.8 | 0.0599 | -765.18 | -690.80 | -1.11 |
| 0.08806 | 0.05388 | 0.4484 | 84.65 | 91.45 | 105.5 | 0.0589 | -774.33 | -699.94 | -1.04 |
| 0.06612 | 0.0496 | 0.4546 | 81.04 | 88.52 | 102.2 | 0.0581 | -781.87 | -707.48 | -0.988 |
| 0.05506 | 0.04725 | 0.4587 | 78.52 | 86.64 | 100.1 | 0.0577 | -786.09 | -711.70 | -0.958 |
| 0.04946 | 0.04589 | 0.4611 | 76.97 | 85.55 | 98.92 | 0.0574 | -788.46 | -714.07 | -0.941 |
| 0.04376 | 0.04441 | 0.4638 | 75.16 | 84.37 | 97.59 | 0.0572 | -790.97 | -716.59 | -0.924 |
| 0.03178 | 0.04081 | 0.4709 | 70.47 | 81.99 | 94.84 | 0.0566 | -796.53 | -722.15 | -0.885 |
| 0.01551 | 0.03396 | 0.487 | 65.54 | 88.21 | 100.6 | 0.0561 | -801.69 | -727.31 | -0.849 |
| 0.01242 | 0.0322 | 0.4911 | 68.06 | 96.58 | 109 | 0.0561 | -800.74 | -726.36 | -0.854 |

Table of Ideal-Filter Diagnostics for Series: 'Imports'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 2.524 | 0.2788 | 0.3975 | 111.3 | 126.2 | 133.2 | 0.0687 | -687.31 | -612.92 | -2.3 |
| 0.7504 | 0.1244 | 0.4197 | 102.8 | 121.4 | 130.7 | 0.0626 | -740.09 | -665.71 | -1.74 |
| 0.3793 | 0.08929 | 0.4197 | 100.1 | 120.6 | 131.3 | 0.0599 | -765.78 | -691.40 | -1.5 |
| 0.1213 | 0.05927 | 0.4416 | 94.73 | 117.2 | 128.9 | 0.0562 | -801.39 | -727.00 | -1.2 |
| 0.08806 | 0.05388 | 0.4484 | 91.85 | 114.8 | 126.5 | 0.0553 | -809.85 | -735.46 | -1.14 |
| 0.06612 | 0.0496 | 0.4546 | 88.62 | 112 | 123.6 | 0.0547 | -816.74 | -742.35 | -1.08 |
| 0.05506 | 0.04725 | 0.4587 | 86.31 | 110.1 | 121.6 | 0.0543 | -820.52 | -746.13 | -1.06 |
| 0.04946 | 0.04589 | 0.4611 | 84.86 | 108.9 | 120.4 | 0.0541 | -822.60 | -748.22 | -1.04 |
| 0.04376 | 0.04441 | 0.4638 | 83.17 | 107.6 | 119 | 0.0539 | -824.77 | -750.38 | -1.03 |
| 0.03178 | 0.04081 | 0.4709 | 78.82 | 104.6 | 115.7 | 0.0534 | -829.28 | -754.89 | -0.992 |
| 0.01551 | 0.03396 | 0.487 | 75.61 | 108.6 | 119 | 0.0532 | -831.00 | -756.61 | -0.978 |
| 0.01242 | 0.0322 | 0.4911 | 79.07 | 116.3 | 126.3 | 0.0534 | -828.72 | -754.33 | -0.991 |

Note:  AIC = -2*LogL_Max +2*k, where k is number of parameters. SIC = -2*LogL_Max +2*log(T)*k for series length T.
R^2_x is the coeff. of determination relative to simple benchmark, a RW (with fixed seasonal dummies for seasonal data.
Specifically, R^2_x = 1-PEV(model)/PEV(benchmark), where PEV is the Prediction Error Variance (in KF steady state).

Table Set D5- Maximum Likelihood Estimates for BEA data

This table contains parameter estimates for various representations of the ideal filter all of order 4
Results are given for 12 different parameter combinations.
Filter class: Generalized Butterworth Band Pass (index n set to 4)

Underlying Model Type for Observations:
Underlying Trend Model: Damped (Order 2) in Standard Form
The underlying cycle model is an n-th order stochastic cycle of the Butterworth ('BW') form.

Results for time series taken from Bureau of Economic Analysis (GDP data)
All series logged, except 'Inventory Change' which is percent of Investment.

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Gross Domestic Product'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 1.245e-005 | 0.007446 |
| 0.4521 | 0.3325 | 0.3537 | 1.372e-005 | 0.007524 |
| 0.3074 | 0.2849 | 0.3636 | 1.452e-005 | 0.007592 |
| 0.2231 | 0.2547 | 0.3725 | 1.512e-005 | 0.007644 |
| 0.1647 | 0.2316 | 0.3813 | 1.565e-005 | 0.007696 |
| 0.1097 | 0.2064 | 0.3937 | 1.643e-005 | 0.007755 |
| 0.05722 | 0.1749 | 0.4146 | 1.797e-005 | 0.00785 |
| 0.03301 | 0.1536 | 0.4326 | 1.996e-005 | 0.007917 |
| 0.02225 | 0.1402 | 0.4456 | 2.208e-005 | 0.007959 |
| 0.01424 | 0.1266 | 0.4602 | 2.556e-005 | 0.008001 |
| 0.01179 | 0.1213 | 0.4662 | 2.749e-005 | 0.008017 |
| 0.007957 | 0.111 | 0.4785 | 3.267e-005 | 0.008045 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 0.0003635 | 0.004666 |
| 0.4521 | 0.3325 | 0.3537 | 0.0003948 | 0.00543 |
| 0.3074 | 0.2849 | 0.3636 | 0.0004136 | 0.005984 |
| 0.223 | 0.2547 | 0.3725 | 0.000427 | 0.006419 |
| 0.1647 | 0.2316 | 0.3813 | 0.0004382 | 0.006795 |
| 0.1097 | 0.2064 | 0.3937 | 0.0004521 | 0.007258 |
| 0.05722 | 0.1749 | 0.4146 | 0.0004762 | 0.007873 |
| 0.03301 | 0.1536 | 0.4326 | 0.0005048 | 0.008284 |
| 0.02225 | 0.1402 | 0.4456 | 0.0005344 | 0.008537 |
| 0.01424 | 0.1266 | 0.4602 | 0.0005821 | 0.008766 |
| 0.01179 | 0.1213 | 0.4662 | 0.0006084 | 0.008854 |
| 0.007957 | 0.111 | 0.4785 | 0.0006777 | 0.009022 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 0.0002689 | 0.004295 |
| 0.4521 | 0.3325 | 0.3537 | 0.0003017 | 0.004933 |
| 0.3074 | 0.2849 | 0.3636 | 0.0003233 | 0.005368 |
| 0.2231 | 0.2547 | 0.3725 | 0.0003414 | 0.00564 |
| 0.1647 | 0.2316 | 0.3813 | 0.0003593 | 0.005759 |
| 0.1097 | 0.2064 | 0.3937 | 0.0003867 | 0.006142 |
| 0.05722 | 0.1749 | 0.4146 | 0.0004492 | 0.006473 |
| 0.03301 | 0.1536 | 0.4326 | 0.0005368 | 0.006674 |
| 0.02225 | 0.1402 | 0.4456 | 0.0006323 | 0.006772 |
| 0.01424 | 0.1266 | 0.4602 | 0.0007899 | 0.006873 |
| 0.01179 | 0.1213 | 0.4662 | 0.0008774 | 0.006909 |
| 0.007957 | 0.1111 | 0.4785 | 0.00111 | 0.006972 |

## Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Non-Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 7.45e-005 | 0.008208 |
| 0.4521 | 0.3325 | 0.3537 | 8.27e-005 | 0.0084 |
| 0.3074 | 0.2849 | 0.3636 | 8.811e-005 | 0.008553 |
| 0.223 | 0.2547 | 0.3725 | 9.239e-005 | 0.008677 |
| 0.1647 | 0.2316 | 0.3813 | 9.647e-005 | 0.008791 |
| 0.1097 | 0.2064 | 0.3937 | 0.0001025 | 0.008936 |
| 0.05722 | 0.1749 | 0.4146 | 0.0001154 | 0.00915 |
| 0.03301 | 0.1536 | 0.4326 | 0.0001328 | 0.009314 |
| 0.02225 | 0.1402 | 0.4456 | 0.0001515 | 0.009425 |
| 0.01424 | 0.1266 | 0.4602 | 0.0001821 | 0.009546 |
| 0.01179 | 0.1213 | 0.4662 | 0.000199 | 0.009594 |
| 0.007957 | 0.1111 | 0.4785 | 0.000244 | 0.009698 |

## Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Inventory Change'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 0.0002528 | -7.628e-005 |
| 0.4521 | 0.3325 | 0.3537 | 0.0002699 | -0.0007604 |
| 0.3074 | 0.2849 | 0.3636 | 0.0002794 | -7.635e-005 |
| 0.223 | 0.2547 | 0.3725 | 0.0002857 | -7.651e-005 |
| 0.1647 | 0.2316 | 0.3813 | 0.0002904 | -8.28e-005 |
| 0.1097 | 0.2064 | 0.3937 | 0.0002952 | -7.677e-005 |
| 0.05722 | 0.1749 | 0.4146 | 0.0003006 | -7.646e-005 |
| 0.03301 | 0.1536 | 0.4326 | 0.0003041 | -8.587e-005 |
| 0.02225 | 0.1402 | 0.4456 | 0.0003066 | -7.645e-005 |
| 0.01424 | 0.1266 | 0.4602 | 0.0003098 | -0.0002603 |
| 0.01179 | 0.1213 | 0.4662 | 0.0003114 | -8.01e-005 |
| 0.007957 | 0.111 | 0.4785 | 0.0003155 | -0.0001488 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 1.073e-005 | 0.008459 |
| 0.4521 | 0.3325 | 0.3537 | 1.154e-005 | 0.008379 |
| 0.3074 | 0.2849 | 0.3636 | 1.198e-005 | 0.008343 |
| 0.2231 | 0.2547 | 0.3725 | 1.236e-005 | 0.008325 |
| 0.1647 | 0.2316 | 0.3813 | 1.266e-005 | 0.008287 |
| 0.1097 | 0.2064 | 0.3937 | 1.318e-005 | 0.008275 |
| 0.05722 | 0.1749 | 0.4146 | 1.418e-005 | 0.008226 |
| 0.03301 | 0.1536 | 0.4326 | 1.556e-005 | 0.008202 |
| 0.02225 | 0.1402 | 0.4456 | 1.706e-005 | 0.008191 |
| 0.01424 | 0.1266 | 0.4602 | 1.952e-005 | 0.008181 |
| 0.01179 | 0.1213 | 0.4662 | 2.088e-005 | 0.008179 |
| 0.007957 | 0.1111 | 0.4785 | 2.454e-005 | 0.008172 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Durables'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 0.0002685 | 0.01374 |
| 0.4521 | 0.3325 | 0.3537 | 0.000288 | 0.0137 |
| 0.3074 | 0.2849 | 0.3636 | 0.0002996 | 0.01366 |
| 0.2231 | 0.2547 | 0.3725 | 0.0003078 | 0.01365 |
| 0.1647 | 0.2316 | 0.3813 | 0.0003147 | 0.0136 |
| 0.1097 | 0.2064 | 0.3937 | 0.0003233 | 0.01354 |
| 0.05722 | 0.1749 | 0.4146 | 0.0003384 | 0.01346 |
| 0.03301 | 0.1536 | 0.4326 | 0.0003561 | 0.0134 |
| 0.02225 | 0.1402 | 0.4456 | 0.0003743 | 0.01335 |
| 0.01424 | 0.1266 | 0.4602 | 0.0004037 | 0.01331 |
| 0.01179 | 0.1213 | 0.4662 | 0.0004199 | 0.01329 |
| 0.007957 | 0.1111 | 0.4785 | 0.0004632 | 0.01325 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Non-Durables'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 1.175e-005 | 0.006687 |
| 0.4521 | 0.3325 | 0.3537 | 1.289e-005 | 0.006684 |
| 0.3074 | 0.2849 | 0.3636 | 1.334e-005 | 0.006505 |
| 0.223 | 0.2547 | 0.3725 | 1.376e-005 | 0.006447 |
| 0.1647 | 0.2316 | 0.3813 | 1.422e-005 | 0.006403 |
| 0.1097 | 0.2064 | 0.3937 | 1.464e-005 | 0.006345 |
| 0.05722 | 0.1749 | 0.4146 | 1.557e-005 | 0.006281 |
| 0.03301 | 0.1536 | 0.4326 | 1.672e-005 | 0.006246 |
| 0.02225 | 0.1402 | 0.4456 | 1.793e-005 | 0.006227 |
| 0.01424 | 0.1266 | 0.4602 | 1.989e-005 | 0.006203 |
| 0.01179 | 0.1213 | 0.4662 | 2.102e-005 | 0.006199 |
| 0.007957 | 0.111 | 0.4785 | 2.393e-005 | 0.006188 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Services'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 3.468e-006 | 0.008488 |
| 0.4521 | 0.3325 | 0.3537 | 3.801e-006 | 0.008443 |
| 0.3074 | 0.2849 | 0.3636 | 3.985e-006 | 0.008509 |
| 0.2231 | 0.2547 | 0.3725 | 4.178e-006 | 0.008361 |
| 0.1647 | 0.2316 | 0.3813 | 4.366e-006 | 0.008456 |
| 0.1097 | 0.2064 | 0.3937 | 4.529e-006 | 0.008371 |
| 0.05722 | 0.1749 | 0.4146 | 4.994e-006 | 0.00835 |
| 0.03301 | 0.1536 | 0.4326 | 5.539e-006 | 0.008356 |
| 0.02225 | 0.1402 | 0.4456 | 6.276e-006 | 0.008358 |
| 0.01424 | 0.1266 | 0.4602 | 7.387e-006 | 0.008366 |
| 0.01179 | 0.1213 | 0.4662 | 8.018e-006 | 0.008368 |
| 0.007957 | 0.1111 | 0.4785 | 9.706e-006 | 0.008372 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Government Expenditures'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 2.719e-005 | 0.007974 |
| 0.4521 | 0.3325 | 0.3537 | 3.027e-005 | 0.007972 |
| 0.3074 | 0.2849 | 0.3636 | 3.248e-005 | 0.007993 |
| 0.2231 | 0.2547 | 0.3725 | 3.442e-005 | 0.008022 |
| 0.1647 | 0.2316 | 0.3813 | 3.644e-005 | 0.008042 |
| 0.1097 | 0.2064 | 0.3937 | 3.975e-005 | 0.00809 |
| 0.05722 | 0.1749 | 0.4146 | 4.77e-005 | 0.008166 |
| 0.03301 | 0.1536 | 0.4326 | 5.908e-005 | 0.008226 |
| 0.02225 | 0.1402 | 0.4456 | 7.164e-005 | 0.008262 |
| 0.01424 | 0.1266 | 0.4602 | 9.255e-005 | 0.008293 |
| 0.01179 | 0.1213 | 0.4662 | 0.0001043 | 0.008301 |
| 0.007957 | 0.1111 | 0.4785 | 0.0001359 | 0.008309 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Exports'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 0.0003385 | 0.009777 |
| 0.4521 | 0.3325 | 0.3537 | 0.0003613 | 0.009626 |
| 0.3074 | 0.2849 | 0.3636 | 0.0003745 | 0.009515 |
| 0.2231 | 0.2547 | 0.3725 | 0.0003837 | 0.009466 |
| 0.1647 | 0.2316 | 0.3813 | 0.0003912 | 0.009424 |
| 0.1097 | 0.2064 | 0.3937 | 0.0004004 | 0.009398 |
| 0.05722 | 0.1749 | 0.4146 | 0.0004156 | 0.009412 |
| 0.03301 | 0.1536 | 0.4326 | 0.0004328 | 0.009472 |
| 0.02225 | 0.1402 | 0.4456 | 0.0004502 | 0.009536 |
| 0.01424 | 0.1266 | 0.4602 | 0.0004781 | 0.009628 |
| 0.01179 | 0.1213 | 0.4662 | 0.0004935 | 0.009671 |
| 0.007957 | 0.111 | 0.4785 | 0.0005341 | 0.009775 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Imports'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 0.000293 | 0.01368 |
| 0.4521 | 0.3325 | 0.3537 | 0.000316 | 0.01348 |
| 0.3074 | 0.2849 | 0.3636 | 0.0003293 | 0.01341 |
| 0.2231 | 0.2547 | 0.3725 | 0.0003387 | 0.01344 |
| 0.1647 | 0.2316 | 0.3813 | 0.0003462 | 0.01347 |
| 0.1097 | 0.2064 | 0.3937 | 0.0003555 | 0.01353 |
| 0.05722 | 0.1749 | 0.4146 | 0.0003708 | 0.01365 |
| 0.03301 | 0.1536 | 0.4326 | 0.0003882 | 0.01374 |
| 0.02225 | 0.1402 | 0.4456 | 0.0004058 | 0.01381 |
| 0.01424 | 0.1266 | 0.4602 | 0.0004343 | 0.01387 |
| 0.01179 | 0.1213 | 0.4662 | 0.0004499 | 0.0139 |
| 0.007957 | 0.111 | 0.4785 | 0.0004914 | 0.01394 |

Note: 'ZetaVar' is the variance of the core Trend disturbance. 'EpsVar' is the variance of the Irregular. 'Q_Zeta' equals ZetaVar/EpsVar, the Trend's Signal-Noise ratio.

'BetaMean' is the Mean of the slope [AR(1)] at the core of the Trend and is only applicable to models with |Phi| < 1, where Phi is damping coefficient of the Slope.

'KappaVar' is the variance of the core Cycle disturbance. 'Q_Kappa' equals KappaVar/EpsVar, the Cycle's Signal-Noise ratio.

'Rho' is the damping rate of shocks to the Cycle. Lambda_C is the cycle's central frequency.

Table Set D6 - These tables contain fit statistics and diagnostics are reported for various representations of the ideal filter given by 12 different parameter combinations.

Filter class: Generalized Butterworth Band Pass (index n set to 4)
Extension to allow for Damped Trend is incorporated.

Results for time series taken from Bureau of Economic Analysis (GDP data)
All series logged, except 'Inventory Change' which is percent of Investment.

Table of Ideal-Filter Diagnostics for Series: 'Gross Domestic Product'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 63.7 | 69.39 | 88.51 | 0.0103 | -1768.35 | -1693.97 | -0.198 |
| 0.4521 | 0.3325 | 0.3537 | 65.26 | 70.71 | 89.77 | 0.0101 | -1779.03 | -1704.65 | -0.154 |
| 0.3074 | 0.2849 | 0.3636 | 65.36 | 70.82 | 89.67 | 0.00997 | -1785.87 | -1711.49 | -0.127 |
| 0.2231 | 0.2547 | 0.3725 | 64.55 | 70.22 | 88.75 | 0.00988 | -1790.82 | -1716.44 | -0.106 |
| 0.1647 | 0.2316 | 0.3813 | 63.12 | 69.25 | 87.35 | 0.0098 | -1794.69 | -1720.31 | -0.0898 |
| 0.1097 | 0.2064 | 0.3937 | 60.82 | 68.28 | 85.55 | 0.00975 | -1798.16 | -1723.78 | -0.0775 |
| 0.05722 | 0.1749 | 0.4146 | 61.78 | 74.71 | 90.03 | 0.00976 | -1797.34 | -1722.96 | -0.0808 |
| 0.03301 | 0.1536 | 0.4326 | 80.5 | 104.5 | 117.8 | 0.00994 | -1787.08 | -1712.70 | -0.12 |
| 0.02225 | 0.1402 | 0.4456 | 116.1 | 154.3 | 166.1 | 0.0102 | -1772.19 | -1697.81 | -0.18 |
| 0.01424 | 0.1266 | 0.4602 | 189.7 | 252.1 | 262.8 | 0.0107 | -1746.19 | -1671.80 | -0.293 |
| 0.01179 | 0.1213 | 0.4662 | 232.8 | 308.3 | 318.9 | 0.0109 | -1732.05 | -1657.66 | -0.358 |
| 0.007957 | 0.111 | 0.4785 | 343.6 | 451.1 | 462.4 | 0.0116 | -1696.71 | -1622.33 | -0.538 |

