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Series on Trend Estimation and Seasonal Adjustment**

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The Effects of Seasonal Heteroskedasticity in Time Series on Trend Estimation and Seasonal Adjustment

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Abstract

Seasonal heteroskedasticity refers to seasonal changes in variability in a time series occurring over calendar years. When present in economic indicators, it can affect seasonal adjustments and trend estimates used for understanding historical patterns in the data, analysis of current trends, and policy making. In this paper, we investigate how seasonal heteroskedasticity affects signal extraction results in the forms of trend estimates and seasonal adjustments from some standard seasonal time series models enhanced to include a seasonally heteroskedastic irregular component. We apply these models to a regional time series of U.S. housing starts that shows higher levels of variability in the winter months. Comparing signal extraction results from the original and enhanced forms of the models shows the importance of accounting for seasonal heteroskedasticity in these time series to both signal extraction estimates and their error variances.

KEYWORDS: periodic models, seasonality, stochastic trends, unobserved components.

JEL classification: C22, C51, C82

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1 Introduction

Seasonal heteroskedasticity refers to systematic seasonal patterns in variability repeated over calendar years. Occurring in a number of economic time series, this property can have a significant impact on signal extraction results for estimating trends and seasonality, on the prediction of future values of a series, and on other substantive analyses of interest to economists, statisticians, and policy-makers. Such results provide important information for current analyses of the macroeconomy used to inform policy at central banks and other government institutions.

Various models accounting for seasonal heteroskedasticity have been proposed in the literature, some pertaining to general kinds of periodic behavior, and some targeting the dynamic forms most relevant for economic series. Bell (2004) proposed an extension of Box and Jenkins's (1976, Ch. 9) airline model to include an additive seasonally heteroskedastic white noise component. Proietti (2004) proposed a contrasting type of seasonal specific model, extending models given earlier in Proietti (1998). These models represent an extension of Harvey's (1989) basic structural model (BSM) to embed seasonal heteroskedasticity within the trend-seasonal part of the model, in contrast to Bell's specification of a seasonally heteroskedastic irregular component. Proietti's models are also related to the periodic models of West and Harrison (1989). Tripodis and Penzer (2007) also used Proietti's (1998) form, as well as models with a seasonally heteroskedastic irregular similar to that of Bell (2004). Tripodis and Penzer concentrated on the case where only a single month or quarter has a different variance from the others, a special case of the kind of seasonal heteroskedasticity we consider here. Finally, Trimbur and Bell (2012) considered the extension of Harvey's structural model to include a seasonally heteroskedastic irregular component.

To address the first question of whether or not seasonal heteroskedasticity is present in a given indicator series, Trimbur and Bell (2012) proposed using a likelihood ratio (LR) test and studied its finite sample behavior via simulation. A step-wise algorithm was set out to classify the months into two groups of either high or low variability. The paper also presented empirical analyses of housing starts and building permits time series data for the four U.S. Census regions. The LR tests strongly confirmed the presence of seasonal heteroskedasticity in monthly starts and permits for the Northeast (NE) and Midwest (MW) regions, the two regions expected to show seasonal heteroskedasticity due to effects of winter weather. The selected groupings were

consistent with expectations about temporary increases in variability occurring systematically in the winter months. Model comparisons given in Trimbur and Bell (2012) for the construction indicators showed that for the NE and MW regions the heteroskedastic seasonal specific trend model was generally outperformed by the models with seasonally heteroskedastic additive noise, so it is these latter models that we consider here.

Our focus in this paper is on using the monthly housing starts series for the MW to examine how seasonal heteroskedasticity affects signal extraction results for trend estimation and seasonal adjustment. This includes comparing signal extraction results from the homoskedastic and heteroskedastic models to show inadequacies of the homoskedastic models. We also show how the signal extraction filters from the heteroskedastic models effectively discount the data from the high variance months when estimating the series' components.