Table of Ideal-Filter Diagnostics for Series: 'Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 54.57 | 63.8 | 73 | 0.0556 | -813.43 | -739.05 | -0.335 |
| 0.4521 | 0.3325 | 0.3537 | 52.55 | 62.03 | 70.78 | 0.0541 | -828.17 | -753.79 | -0.267 |
| 0.3074 | 0.2849 | 0.3636 | 51.44 | 61.04 | 69.54 | 0.0532 | -837.95 | -763.57 | -0.224 |
| 0.2231 | 0.2547 | 0.3725 | 50.41 | 60.04 | 68.39 | 0.0525 | -845.52 | -771.14 | -0.192 |
| 0.1647 | 0.2316 | 0.3813 | 49.16 | 58.79 | 67.04 | 0.0519 | -852.20 | -777.82 | -0.164 |
| 0.1097 | 0.2064 | 0.3937 | 46.97 | 56.6 | 64.79 | 0.0511 | -860.25 | -785.87 | -0.131 |
| 0.05722 | 0.1749 | 0.4146 | 43.13 | 53.05 | 61.5 | 0.0503 | -869.89 | -795.51 | -0.0922 |
| 0.03301 | 0.1536 | 0.4326 | 43.69 | 55.01 | 64.37 | 0.05 | -872.81 | -798.43 | -0.0803 |
| 0.02225 | 0.1402 | 0.4456 | 51.68 | 65.56 | 76.38 | 0.0502 | -870.39 | -796.00 | -0.0891 |
| 0.01424 | 0.1266 | 0.4602 | 76.55 | 96.15 | 109.9 | 0.051 | -861.60 | -787.22 | -0.123 |
| 0.01179 | 0.1213 | 0.4662 | 94.45 | 117.7 | 133.2 | 0.0515 | -855.64 | -781.26 | -0.146 |
| 0.007957 | 0.111 | 0.4785 | 149.2 | 182.8 | 202.8 | 0.053 | -838.63 | -764.25 | -0.216 |

Table of Ideal-Filter Diagnostics for Series: 'Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 66.59 | 75.54 | 80.18 | 0.0478 | -898.84 | -824.46 | 0.0321 |
| 0.4521 | 0.3325 | 0.3537 | 66.13 | 76.04 | 80.87 | 0.0473 | -904.71 | -830.33 | 0.0507 |
| 0.3074 | 0.2849 | 0.3636 | 64.75 | 75.97 | 80.94 | 0.047 | -907.68 | -833.30 | 0.0621 |
| 0.2231 | 0.2547 | 0.3725 | 63.5 | 76.53 | 81.66 | 0.0469 | -908.86 | -834.47 | 0.0658 |
| 0.1647 | 0.2316 | 0.3813 | 63.35 | 79.1 | 84.5 | 0.047 | -908.42 | -834.03 | 0.0643 |
| 0.1097 | 0.2064 | 0.3937 | 68.07 | 90.04 | 96.24 | 0.0473 | -904.47 | -830.09 | 0.0517 |
| 0.05722 | 0.1749 | 0.4146 | 106.1 | 149.2 | 158.9 | 0.0488 | -886.37 | -811.98 | -0.00998 |
| 0.03301 | 0.1536 | 0.4326 | 195.8 | 274.1 | 290.5 | 0.0515 | -855.43 | -781.05 | -0.126 |
| 0.02225 | 0.1402 | 0.4456 | 304.6 | 420 | 443.6 | 0.0546 | -822.76 | -748.37 | -0.263 |
| 0.01424 | 0.1266 | 0.4602 | 468.2 | 635.4 | 668.4 | 0.0594 | -775.21 | -700.83 | -0.494 |
| 0.01179 | 0.1213 | 0.4662 | 546 | 736.9 | 773.9 | 0.0618 | -752.01 | -677.62 | -0.621 |
| 0.007957 | 0.1111 | 0.4785 | 711.8 | 952.3 | 996 | 0.0679 | -699.02 | -624.64 | -0.953 |

Table of Ideal-Filter Diagnostics for Series: 'Non-Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 93.79 | 111.5 | 116.5 | 0.0252 | -1262.00 | -1187.62 | -0.114 |
| 0.4521 | 0.3325 | 0.3537 | 94.92 | 111.1 | 116.4 | 0.0248 | -1270.59 | -1196.20 | -0.081 |
| 0.3074 | 0.2849 | 0.3636 | 94.5 | 109.9 | 115.3 | 0.0246 | -1275.61 | -1201.23 | -0.0619 |
| 0.2231 | 0.2547 | 0.3725 | 93.26 | 108.3 | 113.8 | 0.0244 | -1278.73 | -1204.34 | -0.0503 |
| 0.1647 | 0.2316 | 0.3813 | 91.72 | 106.8 | 112.3 | 0.0243 | -1280.47 | -1206.09 | -0.0437 |
| 0.1097 | 0.2064 | 0.3937 | 90.7 | 106.8 | 112.3 | 0.0243 | -1280.30 | -1205.91 | -0.0441 |
| 0.05722 | 0.1749 | 0.4146 | 102.6 | 125 | 130.7 | 0.0247 | -1271.05 | -1196.67 | -0.0781 |
| 0.03301 | 0.1536 | 0.4326 | 147.4 | 183.7 | 190.3 | 0.0256 | -1250.74 | -1176.35 | -0.158 |
| 0.02225 | 0.1402 | 0.4456 | 213.5 | 267 | 275 | 0.0267 | -1227.12 | -1152.73 | -0.258 |
| 0.01424 | 0.1266 | 0.4602 | 328.5 | 409.8 | 419.8 | 0.0285 | -1190.44 | -1116.06 | -0.431 |
| 0.01179 | 0.1213 | 0.4662 | 388.9 | 484.1 | 495 | 0.0295 | -1171.79 | -1097.40 | -0.528 |
| 0.007957 | 0.1111 | 0.4785 | 528.7 | 655.6 | 667.9 | 0.0318 | -1127.72 | -1053.33 | -0.784 |

Table of Ideal-Filter Diagnostics for Series: 'Inventory Change'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 74.04 | 87.24 | 96.14 | 0.0463 | -916.14 | -841.76 | -0.989 |
| 0.4521 | 0.3325 | 0.3537 | 70.08 | 82.8 | 91.81 | 0.0448 | -935.92 | -861.53 | -0.856 |
| 0.3074 | 0.2849 | 0.3636 | 68.42 | 80.89 | 89.97 | 0.0437 | -949.01 | -874.63 | -0.772 |
| 0.2231 | 0.2547 | 0.3725 | 67.55 | 79.85 | 88.98 | 0.0429 | -959.28 | -884.89 | -0.709 |
| 0.1647 | 0.2316 | 0.3813 | 66.97 | 79.15 | 88.33 | 0.0422 | -968.63 | -894.25 | -0.653 |
| 0.1097 | 0.2064 | 0.3937 | 66.39 | 78.42 | 87.64 | 0.0413 | -980.85 | -906.47 | -0.583 |
| 0.05722 | 0.1749 | 0.4146 | 65.36 | 77.18 | 86.42 | 0.0399 | -1000.12 | -925.73 | -0.477 |
| 0.03301 | 0.1536 | 0.4326 | 63.93 | 75.55 | 84.78 | 0.0388 | -1016.25 | -941.87 | -0.395 |
| 0.02225 | 0.1402 | 0.4456 | 62.36 | 73.82 | 83.02 | 0.038 | -1027.64 | -953.25 | -0.339 |
| 0.01424 | 0.1266 | 0.4602 | 59.9 | 71.16 | 80.31 | 0.0372 | -1040.08 | -965.70 | -0.28 |
| 0.01179 | 0.1213 | 0.4662 | 58.63 | 69.81 | 78.94 | 0.0368 | -1045.13 | -970.74 | -0.257 |
| 0.007957 | 0.111 | 0.4785 | 55.61 | 66.63 | 75.71 | 0.0362 | -1054.98 | -980.60 | -0.214 |

### Table of Ideal-Filter Diagnostics for Series: 'Consumption'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 92.98 | 98.04 | 109 | 0.00954 | -1810.54 | -1736.16 | -0.396 |
| 0.4521 | 0.3325 | 0.3537 | 84.5 | 89.92 | 101.7 | 0.00926 | -1829.00 | -1754.62 | -0.313 |
| 0.3074 | 0.2849 | 0.3636 | 79.17 | 84.87 | 97.17 | 0.00906 | -1840.26 | -1765.88 | -0.256 |
| 0.2231 | 0.2547 | 0.3725 | 74.87 | 80.84 | 93.4 | 0.00893 | -1848.19 | -1773.80 | -0.223 |
| 0.1647 | 0.2316 | 0.3813 | 70.72 | 77 | 89.69 | 0.00882 | -1854.43 | -1780.05 | -0.192 |
| 0.1097 | 0.2064 | 0.3937 | 65.13 | 72.07 | 84.7 | 0.00873 | -1860.63 | -1786.25 | -0.169 |
| 0.05722 | 0.1749 | 0.4146 | 59.12 | 68.71 | 80.76 | 0.00867 | -1863.80 | -1789.41 | -0.153 |
| 0.03301 | 0.1536 | 0.4326 | 66.67 | 82.26 | 93.52 | 0.00878 | -1857.12 | -1782.73 | -0.18 |
| 0.02225 | 0.1402 | 0.4456 | 89.54 | 113.5 | 124.3 | 0.00897 | -1845.07 | -1770.68 | -0.232 |
| 0.01424 | 0.1266 | 0.4602 | 144.6 | 184.2 | 194.9 | 0.00933 | -1822.49 | -1748.10 | -0.334 |
| 0.01179 | 0.1213 | 0.4662 | 179.4 | 227.9 | 238.8 | 0.00954 | -1809.81 | -1735.43 | -0.394 |
| 0.007957 | 0.1111 | 0.4785 | 274.3 | 345.4 | 357.8 | 0.0101 | -1777.42 | -1703.03 | -0.561 |