The presence of seasonally heteroskedastic noise in a model raises the question of how this component should be treated for seasonal adjustment. That is, should "seasonal adjustment" include removing the estimate of this seasonally heteroskedastic noise component, or should it remove only the estimate of the homoskedastic seasonal component? We suggest that there is no uniformly right answer to this question. Rather, the answer should, ideally, depend on the use to be made of the adjusted series. We can, however, assess for the housing starts series how the adjustments differ between these two options, to provide some guidance on their use as signals with different types of seasonal variation removed.

The paper proceeds as follows. Section 2 sets out the two models we consider for seasonal heteroskedasticity: the airline plus seasonally heteroskedastic noise model of Bell (2004) and the structural time series model of Harvey (1989) with a seasonally heteroskedastic irregular component. Section 3 examines signal extraction for models with seasonal heteroskedasticity compared to signal extraction for homoskedastic models. In Section 4 the two seasonally heteroskedastic models, along with their two homoskedastic counterparts, are applied to the time series of monthly housing starts for the MW region of the United States. Section 5 then offers some conclusions.

2 Models and testing for seasonal heteroskedasticity

A generic model form for a seasonal nonstationary time series is

$$y_t = \mu_t + \gamma_t + \varepsilon_t, \quad \varepsilon_t \sim i.i.d. N(0, \sigma_\varepsilon^2), \quad t = 1, \dots, T \quad (1)$$

where μ_t is the trend, γ_t is a stochastic seasonal component, and the irregular or noise component ε_t absorbs any remaining variation due to measurement error or to idiosyncratic influences. Observations occur at time points $t = 1, \dots, T$, and the complete sample is arranged in the vector $\mathbf{y} = (y_1, \dots, y_T)'$. The model depends on a parameter vector θ whose elements we estimate by Gaussian maximum likelihood.

The structural approach to modeling (Harvey 1989, Durbin and Koopman 2001) directly specifies models for the components μ_t , γ_t , and ε_t , subject to unknown variance parameters. The ARIMA model-based approach to seasonal adjustment (Hillmer and Tiao 1982, Burman 1980, Gomez and Maravall 1997) starts by developing a seasonal ARIMA (autoregressive-integrated-moving average) model for the observed series y_t (Box and Jenkins 1976), and then decomposes the pseudo-spectral density of y_t into pseudo-spectral densities for the components, implying ARIMA models for μ_t , γ_t , and ε_t . We discuss basic seasonal models from these two approaches in Sections 2.1 and 2.2.

Regardless of the choice of models for μ_t and γ_t , we can generalize the usual specification in (1) of a white noise irregular ε_t to a seasonally heteroskedastic model that decomposes the irregular into homoskedastic and seasonally heteroskedastic parts. We write this model as

$$y_t = \mu_t + \gamma_t + \varepsilon_{n,t} + \varepsilon_{s,t}, \quad t = 1, \dots, T \quad (2)$$

$$\varepsilon_{n,t} \sim i.i.d. N(0, \sigma_{\varepsilon,n}^2), \quad \varepsilon_{s,t} \sim i.i.d. N(0, \sigma_{\varepsilon,s}^2(j))$$

with season $j = 1 + (t - 1) \bmod s$ at time period t , where s is the number of seasons or observations per year. (In what follows we use “season” and “month” interchangeably, since the application considered here involves monthly data.) Note that only $\sigma_{\varepsilon,s}^2(j)$ varies explicitly with the seasonal index j . We set the minimum $\sigma_{\varepsilon,s}^2(j)$ to zero, which serves to identify the model and gives ε_t^s the interpretation of excess volatility occurring in some months over each calendar year. In fact, in our application here we specify just two values for $\sigma_{\varepsilon,s}^2(j)$ corresponding to “high variance” and “low variance” months. The high variance months for the application to monthly housing starts roughly correspond to “winter,” since our previous research (Trimbur

and Bell 2012), and research by other authors cited later in Section 5, found higher variation of housing starts in winter in the NE and MW. Since we set $\sigma_{\varepsilon,s}^2(j) = 0$ for the low variance months, we could write $\varepsilon_{s,t} = h_{j(t)}\xi_t$ where $h_{j(t)} = 1$ when $j(t)$ corresponds to a high variance month and is zero otherwise, and where $\xi_t \sim i.i.d. N(0, \sigma_{\varepsilon,s}^2)$. Then $\sigma_{\varepsilon,s}^2$ quantifies the additional irregular variance due to the seasonal heteroskedasticity which, for our analyses of housing starts data, occurs only in the winter months.

With these definitions a clear distinction is made between the homoskedastic and seasonally heteroskedastic components of the irregular. We call $\varepsilon_{s,t}$ “seasonal noise” to distinguish it from $\varepsilon_{n,t}$, which may be regarded as a traditional homoskedastic irregular component.

We now discuss the two models we consider with their extensions to include seasonal noise.

2.1 Structural time series model with seasonal noise

Harvey’s (1989) structural model specifies a local linear trend component model for μ_t of (1), which is given by

$$\begin{aligned}\mu_{t+1} &= \mu_t + \beta_t + \eta_t, & \eta_t &\sim i.i.d. N(0, \sigma_\eta^2), & t &= 1, \dots, T \\ \beta_{t+1} &= \beta_t + \zeta_t, & \zeta_t &\sim i.i.d. N(0, \sigma_\zeta^2).\end{aligned}\tag{3}$$

The model for the stochastic seasonal component γ_t is specified as:

$$\begin{aligned}\gamma_t &= \sum_{j=1}^{\lfloor s/2 \rfloor} \gamma_{j,t}, & (4) \\ \gamma_{j,t+1} &= \gamma_{j,t} \cos \lambda_j + \gamma_{j,t}^* \sin \lambda_j + \omega_{j,t}, \\ \gamma_{j,t+1}^* &= -\gamma_{j,t} \sin \lambda_j + \gamma_{j,t}^* \cos \lambda_j + \omega_{j,t}^*, \\ \omega_{j,t}, \omega_{j,t}^* &\sim i.i.d. N(0, \sigma_\omega^2)\end{aligned}$$

where the (nonstationary) stochastic seasonal cycles, the $\gamma_{j,t}$ s, have frequencies given by

$$\begin{aligned}\lambda_j &= 2\pi j/s, & j &= 1, \dots, \lfloor s/2 \rfloor - 1, \\ \lambda_j &= \pi, & j &= \lfloor s/2 \rfloor.\end{aligned}$$

Given the constant variance σ_ω^2 for all j , the model (4) is referred to as TRIG-1. This contrasts with a more general model proposed by Harvey (1989), in which $\text{Var}(\omega_{j,t})$ is distinct for each $j = 1, \dots, s/2$. Here we restrict attention to the TRIG-1 model. The white noise disturbances

η_t , ζ_t , $\omega_{j,t}$, and $\omega_{j,t}^*$ are assumed uncorrelated with one another across all time points, and also with the irregular.

Bell (2004, pp. 252–253) provides ARIMA representations of the models (3) and (4), including the more general version of (4) which has distinct $\text{Var}(\omega_{j,t})$ for each j . These representations make clear that (i) $(1 - B)^2\mu_t$ is stationary (μ_t is at most $I(2)$), and (ii) $(1 + B + \dots + B^{s-1})\gamma_t$ is stationary, and so both of these are homoskedastic. (B is the backshift operator defined by $B^j z_t = z_{t-j}$.) The model given by (1), (3), and (4) is thus a homoskedastic model, while the heteroskedastic generalization is given by (2), (3), and (4).