### Table of Ideal-Filter Diagnostics for Series: 'Consumption of Durables'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 84.1 | 91.26 | 107.1 | 0.0478 | -899.16 | -824.78 | -0.782 |
| 0.4521 | 0.3325 | 0.3537 | 81.93 | 88.41 | 103.6 | 0.0462 | -917.43 | -843.05 | -0.671 |
| 0.3074 | 0.2849 | 0.3636 | 80.71 | 86.86 | 101.7 | 0.0453 | -929.24 | -854.85 | -0.603 |
| 0.2231 | 0.2547 | 0.3725 | 79.5 | 85.47 | 100.1 | 0.0446 | -938.15 | -863.77 | -0.553 |
| 0.1647 | 0.2316 | 0.3813 | 77.95 | 83.86 | 98.23 | 0.044 | -945.87 | -871.49 | -0.511 |
| 0.1097 | 0.2064 | 0.3937 | 75.02 | 81.01 | 95.08 | 0.0433 | -955.10 | -880.72 | -0.462 |
| 0.05722 | 0.1749 | 0.4146 | 68.2 | 75.11 | 88.61 | 0.0424 | -966.58 | -892.19 | -0.403 |
| 0.03301 | 0.1536 | 0.4326 | 62.34 | 71.83 | 84.76 | 0.042 | -971.56 | -897.18 | -0.378 |
| 0.02225 | 0.1402 | 0.4456 | 61.57 | 75.16 | 87.72 | 0.042 | -971.15 | -896.77 | -0.379 |
| 0.01424 | 0.1266 | 0.4602 | 70.67 | 93.03 | 105.4 | 0.0424 | -965.20 | -890.82 | -0.408 |
| 0.01179 | 0.1213 | 0.4662 | 79.9 | 107.9 | 120.4 | 0.0428 | -960.58 | -886.19 | -0.43 |
| 0.007957 | 0.1111 | 0.4785 | 113.9 | 158.4 | 171.9 | 0.0439 | -946.38 | -872.00 | -0.503 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption of Non-Durables'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 78.07 | 98.01 | 115.3 | 0.00999 | -1784.65 | -1710.27 | -0.659 |
| 0.4521 | 0.3325 | 0.3537 | 77.38 | 98.2 | 116.7 | 0.00978 | -1800.22 | -1725.84 | -0.591 |
| 0.3074 | 0.2849 | 0.3636 | 76.99 | 98.26 | 117.5 | 0.00955 | -1810.24 | -1735.86 | -0.518 |
| 0.2231 | 0.2547 | 0.3725 | 76.23 | 97.67 | 117.3 | 0.00942 | -1817.68 | -1743.29 | -0.477 |
| 0.1647 | 0.2316 | 0.3813 | 74.95 | 96.35 | 116.2 | 0.00935 | -1823.94 | -1749.56 | -0.453 |
| 0.1097 | 0.2064 | 0.3937 | 72.28 | 93.37 | 113.3 | 0.0092 | -1831.01 | -1756.62 | -0.409 |
| 0.05722 | 0.1749 | 0.4146 | 66.94 | 87.29 | 106.7 | 0.00909 | -1837.87 | -1763.49 | -0.373 |
| 0.03301 | 0.1536 | 0.4326 | 66.67 | 87.51 | 105.9 | 0.0091 | -1837.10 | -1762.72 | -0.376 |
| 0.02225 | 0.1402 | 0.4456 | 75.5 | 98.96 | 116.5 | 0.00919 | -1831.09 | -1756.71 | -0.405 |
| 0.01424 | 0.1266 | 0.4602 | 104.1 | 135.2 | 151.9 | 0.00942 | -1817.10 | -1742.72 | -0.476 |
| 0.01179 | 0.1213 | 0.4662 | 124.8 | 161.2 | 177.7 | 0.00957 | -1808.53 | -1734.15 | -0.523 |
| 0.007957 | 0.111 | 0.4785 | 187.5 | 240 | 256.9 | 0.00997 | -1785.26 | -1710.87 | -0.652 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption of Services'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 76.74 | 80.49 | 89.3 | 0.00543 | -2129.47 | -2055.09 | -0.139 |
| 0.4521 | 0.3325 | 0.3537 | 72.2 | 76.08 | 85.03 | 0.00531 | -2142.29 | -2067.91 | -0.091 |
| 0.3074 | 0.2849 | 0.3636 | 68.42 | 72.41 | 81.38 | 0.00522 | -2150.25 | -2075.87 | -0.0549 |
| 0.2231 | 0.2547 | 0.3725 | 64.86 | 68.97 | 77.86 | 0.00519 | -2155.79 | -2081.40 | -0.043 |
| 0.1647 | 0.2316 | 0.3813 | 61.22 | 65.52 | 74.26 | 0.00518 | -2159.86 | -2085.48 | -0.0373 |
| 0.1097 | 0.2064 | 0.3937 | 56.59 | 61.33 | 69.68 | 0.00512 | -2163.12 | -2088.73 | -0.0134 |
| 0.05722 | 0.1749 | 0.4146 | 55.88 | 62.3 | 69.59 | 0.00515 | -2160.41 | -2086.03 | -0.0245 |
| 0.03301 | 0.1536 | 0.4326 | 76.61 | 86.39 | 92.34 | 0.00524 | -2146.84 | -2072.46 | -0.0604 |
| 0.02225 | 0.1402 | 0.4456 | 116.5 | 130.4 | 135.2 | 0.00544 | -2128.52 | -2054.14 | -0.144 |
| 0.01424 | 0.1266 | 0.4602 | 197.8 | 218.5 | 222.1 | 0.00574 | -2097.56 | -2023.18 | -0.274 |
| 0.01179 | 0.1213 | 0.4662 | 245 | 269.2 | 272.3 | 0.00591 | -2081.02 | -2006.64 | -0.351 |
| 0.007957 | 0.1111 | 0.4785 | 365.3 | 397.2 | 399.6 | 0.00635 | -2040.21 | -1965.82 | -0.558 |

Table of Ideal-Filter Diagnostics for Series: 'Government Expenditures'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 51.96 | 61.08 | 69.7 | 0.0152 | -1547.13 | -1472.74 | 0.154 |
| 0.4521 | 0.3325 | 0.3537 | 46.24 | 56.27 | 64.42 | 0.015 | -1554.99 | -1480.61 | 0.177 |
| 0.3074 | 0.2849 | 0.3636 | 42.99 | 54.82 | 62.82 | 0.0149 | -1558.00 | -1483.61 | 0.186 |
| 0.2231 | 0.2547 | 0.3725 | 42.3 | 56.96 | 64.99 | 0.0149 | -1558.21 | -1483.83 | 0.186 |
| 0.1647 | 0.2316 | 0.3813 | 45.26 | 64.42 | 72.59 | 0.015 | -1556.00 | -1481.61 | 0.18 |
| 0.1097 | 0.2064 | 0.3937 | 59.68 | 89.22 | 97.8 | 0.0152 | -1548.33 | -1473.94 | 0.158 |
| 0.05722 | 0.1749 | 0.4146 | 130.2 | 193.3 | 203.3 | 0.0159 | -1521.10 | -1446.71 | 0.0731 |
| 0.03301 | 0.1536 | 0.4326 | 260 | 373.7 | 385.8 | 0.0171 | -1479.94 | -1405.56 | -0.0713 |
| 0.02225 | 0.1402 | 0.4456 | 395.2 | 557.5 | 571.7 | 0.0184 | -1439.10 | -1364.72 | -0.237 |
| 0.01424 | 0.1266 | 0.4602 | 575.9 | 800.2 | 817.3 | 0.0203 | -1382.01 | -1307.63 | -0.512 |
| 0.01179 | 0.1213 | 0.4662 | 655.4 | 906.4 | 924.9 | 0.0213 | -1354.77 | -1280.38 | -0.665 |
| 0.007957 | 0.1111 | 0.4785 | 815.2 | 1119 | 1141 | 0.0238 | -1293.36 | -1218.98 | -1.07 |

Table of Ideal-Filter Diagnostics for Series: 'Exports'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 92.66 | 99.7 | 116.3 | 0.0536 | -833.58 | -759.20 | -0.691 |
| 0.4521 | 0.3325 | 0.3537 | 85.29 | 91.64 | 107.5 | 0.0518 | -853.33 | -778.95 | -0.577 |
| 0.3074 | 0.2849 | 0.3636 | 81.05 | 87.12 | 102.5 | 0.0506 | -866.11 | -791.72 | -0.508 |
| 0.2231 | 0.2547 | 0.3725 | 77.9 | 83.88 | 98.96 | 0.0498 | -875.80 | -801.41 | -0.457 |
| 0.1647 | 0.2316 | 0.3813 | 74.98 | 81.01 | 95.79 | 0.049 | -884.27 | -809.89 | -0.414 |
| 0.1097 | 0.2064 | 0.3937 | 70.81 | 77.13 | 91.52 | 0.0481 | -894.63 | -820.24 | -0.363 |
| 0.05722 | 0.1749 | 0.4146 | 62.85 | 70.54 | 84.21 | 0.047 | -908.46 | -834.08 | -0.297 |
| 0.03301 | 0.1536 | 0.4326 | 55.64 | 66.18 | 79.11 | 0.0463 | -916.38 | -842.00 | -0.26 |
| 0.02225 | 0.1402 | 0.4456 | 52.36 | 66.81 | 79.16 | 0.0461 | -918.88 | -844.50 | -0.248 |
| 0.01424 | 0.1266 | 0.4602 | 55.13 | 77.23 | 88.97 | 0.0462 | -917.28 | -842.89 | -0.255 |
| 0.01179 | 0.1213 | 0.4662 | 60.04 | 86.8 | 98.35 | 0.0464 | -914.87 | -840.49 | -0.265 |
| 0.007957 | 0.111 | 0.4785 | 81.12 | 121.1 | 132.4 | 0.0471 | -906.06 | -831.67 | -0.304 |