2.2 Airline model with seasonal noise

The airline model (Box and Jenkins 1976, ch. 9) is given by

$$(1 - B)(1 - B^{12})y_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12})a_t, \quad a_t \sim i.i.d. N(0, \sigma_a^2). \quad (5)$$

This model can be mapped into a components model via the canonical decomposition of Hillmer and Tiao (1982) and Burman (1980). Thus, for given values of θ_1 , $\theta_{12} > 0$, and $\sigma_a^2 > 0$, we can write, in the form of (1),

$$y_t = \mu_t^A + \gamma_t^A + \varepsilon_t^A, \quad \varepsilon_t^A \sim i.i.d. N(0, \sigma_{\varepsilon,A}^2). \quad (6)$$

where the components follow the ARIMA models

$$\begin{aligned} (1 - B)^2\mu_t^A &= (1 - \psi_1 B - \psi_2 B^2)b_t, & b_t &\sim i.i.d. N(0, \sigma_b^2) \\ (1 + B + \dots + B^{s-1})\gamma_t^A &= (1 + \alpha_1 B + \dots + \alpha_{s-1} B^{s-1})c_t, & c_t &\sim i.i.d. N(0, \sigma_c^2). \end{aligned} \quad (7)$$

That is, the canonical decomposition determines values of the parameters ψ_1 , ψ_2 , $\alpha_1, \dots, \alpha_{s-1}$, σ_b^2 , σ_c^2 , and $\sigma_{\varepsilon,A}^2$ that yield agreement (in terms of autocovariance generating functions or pseudo-spectral densities) between the component models (6)–(7) and the original airline model (5). The term “canonical” refers to determining the irregular variance $\sigma_{\varepsilon,A}^2$ to be as large as possible subject to the component models agreeing with the airline model, so the remaining trend-seasonal part could be regarded as fundamental signal in the sense of having the greatest regularity among all possibilities.

Bell (2004) proposed adding seasonally heteroskedastic noise to the model by letting $y_t = Y_t + \varepsilon_{s,t}^A$, where Y_t continues to follow the airline model (5) and corresponding canonical decomposition (6)–(7), now with ε_t^A replaced by a baseline irregular $\varepsilon_{n,t}^A$. Thus, with the addition of

the seasonal noise $\varepsilon_{s,t}^A$, analogously to (2),

$$y_t = \mu_t^A + \gamma_t^A + \varepsilon_{n,t}^A + \varepsilon_{s,t}^A \quad (8)$$

$$\varepsilon_{n,t}^A \sim i.i.d. N(0, \sigma_{\varepsilon,n,A}^2), \quad \varepsilon_{s,t}^A \sim i.i.d. N(0, \sigma_{\varepsilon,s,A}^2(j)).$$

We see that $(1 - B)(1 - B^{12})Y_t$ follows the stationary moving-average model shown on the right hand side of (5), and so is homoskedastic. The components in (7) are now defined in a way that maximizes the variance of $\varepsilon_{n,t}^A$. If there is no seasonal noise, then $\varepsilon_{s,t}^A$ is not present in (8), which is then a homoskedastic model.

We use the “A” superscript on the components in (6)–(8) to emphasize that these components correspond to the canonical decomposition of the airline model and so differ from the components of a structural model applied to the same time series. This is because the components are effectively defined by their stochastic structures, i.e., by their models, and the component models differ between the structural and airline plus seasonal noise models. Because of this, part of the difference in the component estimates between the two model forms will be due to this definitional difference, a point that must be kept in mind when comparing the estimates. See Bell and Hillmer (1984) for further related discussion.

2.3 Testing for seasonal heteroskedasticity

For either of the two models just discussed, and given a specification of which months have seasonal noise (the high variance months), we can test for the presence of seasonal heteroskedasticity via a likelihood ratio (LR) test of $H_0: \sigma_{\varepsilon,s}^2 = 0$ (or, for the airline plus seasonal noise model, of $H_0: \sigma_{\varepsilon,s,A}^2 = 0$.) Note that this null hypothesis of homoskedasticity constrains $\sigma_{\varepsilon,s}^2$ to be on the boundary of the parameter space ($\sigma_{\varepsilon,s}^2 = 0$), a nonstandard case for hypothesis testing. Trimbur and Bell (2012) thus used a result of Self and Liang (1987) on the asymptotic distribution of LR statistics when parameters are on the boundary. (Self and Liang’s results, which apply for the case of independent observations, extended results of Chernoff (1954).) For the case of testing whether a single parameter is on the boundary with other (nuisance) parameters away from the boundary (Self and Liang’s Case 5), the asymptotic distribution of LR is $\frac{1}{2}\chi_0^2 + \frac{1}{2}\chi_1^2$, where $\chi_0^2 = 0$ with probability 1. Harvey (1989, pp. 236-237) discusses the application of such results to time series component models related to those considered here.

Following this prescription, to perform a 5% test of $H_0: \sigma_{\varepsilon,s}^2 = 0$ we compare LR to the 10% critical value from the χ_1^2 distribution, which is 2.71.

Trimbur and Bell (2012, pp. 47–53) used simulation to study the finite sample performance of the LR test for the airline plus seasonal noise model. Simulation settings allowed for different values of model parameters, different series lengths (10 or 20 years), and different patterns of seasonal heteroskedasticity (high variance only in January, or only in January-February, or in January-April, or in January-June). Under the null hypothesis, the simulation results showed very good conformance of the finite sample distribution of LR to the asymptotic distribution, with actual sizes of nominal 5% tests ranging from about 4% – 6%. (We also found similar performance for the LR test of seasonal heteroskedasticity with Proietti’s (2004) seasonal specific irregular model, though the different formulation of this model implies application of the standard asymptotic results for the LR test, that is, the modification of Self and Liang (1987) is not needed.) Although we have not done simulations for the LR test for the structural time series model with seasonal noise, based on our previous results for the airline plus seasonal noise and seasonal specific irregular models, we would expect that the heteroskedasticity test for the structural time series model with seasonal noise would also have good size properties.

Application of the LR heteroskedasticity tests with the airline plus seasonal noise model to U.S. regional housing starts data from January 1989 through April 2009 (Trimbur and Bell 2012, Table 2.3) showed strong evidence of heteroskedasticity for the NE ($LR = 9.8$, compared to $\chi_{1,10}^2 = 2.71$) and very strong evidence for the MW ($LR = 42.3$). The test for the SO region was significant but weaker ($LR = 4.2$), and for the WE region the test was insignificant ($LR = 0.2$). The corresponding tests from data on building permits showed a similar pattern but were even more significant for the MW, NE, and SO, while again insignificant for the WE. In the remainder of the paper we focus on analyzing trend estimation and seasonal adjustment for housing starts in the MW.

3 Trend estimation and seasonal adjustment accounting for seasonal heteroskedasticity

It is clear that, to attain the most relevant and useful indication of trend, one should extract out both predictable seasonal movements and temporary or idiosyncratic fluctuations. Explicit modelling of seasonal noise allows one to remove this additional source of short-lived variation that occurs in certain months of the year, and that can be due to sources such as unusual weather that one would not want to assign to the trend component. Failure to model and remove seasonal noise will allow it to leak into and distort the trend estimate.

As noted in the Introduction, the treatment of seasonal noise for seasonal adjustment is less clear. While our models assume that the pattern of months subject to seasonal noise remains fixed year-to-year, a property that can be considered “seasonal,” they also assume that the actual seasonal noise effects (their magnitudes and signs) are independent year-to-year. We thus suggest that the right answer to the question of whether one should adjust for seasonal noise should, ideally, depend on the use to be made of the adjusted data. Consider, for instance, the case where the seasonal noise is linked to unusual weather. If we desire an indicator reflecting the persistent state of underlying economic patterns, the removal of such noise is called for. However, if there is interest in preserving the effects of unusual weather on the series, this would suggest not removing the seasonal noise. Thus, one can conceive of situations where the removal of seasonal noise would be seen as desirable, and others where it would be seen as undesirable. For this reason we refrain from making general judgments and focus on analyzing and illustrating the consequences for seasonal adjustment of removing seasonal noise versus not doing so.