Table of Ideal-Filter Diagnostics for Series: 'Imports'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.7635 | 0.4276 | 0.3421 | 82.89 | 98.74 | 106.9 | 0.0499 | -874.39 | -800.00 | -0.737 |
| 0.4521 | 0.3325 | 0.3537 | 79.16 | 96.04 | 105.1 | 0.0484 | -891.26 | -816.87 | -0.637 |
| 0.3074 | 0.2849 | 0.3636 | 76.99 | 94.49 | 104.1 | 0.0475 | -902.46 | -828.08 | -0.573 |
| 0.2231 | 0.2547 | 0.3725 | 75.2 | 93.13 | 103.1 | 0.0468 | -911.13 | -836.75 | -0.526 |
| 0.1647 | 0.2316 | 0.3813 | 73.33 | 91.61 | 101.9 | 0.0461 | -918.81 | -844.43 | -0.485 |
| 0.1097 | 0.2064 | 0.3937 | 70.28 | 88.99 | 99.48 | 0.0454 | -928.26 | -853.88 | -0.436 |
| 0.05722 | 0.1749 | 0.4146 | 63.81 | 83.43 | 94.03 | 0.0444 | -940.72 | -866.34 | -0.373 |
| 0.03301 | 0.1536 | 0.4326 | 58.25 | 79.51 | 90.06 | 0.0438 | -947.18 | -872.80 | -0.341 |
| 0.02225 | 0.1402 | 0.4456 | 57.06 | 80.78 | 91.31 | 0.0437 | -948.26 | -873.88 | -0.335 |
| 0.01424 | 0.1266 | 0.4602 | 64.04 | 93.03 | 103.7 | 0.044 | -944.52 | -870.14 | -0.352 |
| 0.01179 | 0.1213 | 0.4662 | 71.56 | 104 | 114.7 | 0.0443 | -941.02 | -866.64 | -0.368 |
| 0.007957 | 0.111 | 0.4785 | 99.97 | 142.4 | 153.5 | 0.0452 | -929.59 | -855.21 | -0.424 |

Note: AIC = -2*LogL_Max +2*k, where k is number of parameters. SIC = -2*LogL_Max +2*log(T)*k for series length T.
R^2_x is the coeff. of determination relative to simple benchmark, a RW (with fixed seasonal dummies for seasonal data.
Specifically, R^2_x = 1-PEV(model)/PEV(benchmark), where PEV is the Prediction Error Variance (in KF steady state).

Table Set D7 - Maximum Likelihood Estimates for BEA data

This table contains parameter estimates for various representations of the ideal filter all of order 8
Results are given for 12 different parameter combinations.
Filter class: Generalized Butterworth Band Pass (index n set to 8)

Underlying Model Type for Observations:
Underlying Trend Model: Damped (Order 2) in Standard Form
The underlying cycle model is an n-th order stochastic cycle of the Butterworth ('BW') form.

Results for time series taken from Bureau of Economic Analysis (GDP data)
All series logged, except 'Inventory Change' which is percent of Investment.

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Gross Domestic Product'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 1.191e-005 | 0.008187 |
| 0.5935 | 0.03031 | 0.4455 | 1.276e-005 | 0.008196 |
| 0.4445 | 0.02627 | 0.4489 | 1.332e-005 | 0.008204 |
| 0.3383 | 0.02324 | 0.4523 | 1.38e-005 | 0.008211 |
| 0.245 | 0.02037 | 0.4568 | 1.433e-005 | 0.008221 |
| 0.1618 | 0.0175 | 0.463 | 1.495e-005 | 0.00823 |
| 0.1005 | 0.01495 | 0.4705 | 1.571e-005 | 0.00824 |
| 0.07047 | 0.01341 | 0.4764 | 1.638e-005 | 0.008252 |
| 0.05188 | 0.01226 | 0.4815 | 1.713e-005 | 0.008251 |
| 0.03447 | 0.01092 | 0.4885 | 1.837e-005 | 0.008255 |
| 0.02497 | 0.009992 | 0.494 | 1.974e-005 | 0.008256 |
| 0.01839 | 0.009199 | 0.4992 | 2.147e-005 | 0.008255 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 0.0003396 | 0.009068 |
| 0.5935 | 0.03031 | 0.4455 | 0.0003596 | 0.009148 |
| 0.4445 | 0.02627 | 0.4489 | 0.0003725 | 0.009201 |
| 0.3383 | 0.02324 | 0.4523 | 0.000383 | 0.009262 |
| 0.245 | 0.02037 | 0.4568 | 0.0003938 | 0.009336 |
| 0.1618 | 0.01749 | 0.463 | 0.0004059 | 0.009432 |
| 0.1005 | 0.01495 | 0.4705 | 0.0004188 | 0.009538 |
| 0.07047 | 0.01341 | 0.4764 | 0.000429 | 0.009616 |
| 0.05188 | 0.01226 | 0.4815 | 0.0004392 | 0.009677 |
| 0.03447 | 0.01092 | 0.4885 | 0.0004563 | 0.009757 |
| 0.02497 | 0.009992 | 0.494 | 0.0004742 | 0.009815 |
| 0.01839 | 0.009199 | 0.4992 | 0.0004961 | 0.009865 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 0.0002384 | 0.005008 |
| 0.5935 | 0.03031 | 0.4455 | 0.0002583 | 0.005361 |
| 0.4445 | 0.02627 | 0.4489 | 0.0002722 | 0.005612 |
| 0.3383 | 0.02324 | 0.4523 | 0.0002848 | 0.005796 |
| 0.245 | 0.02037 | 0.4568 | 0.0002997 | 0.006052 |
| 0.1618 | 0.0175 | 0.463 | 0.0003201 | 0.006335 |
| 0.1005 | 0.01495 | 0.4705 | 0.000349 | 0.006594 |
| 0.07047 | 0.01341 | 0.4764 | 0.0003776 | 0.00676 |
| 0.05188 | 0.01226 | 0.4815 | 0.0004098 | 0.006888 |
| 0.03447 | 0.01092 | 0.4885 | 0.0004687 | 0.007028 |
| 0.02497 | 0.009992 | 0.494 | 0.0005333 | 0.007125 |
| 0.01839 | 0.009199 | 0.4992 | 0.0006145 | 0.007202 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Non-Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 7.249e-005 | 0.00948 |
| 0.5935 | 0.03031 | 0.4455 | 7.82e-005 | 0.009582 |
| 0.4445 | 0.02627 | 0.4489 | 8.206e-005 | 0.009686 |
| 0.3383 | 0.02324 | 0.4523 | 8.544e-005 | 0.009758 |
| 0.245 | 0.02037 | 0.4568 | 8.919e-005 | 0.009862 |
| 0.1618 | 0.0175 | 0.463 | 9.394e-005 | 0.009995 |
| 0.1005 | 0.01495 | 0.4705 | 9.999e-005 | 0.01013 |
| 0.07047 | 0.01341 | 0.4764 | 0.0001055 | 0.01024 |
| 0.05188 | 0.01226 | 0.4815 | 0.0001116 | 0.01031 |
| 0.03447 | 0.01092 | 0.4885 | 0.0001224 | 0.01041 |
| 0.02497 | 0.009992 | 0.494 | 0.000134 | 0.01048 |
| 0.01839 | 0.009199 | 0.4992 | 0.0001486 | 0.01054 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Inventory Change'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 0.0002473 | -7.615e-005 |
| 0.5935 | 0.03031 | 0.4455 | 0.0002595 | -7.614e-005 |
| 0.4445 | 0.02627 | 0.4489 | 0.0002671 | -7.614e-005 |
| 0.3383 | 0.02324 | 0.4523 | 0.000273 | -7.613e-005 |
| 0.245 | 0.02037 | 0.4568 | 0.0002787 | -7.613e-005 |
| 0.1618 | 0.0175 | 0.463 | 0.0002842 | -7.636e-006 |
| 0.1005 | 0.01495 | 0.4705 | 0.0002888 | -7.608e-005 |
| 0.07047 | 0.01341 | 0.4764 | 0.0002912 | -7.678e-006 |
| 0.05188 | 0.01226 | 0.4815 | 0.000293 | -7.697e-006 |
| 0.03447 | 0.01092 | 0.4885 | 0.000295 | -7.579e-005 |
| 0.02497 | 0.009992 | 0.494 | 0.0002965 | -7.651e-006 |
| 0.01839 | 0.009199 | 0.4992 | 0.000298 | -7.632e-006 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 1.051e-005 | 0.008425 |
| 0.5935 | 0.03031 | 0.4455 | 1.106e-005 | 0.00839 |
| 0.4445 | 0.02627 | 0.4489 | 1.143e-005 | 0.008364 |
| 0.3383 | 0.02324 | 0.4523 | 1.174e-005 | 0.008342 |
| 0.245 | 0.02037 | 0.4568 | 1.208e-005 | 0.008321 |
| 0.1618 | 0.0175 | 0.463 | 1.25e-005 | 0.008291 |
| 0.1005 | 0.01495 | 0.4705 | 1.303e-005 | 0.008267 |
| 0.07047 | 0.01341 | 0.4764 | 1.35e-005 | 0.00825 |
| 0.05188 | 0.01226 | 0.4815 | 1.403e-005 | 0.008237 |
| 0.03447 | 0.01092 | 0.4885 | 1.496e-005 | 0.008221 |
| 0.02497 | 0.009992 | 0.494 | 1.596e-005 | 0.008212 |
| 0.01839 | 0.009199 | 0.4992 | 1.723e-005 | 0.008202 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Durables'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 0.00027 | 0.01276 |
| 0.5935 | 0.03031 | 0.4455 | 0.000285 | 0.01278 |
| 0.4445 | 0.02627 | 0.4489 | 0.0002945 | 0.01279 |
| 0.3383 | 0.02324 | 0.4523 | 0.0003023 | 0.0128 |
| 0.245 | 0.02037 | 0.4568 | 0.0003102 | 0.01282 |
| 0.1618 | 0.01749 | 0.463 | 0.0003189 | 0.01284 |
| 0.1005 | 0.01495 | 0.4705 | 0.000328 | 0.01287 |
| 0.07047 | 0.01341 | 0.4764 | 0.0003351 | 0.01288 |
| 0.05188 | 0.01226 | 0.4815 | 0.0003421 | 0.0129 |
| 0.03447 | 0.01092 | 0.4885 | 0.0003538 | 0.01291 |
| 0.02497 | 0.009992 | 0.494 | 0.000366 | 0.01292 |
| 0.01839 | 0.009199 | 0.4992 | 0.000381 | 0.01292 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Non-Durables'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 1.16e-005 | 0.006538 |
| 0.5935 | 0.03031 | 0.4455 | 1.234e-005 | 0.006507 |
| 0.4445 | 0.02627 | 0.4489 | 1.28e-005 | 0.00648 |
| 0.3383 | 0.02324 | 0.4523 | 1.319e-005 | 0.006457 |
| 0.245 | 0.02037 | 0.4568 | 1.36e-005 | 0.006428 |
| 0.1618 | 0.01749 | 0.463 | 1.408e-005 | 0.006398 |
| 0.1005 | 0.01495 | 0.4705 | 1.461e-005 | 0.006363 |
| 0.07047 | 0.01341 | 0.4764 | 1.505e-005 | 0.006343 |
| 0.05188 | 0.01226 | 0.4815 | 1.549e-005 | 0.006326 |
| 0.03447 | 0.01092 | 0.4885 | 1.626e-005 | 0.006306 |
| 0.02497 | 0.009992 | 0.494 | 1.708e-005 | 0.006291 |
| 0.01839 | 0.009199 | 0.4992 | 1.809e-005 | 0.00628 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Consumption of Services'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 3.165e-006 | 0.008824 |
| 0.5935 | 0.03031 | 0.4455 | 3.372e-006 | 0.00879 |
| 0.4445 | 0.02627 | 0.4489 | 3.508e-006 | 0.008728 |
| 0.3383 | 0.02324 | 0.4523 | 3.624e-006 | 0.008697 |
| 0.245 | 0.02037 | 0.4568 | 3.756e-006 | 0.008662 |
| 0.1618 | 0.0175 | 0.463 | 3.905e-006 | 0.00862 |
| 0.1005 | 0.01495 | 0.4705 | 4.161e-006 | 0.008585 |
| 0.07047 | 0.01341 | 0.4764 | 4.356e-006 | 0.00856 |
| 0.05188 | 0.01226 | 0.4815 | 4.582e-006 | 0.00854 |
| 0.03447 | 0.01092 | 0.4885 | 4.999e-006 | 0.008519 |
| 0.02497 | 0.009992 | 0.494 | 5.451e-006 | 0.008507 |
| 0.01839 | 0.009199 | 0.4992 | 6.031e-006 | 0.008494 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Government Expenditures'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 2.473e-005 | 0.009022 |
| 0.5935 | 0.03031 | 0.4455 | 2.65e-005 | 0.008965 |
| 0.4445 | 0.02627 | 0.4489 | 2.776e-005 | 0.008926 |
| 0.3383 | 0.02324 | 0.4523 | 2.894e-005 | 0.008888 |
| 0.245 | 0.02037 | 0.4568 | 3.04e-005 | 0.008835 |
| 0.1618 | 0.01749 | 0.463 | 3.254e-005 | 0.008787 |
| 0.1005 | 0.01495 | 0.4705 | 3.581e-005 | 0.008724 |
| 0.07047 | 0.01341 | 0.4764 | 3.916e-005 | 0.008678 |
| 0.05188 | 0.01226 | 0.4815 | 4.303e-005 | 0.008638 |
| 0.03447 | 0.01092 | 0.4885 | 5.021e-005 | 0.008579 |
| 0.02497 | 0.009992 | 0.494 | 5.817e-005 | 0.008534 |
| 0.01839 | 0.009199 | 0.4992 | 6.831e-005 | 0.008482 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Exports'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 0.0003318 | 0.009258 |
| 0.5935 | 0.03031 | 0.4455 | 0.0003482 | 0.009387 |
| 0.4445 | 0.02627 | 0.4489 | 0.0003585 | 0.009487 |
| 0.3383 | 0.02324 | 0.4523 | 0.0003668 | 0.009594 |
| 0.245 | 0.02037 | 0.4568 | 0.0003751 | 0.009722 |
| 0.1618 | 0.0175 | 0.463 | 0.0003843 | 0.009883 |
| 0.1005 | 0.01495 | 0.4705 | 0.0003937 | 0.01006 |
| 0.07047 | 0.01341 | 0.4764 | 0.0004008 | 0.01018 |
| 0.05188 | 0.01226 | 0.4815 | 0.0004078 | 0.01028 |
| 0.03447 | 0.01092 | 0.4885 | 0.0004192 | 0.0104 |
| 0.02497 | 0.009992 | 0.494 | 0.0004309 | 0.0105 |
| 0.01839 | 0.009199 | 0.4992 | 0.0004451 | 0.01059 |

Table of Ideal Filter-Maximum likelihood estimates of parameters for Series: 'Imports'

| Q_Zeta | Q_Kappa | Lambda_C | EpsVar | BetaMean |
|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 0.0002832 | 0.01731 |
| 0.5935 | 0.03031 | 0.4455 | 0.0002995 | 0.0167 |
| 0.4445 | 0.02627 | 0.4489 | 0.0003096 | 0.01629 |
| 0.3383 | 0.02324 | 0.4523 | 0.0003181 | 0.01598 |
| 0.245 | 0.02037 | 0.4568 | 0.0003264 | 0.01566 |
| 0.1618 | 0.0175 | 0.463 | 0.0003355 | 0.01534 |
| 0.1005 | 0.01495 | 0.4705 | 0.000345 | 0.01505 |
| 0.07047 | 0.01341 | 0.4764 | 0.000352 | 0.0149 |
| 0.05188 | 0.01226 | 0.4815 | 0.0003589 | 0.01479 |
| 0.03447 | 0.01092 | 0.4885 | 0.0003704 | 0.01467 |
| 0.02497 | 0.009992 | 0.494 | 0.0003822 | 0.01459 |
| 0.01839 | 0.009199 | 0.4992 | 0.0003967 | 0.01453 |

Note: 'ZetaVar' is the variance of the core Trend disturbance. 'EpsVar' is the variance of the Irregular. 'Q_Zeta' equals ZetaVar/EpsVar, the Trend's Signal-Noise ratio.

'BetaMean' is the Mean of the slope [AR(1)] at the core of the Trend and is only applicable to models with |Phi| < 1, where Phi is damping coefficient of the Slope.

'KappaVar' is the variance of the core Cycle disturbance. 'Q_Kappa' equals KappaVar/EpsVar, the Cycle's Signal-Noise ratio.

'Rho' is the damping rate of shocks to the Cycle. Lambda_C is the cycle's central frequency.

Table Set D8 - These tables contain fit statistics and diagnostics are reported for various representations of the ideal filter given by 12 different parameter combinations.

Filter class: Generalized Butterworth Band Pass (index n set to 8)
Extension to allow for Damped Trend is incorporated.

Results for time series taken from Bureau of Economic Analysis (GDP data)
All series logged, except 'Inventory Change' which is percent of Investment.

Table of Ideal-Filter Diagnostics for Series: 'Gross Domestic Product'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 106.5 | 112.3 | 132.9 | 0.0163 | -1494.44 | -1420.05 | -2 |
| 0.5935 | 0.03031 | 0.4455 | 112.2 | 117.7 | 138.5 | 0.0159 | -1506.55 | -1432.16 | -1.88 |
| 0.4445 | 0.02627 | 0.4489 | 115.4 | 120.8 | 141.6 | 0.0157 | -1515.16 | -1440.78 | -1.79 |
| 0.3383 | 0.02324 | 0.4523 | 117.3 | 122.7 | 143.5 | 0.0155 | -1522.85 | -1448.47 | -1.72 |
| 0.245 | 0.02037 | 0.4568 | 117.9 | 123.5 | 144.2 | 0.0152 | -1531.32 | -1456.94 | -1.63 |
| 0.1618 | 0.0175 | 0.463 | 116.1 | 122.4 | 142.7 | 0.015 | -1540.93 | -1466.55 | -1.55 |
| 0.1005 | 0.01495 | 0.4705 | 110.9 | 119.2 | 139.1 | 0.0148 | -1549.43 | -1475.04 | -1.47 |
| 0.07047 | 0.01341 | 0.4764 | 106.2 | 117.6 | 137.2 | 0.0147 | -1553.18 | -1478.80 | -1.43 |
| 0.05188 | 0.01226 | 0.4815 | 103.5 | 119.6 | 139 | 0.0146 | -1554.00 | -1479.61 | -1.43 |
| 0.03447 | 0.01092 | 0.4885 | 106.5 | 133.6 | 153.1 | 0.0147 | -1550.49 | -1476.11 | -1.46 |
| 0.02497 | 0.009992 | 0.494 | 119.5 | 161 | 181.2 | 0.0149 | -1543.10 | -1468.71 | -1.52 |
| 0.01839 | 0.009199 | 0.4992 | 145.8 | 207.2 | 229.1 | 0.0152 | -1531.55 | -1457.17 | -1.62 |

Table of Ideal-Filter Diagnostics for Series: 'Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 85.81 | 93.73 | 103.5 | 0.0869 | -546.24 | -471.85 | -2.26 |
| 0.5935 | 0.03031 | 0.4455 | 86.23 | 94.16 | 103.7 | 0.0846 | -561.72 | -487.34 | -2.09 |
| 0.4445 | 0.02627 | 0.4489 | 86.54 | 94.46 | 103.9 | 0.0829 | -572.67 | -498.28 | -1.97 |
| 0.3383 | 0.02324 | 0.4523 | 86.62 | 94.53 | 103.8 | 0.0815 | -582.48 | -508.10 | -1.87 |
| 0.245 | 0.02037 | 0.4568 | 86.25 | 94.14 | 103.2 | 0.0799 | -593.49 | -519.11 | -1.76 |
| 0.1618 | 0.01749 | 0.463 | 84.77 | 92.67 | 101.6 | 0.0781 | -606.70 | -532.32 | -1.64 |
| 0.1005 | 0.01495 | 0.4705 | 81.43 | 89.48 | 98.22 | 0.0762 | -620.36 | -545.97 | -1.51 |
| 0.07047 | 0.01341 | 0.4764 | 77.83 | 86.22 | 94.87 | 0.075 | -629.14 | -554.75 | -1.43 |
| 0.05188 | 0.01226 | 0.4815 | 74.25 | 83.24 | 91.84 | 0.0742 | -635.44 | -561.06 | -1.38 |
| 0.03447 | 0.01092 | 0.4885 | 69.81 | 80.42 | 89.08 | 0.0733 | -641.40 | -567.02 | -1.33 |
| 0.02497 | 0.009992 | 0.494 | 68.2 | 81.23 | 90.09 | 0.0731 | -643.50 | -569.12 | -1.31 |
| 0.01839 | 0.009199 | 0.4992 | 70.39 | 87.2 | 96.44 | 0.0731 | -642.81 | -568.42 | -1.31 |

Table of Ideal-Filter Diagnostics for Series: 'Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 118.1 | 126 | 130 | 0.0728 | -646.36 | -571.98 | -1.25 |
| 0.5935 | 0.03031 | 0.4455 | 120.4 | 128.4 | 132.4 | 0.0717 | -655.38 | -580.99 | -1.18 |
| 0.4445 | 0.02627 | 0.4489 | 120.5 | 129 | 132.9 | 0.0709 | -661.41 | -587.03 | -1.13 |
| 0.3383 | 0.02324 | 0.4523 | 119.3 | 128.6 | 132.6 | 0.0703 | -666.33 | -591.94 | -1.09 |
| 0.245 | 0.02037 | 0.4568 | 116.7 | 127.6 | 131.7 | 0.0697 | -670.85 | -596.47 | -1.06 |
| 0.1618 | 0.0175 | 0.463 | 112.7 | 127.4 | 131.9 | 0.0693 | -673.89 | -599.50 | -1.04 |
| 0.1005 | 0.01495 | 0.4705 | 112 | 135.6 | 141.2 | 0.0696 | -671.90 | -597.52 | -1.05 |
| 0.07047 | 0.01341 | 0.4764 | 120.4 | 156 | 163.1 | 0.0704 | -665.26 | -590.88 | -1.1 |
| 0.05188 | 0.01226 | 0.4815 | 139.4 | 191 | 200.1 | 0.0716 | -655.05 | -580.67 | -1.17 |
| 0.03447 | 0.01092 | 0.4885 | 190.8 | 274.2 | 287.4 | 0.0743 | -633.80 | -559.42 | -1.34 |
| 0.02497 | 0.009992 | 0.494 | 258 | 376.4 | 393.9 | 0.0775 | -610.28 | -535.89 | -1.54 |
| 0.01839 | 0.009199 | 0.4992 | 345 | 504.6 | 527 | 0.0814 | -582.24 | -507.86 | -1.81 |

Table of Ideal-Filter Diagnostics for Series: 'Non-Residential Investment'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 141.2 | 168.1 | 173.9 | 0.0401 | -983.30 | -908.92 | -1.84 |
| 0.5935 | 0.03031 | 0.4455 | 147.1 | 173.5 | 179.6 | 0.0394 | -993.53 | -919.14 | -1.74 |
| 0.4445 | 0.02627 | 0.4489 | 150.4 | 176.4 | 182.8 | 0.0389 | -1000.73 | -926.34 | -1.67 |
| 0.3383 | 0.02324 | 0.4523 | 152.3 | 177.9 | 184.6 | 0.0385 | -1007.05 | -932.67 | -1.61 |
| 0.245 | 0.02037 | 0.4568 | 152.6 | 177.9 | 184.9 | 0.038 | -1013.80 | -939.42 | -1.55 |
| 0.1618 | 0.0175 | 0.463 | 150 | 175.3 | 182.5 | 0.0376 | -1020.88 | -946.49 | -1.48 |
| 0.1005 | 0.01495 | 0.4705 | 144.5 | 170.6 | 178 | 0.0372 | -1025.72 | -951.33 | -1.44 |
| 0.07047 | 0.01341 | 0.4764 | 140.9 | 169.2 | 176.8 | 0.0372 | -1026.02 | -951.64 | -1.44 |
| 0.05188 | 0.01226 | 0.4815 | 141.2 | 173.5 | 181.3 | 0.0374 | -1023.22 | -948.84 | -1.46 |
| 0.03447 | 0.01092 | 0.4885 | 152.8 | 195.5 | 203.8 | 0.038 | -1013.90 | -939.52 | -1.54 |
| 0.02497 | 0.009992 | 0.494 | 177.1 | 234 | 242.9 | 0.0388 | -1001.16 | -926.77 | -1.66 |
| 0.01839 | 0.009199 | 0.4992 | 217.7 | 294.3 | 304.2 | 0.04 | -984.07 | -909.69 | -1.82 |

Table of Ideal-Filter Diagnostics for Series: 'Inventory Change'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 87.32 | 98.96 | 107.2 | 0.0742 | -636.02 | -561.64 | -4.09 |
| 0.5935 | 0.03031 | 0.4455 | 85.16 | 96.43 | 104.8 | 0.0718 | -654.05 | -579.67 | -3.78 |
| 0.4445 | 0.02627 | 0.4489 | 84.11 | 95.16 | 103.5 | 0.0702 | -666.82 | -592.43 | -3.57 |
| 0.3383 | 0.02324 | 0.4523 | 83.46 | 94.31 | 102.7 | 0.0688 | -678.32 | -603.94 | -3.39 |
| 0.245 | 0.02037 | 0.4568 | 82.96 | 93.62 | 102.1 | 0.0672 | -691.40 | -617.01 | -3.19 |
| 0.1618 | 0.0175 | 0.463 | 82.57 | 93.01 | 101.5 | 0.0653 | -707.58 | -633.20 | -2.95 |
| 0.1005 | 0.01495 | 0.4705 | 82.2 | 92.44 | 101 | 0.0633 | -725.62 | -651.23 | -2.71 |
| 0.07047 | 0.01341 | 0.4764 | 81.87 | 91.98 | 100.5 | 0.0618 | -738.79 | -664.41 | -2.54 |
| 0.05188 | 0.01226 | 0.4815 | 81.48 | 91.48 | 99.95 | 0.0606 | -750.04 | -675.65 | -2.4 |
| 0.03447 | 0.01092 | 0.4885 | 80.71 | 90.58 | 99.01 | 0.059 | -764.87 | -690.49 | -2.22 |
| 0.02497 | 0.009992 | 0.494 | 79.83 | 89.61 | 98 | 0.0578 | -776.38 | -702.00 | -2.09 |
| 0.01839 | 0.009199 | 0.4992 | 78.72 | 88.44 | 96.78 | 0.0567 | -787.08 | -712.69 | -1.97 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 118.7 | 123.8 | 134.4 | 0.0153 | -1529.91 | -1455.53 | -2.58 |
| 0.5935 | 0.03031 | 0.4455 | 115.3 | 120.8 | 132.1 | 0.0148 | -1546.99 | -1472.60 | -2.37 |
| 0.4445 | 0.02627 | 0.4489 | 113 | 118.9 | 130.6 | 0.0145 | -1558.66 | -1484.28 | -2.23 |
| 0.3383 | 0.02324 | 0.4523 | 110.8 | 117.1 | 129.2 | 0.0143 | -1568.78 | -1494.40 | -2.12 |
| 0.245 | 0.02037 | 0.4568 | 107.9 | 114.5 | 127.1 | 0.014 | -1579.63 | -1505.24 | -2 |
| 0.1618 | 0.0175 | 0.463 | 103.1 | 110.4 | 123.4 | 0.0137 | -1591.71 | -1517.32 | -1.88 |
| 0.1005 | 0.01495 | 0.4705 | 96.2 | 104.8 | 118 | 0.0134 | -1602.54 | -1528.16 | -1.77 |
| 0.07047 | 0.01341 | 0.4764 | 90.83 | 101.3 | 114.5 | 0.0133 | -1607.91 | -1533.53 | -1.71 |
| 0.05188 | 0.01226 | 0.4815 | 87.35 | 100.5 | 113.9 | 0.0133 | -1610.17 | -1535.78 | -1.69 |
| 0.03447 | 0.01092 | 0.4885 | 87.95 | 107.7 | 121.4 | 0.0133 | -1608.76 | -1534.38 | -1.7 |
| 0.02497 | 0.009992 | 0.494 | 97.01 | 125.6 | 140 | 0.0134 | -1603.25 | -1528.86 | -1.75 |
| 0.01839 | 0.009199 | 0.4992 | 117.2 | 158.6 | 174.2 | 0.0136 | -1593.69 | -1519.31 | -1.85 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption of Durables'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 101.4 | 109.2 | 125 | 0.0775 | -611.15 | -536.77 | -3.69 |
| 0.5935 | 0.03031 | 0.4455 | 102.1 | 109.4 | 125 | 0.0753 | -627.55 | -553.17 | -3.43 |
| 0.4445 | 0.02627 | 0.4489 | 102.8 | 109.8 | 125.2 | 0.0738 | -639.13 | -564.74 | -3.25 |
| 0.3383 | 0.02324 | 0.4523 | 103.3 | 110.1 | 125.4 | 0.0724 | -649.48 | -575.10 | -3.1 |
| 0.245 | 0.02037 | 0.4568 | 103.3 | 110 | 125.2 | 0.0709 | -661.09 | -586.71 | -2.93 |
| 0.1618 | 0.01749 | 0.463 | 102.3 | 109 | 124 | 0.0692 | -675.02 | -600.64 | -2.74 |
| 0.1005 | 0.01495 | 0.4705 | 99.2 | 106.1 | 121.1 | 0.0674 | -689.55 | -615.17 | -2.55 |
| 0.07047 | 0.01341 | 0.4764 | 95.5 | 102.9 | 117.8 | 0.0663 | -699.09 | -624.70 | -2.43 |
| 0.05188 | 0.01226 | 0.4815 | 91.46 | 99.81 | 114.7 | 0.0654 | -706.18 | -631.79 | -2.35 |
| 0.03447 | 0.01092 | 0.4885 | 85.53 | 96.35 | 111.4 | 0.0646 | -713.44 | -639.06 | -2.26 |
| 0.02497 | 0.009992 | 0.494 | 81.66 | 96.2 | 111.6 | 0.0642 | -716.82 | -642.44 | -2.22 |
| 0.01839 | 0.009199 | 0.4992 | 80.44 | 100.9 | 117 | 0.0641 | -717.56 | -643.17 | -2.21 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption of Non-Durables'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 92.23 | 112.6 | 130.4 | 0.0161 | -1501.67 | -1427.29 | -3.29 |
| 0.5935 | 0.03031 | 0.4455 | 93.31 | 114.7 | 133.3 | 0.0157 | -1516.27 | -1441.88 | -3.08 |
| 0.4445 | 0.02627 | 0.4489 | 94.18 | 116.2 | 135.4 | 0.0154 | -1526.56 | -1452.17 | -2.93 |
| 0.3383 | 0.02324 | 0.4523 | 94.75 | 117.2 | 137 | 0.0151 | -1535.75 | -1461.36 | -2.81 |
| 0.245 | 0.02037 | 0.4568 | 94.83 | 117.7 | 137.9 | 0.0149 | -1545.98 | -1471.60 | -2.67 |
| 0.1618 | 0.01749 | 0.463 | 93.67 | 116.6 | 137.3 | 0.0145 | -1558.08 | -1483.70 | -2.52 |
| 0.1005 | 0.01495 | 0.4705 | 90.39 | 113.1 | 133.9 | 0.0142 | -1570.14 | -1495.76 | -2.37 |
| 0.07047 | 0.01341 | 0.4764 | 86.88 | 109.2 | 130.1 | 0.014 | -1577.38 | -1503.00 | -2.28 |
| 0.05188 | 0.01226 | 0.4815 | 83.72 | 106.1 | 126.8 | 0.0139 | -1582.01 | -1507.62 | -2.23 |
| 0.03447 | 0.01092 | 0.4885 | 81.22 | 104.8 | 125.3 | 0.0138 | -1585.05 | -1510.67 | -2.19 |
| 0.02497 | 0.009992 | 0.494 | 83.24 | 109.7 | 130.3 | 0.0139 | -1584.17 | -1509.79 | -2.2 |
| 0.01839 | 0.009199 | 0.4992 | 91.62 | 123.7 | 144.5 | 0.014 | -1579.98 | -1505.59 | -2.24 |

Table of Ideal-Filter Diagnostics for Series: 'Consumption of Services'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 105.1 | 114.5 | 123.4 | 0.00839 | -1869.37 | -1794.99 | -1.72 |
| 0.5935 | 0.03031 | 0.4455 | 103.6 | 113.4 | 122.6 | 0.00819 | -1883.25 | -1808.86 | -1.59 |
| 0.4445 | 0.02627 | 0.4489 | 101.9 | 112 | 121.4 | 0.00805 | -1892.86 | -1818.48 | -1.51 |
| 0.3383 | 0.02324 | 0.4523 | 99.92 | 110.2 | 119.7 | 0.00793 | -1901.20 | -1826.81 | -1.43 |
| 0.245 | 0.02037 | 0.4568 | 96.87 | 107.3 | 116.9 | 0.00781 | -1910.00 | -1835.62 | -1.36 |
| 0.1618 | 0.0175 | 0.463 | 92.05 | 102.9 | 112.2 | 0.00766 | -1919.32 | -1844.94 | -1.27 |
| 0.1005 | 0.01495 | 0.4705 | 86.29 | 97.69 | 106.4 | 0.00759 | -1926.36 | -1851.98 | -1.23 |
| 0.07047 | 0.01341 | 0.4764 | 83.8 | 96.03 | 104.1 | 0.00756 | -1928.15 | -1853.77 | -1.21 |
| 0.05188 | 0.01226 | 0.4815 | 85.34 | 98.75 | 106.1 | 0.00757 | -1926.59 | -1852.20 | -1.22 |
| 0.03447 | 0.01092 | 0.4885 | 98.26 | 114.4 | 120.5 | 0.00768 | -1918.83 | -1844.45 | -1.28 |
| 0.02497 | 0.009992 | 0.494 | 123 | 142.5 | 147.8 | 0.00783 | -1907.19 | -1832.81 | -1.37 |
| 0.01839 | 0.009199 | 0.4992 | 163.5 | 187.6 | 192.1 | 0.00806 | -1890.93 | -1816.54 | -1.51 |

Table of Ideal-Filter Diagnostics for Series: 'Government Expenditures'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 88.3 | 98.13 | 108.1 | 0.0234 | -1287.64 | -1213.26 | -1.01 |
| 0.5935 | 0.03031 | 0.4455 | 84.62 | 94.58 | 104.1 | 0.023 | -1299.75 | -1225.37 | -0.93 |
| 0.4445 | 0.02627 | 0.4489 | 81.28 | 91.69 | 101.1 | 0.0226 | -1307.46 | -1233.07 | -0.878 |
| 0.3383 | 0.02324 | 0.4523 | 77.92 | 89.26 | 98.55 | 0.0224 | -1313.39 | -1239.00 | -0.839 |
| 0.245 | 0.02037 | 0.4568 | 74.35 | 87.82 | 97.17 | 0.0222 | -1318.38 | -1243.99 | -0.807 |
| 0.1618 | 0.01749 | 0.463 | 72.79 | 91.83 | 101.6 | 0.0221 | -1320.80 | -1246.42 | -0.79 |
| 0.1005 | 0.01495 | 0.4705 | 82.58 | 115.2 | 126 | 0.0223 | -1316.35 | -1241.97 | -0.818 |
| 0.07047 | 0.01341 | 0.4764 | 106.2 | 157.1 | 169.4 | 0.0227 | -1306.64 | -1232.26 | -0.881 |
| 0.05188 | 0.01226 | 0.4815 | 144.2 | 218.8 | 232.9 | 0.0232 | -1292.95 | -1218.57 | -0.973 |
| 0.03447 | 0.01092 | 0.4885 | 228.3 | 348.3 | 365.7 | 0.0243 | -1265.99 | -1191.61 | -1.17 |
| 0.02497 | 0.009992 | 0.494 | 324 | 490.7 | 511.6 | 0.0256 | -1237.28 | -1162.89 | -1.4 |
| 0.01839 | 0.009199 | 0.4992 | 436 | 654.5 | 679.2 | 0.0271 | -1203.92 | -1129.54 | -1.7 |

Table of Ideal-Filter Diagnostics for Series: 'Exports'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 111.3 | 118.2 | 135.4 | 0.0859 | -552.83 | -478.45 | -3.34 |
| 0.5935 | 0.03031 | 0.4455 | 107.7 | 114 | 130.6 | 0.0832 | -570.88 | -496.50 | -3.07 |
| 0.4445 | 0.02627 | 0.4489 | 105.5 | 111.6 | 127.8 | 0.0814 | -583.52 | -509.13 | -2.89 |
| 0.3383 | 0.02324 | 0.4523 | 103.7 | 109.5 | 125.4 | 0.0798 | -594.75 | -520.37 | -2.74 |
| 0.245 | 0.02037 | 0.4568 | 101.6 | 107.3 | 122.8 | 0.078 | -607.26 | -532.87 | -2.58 |
| 0.1618 | 0.0175 | 0.463 | 98.58 | 104.4 | 119.4 | 0.076 | -622.22 | -547.84 | -2.39 |
| 0.1005 | 0.01495 | 0.4705 | 94.23 | 100.4 | 114.9 | 0.0739 | -637.89 | -563.51 | -2.21 |
| 0.07047 | 0.01341 | 0.4764 | 90.1 | 97 | 111.1 | 0.0725 | -648.39 | -574.00 | -2.09 |
| 0.05188 | 0.01226 | 0.4815 | 85.91 | 93.87 | 107.6 | 0.0714 | -656.46 | -582.08 | -2 |
| 0.03447 | 0.01092 | 0.4885 | 79.75 | 90.19 | 103.4 | 0.0703 | -665.42 | -591.04 | -1.91 |
| 0.02497 | 0.009992 | 0.494 | 75.21 | 89.01 | 102 | 0.0696 | -670.58 | -596.20 | -1.85 |
| 0.01839 | 0.009199 | 0.4992 | 72.39 | 91.16 | 104 | 0.0693 | -673.50 | -599.12 | -1.82 |

Table of Ideal-Filter Diagnostics for Series: 'Imports'

| Q_Zeta | Q_Kappa | Lambda_C | Q(16) | Q(24) | Q(32) | Eq SE | AIC | SIC | R^2_d |
|---|---|---|---|---|---|---|---|---|---|
| 0.8716 | 0.0375 | 0.4415 | 109.1 | 126.5 | 135.7 | 0.0794 | -597.62 | -523.24 | -3.39 |
| 0.5935 | 0.03031 | 0.4455 | 109 | 127.4 | 137.4 | 0.0772 | -613.49 | -539.11 | -3.16 |
| 0.4445 | 0.02627 | 0.4489 | 109 | 128 | 138.6 | 0.0756 | -624.83 | -550.44 | -2.99 |
| 0.3383 | 0.02324 | 0.4523 | 108.8 | 128.3 | 139.4 | 0.0743 | -635.06 | -560.67 | -2.85 |
| 0.245 | 0.02037 | 0.4568 | 108.2 | 128.3 | 139.7 | 0.0728 | -646.60 | -572.22 | -2.7 |
| 0.1618 | 0.0175 | 0.463 | 106.7 | 127.3 | 139 | 0.071 | -660.59 | -586.20 | -2.52 |
| 0.1005 | 0.01495 | 0.4705 | 103.4 | 124.6 | 136.4 | 0.0692 | -675.32 | -600.94 | -2.34 |
| 0.07047 | 0.01341 | 0.4764 | 99.8 | 121.5 | 133.2 | 0.0679 | -685.13 | -610.75 | -2.22 |
| 0.05188 | 0.01226 | 0.4815 | 95.91 | 118.3 | 129.8 | 0.067 | -692.55 | -618.17 | -2.14 |
| 0.03447 | 0.01092 | 0.4885 | 90.02 | 114 | 125.1 | 0.0661 | -700.44 | -626.05 | -2.05 |
| 0.02497 | 0.009992 | 0.494 | 85.69 | 112 | 122.6 | 0.0656 | -704.50 | -630.11 | -2 |
| 0.01839 | 0.009199 | 0.4992 | 83.19 | 113 | 123.1 | 0.0654 | -706.08 | -631.70 | -1.98 |

Note: AIC = -2*LogL_Max +2*k, where k is number of parameters. SIC = -2*LogL_Max +2*log(T)*k for series length T. R^2_x is the coeff. of determination relative to simple benchmark, a RW (with fixed seasonal dummies for seasonal data. Specifically, R^2_x = 1-PEV(model)/PEV(benchmark), where PEV is the Prediction Error Variance (in KF steady state).