3.1 Signal extraction estimates of the components

For any of the models discussed above, and given known values of the model parameters, minimum mean squared error signal extraction estimates of the components are given by their conditional expectations given the data. These can be computed via the Kalman filter with an associated smoother (Harvey 1989, Durbin and Koopman 2001), also making suitable assumptions about the nonstationary initial conditions (Bell 1984, Bell and Hillmer 1991). Another

option would be to use the matrix formulas given by McElroy (2008) or Bell (2004). (Note that McElroy (2008) assumes that differenced components are stationary, but a simple extension to heteroskedastic noise is immediate)

. Either way, for the homoskedastic structural model, we could write the smoothed trend estimate as

$$\hat{\mu}_t^{Hom} = E[\mu_t | \mathbf{y}, \theta^{Hom}] \quad (9)$$

where θ^{Hom} denotes the homoskedastic model parameters, and implies use of the homoskedastic model. We can analogously define $\hat{\mu}_t^{Het}$, as well as the seasonal component estimates $\hat{\gamma}_t^{Hom}$ and $\hat{\gamma}_t^{Het}$. Estimates of the noise components are then obtained by subtracting out trend and seasonal from the observations, e.g. $\hat{\varepsilon}_t^{Hom} = y_t - \hat{\mu}_t^{Hom} - \hat{\gamma}_t^{Hom}$. Statistical uncertainty in the estimates is summarized in the conditional variances, e.g., $\text{var}(\mu_t | \mathbf{y}, \theta^{Hom})$, which can be used in forming prediction intervals in the usual way under the assumption of normality.

In practice, we substitute parameter estimates $\hat{\theta}$ into the formula or algorithm for computing the signal extraction estimate. First order approximations to the prediction error variances are given by, e.g., $\text{var}(\mu_t | \mathbf{y}, \hat{\theta}^{Hom})$. These values can be improved to also recognize uncertainty in the parameter estimates; see, e.g., Ansley and Kohn (1986), Hamilton (1986), and Pfeiffermann and Tiller (2005).

We have analogous definitions of component estimators and associated error variances for the airline plus seasonal noise model. Using the superscript “A” for such estimators, we have

$$\hat{\mu}_t^{A,Hom} = E[\mu_t^A | \mathbf{y}, \theta^{A,Hom}] \quad (10)$$

and so forth, through to $\text{var}(\mu_t^A | \mathbf{y}, \hat{\theta}^{A,Het})$.

Notice that in (9) we write μ_t for the trend being estimated by the homoskedastic structural model rather than writing μ_t^{Hom} . The latter notation would be appropriate if we regarded the trend components being estimated by the homoskedastic and heteroskedastic models as different. We do not; we view both the homoskedastic and heteroskedastic models as attempting to model and estimate the same trend component. If truth is the homoskedastic structural model ($\sigma_{\varepsilon,s}^2 = 0$), then both this and the heteroskedastic structural model are “correct”, but the heteroskedastic version estimates an unnecessary variance $\sigma_{\varepsilon,s}^2$, which is actually zero. This will have some effect on the accuracy of the trend estimate, at least for small to moderate T . If truth is the heteroskedastic structural model ($\sigma_{\varepsilon,s}^2 > 0$), then the homoskedastic version is

misspecified, which will increase the true error variances of its signal extraction estimates and predictions into the future. Also, $\text{var}(\mu_t | \mathbf{y}, \theta^{Hom})$, which is computed assuming the homoskedastic model is true, will misstate the associated error variances. Analogous comments apply to the homoskedastic and heteroskedastic versions of the airline plus seasonal noise model.

This choice not to distinguish trend components in homoskedastic versus corresponding heteroskedastic models differs from our explicit choice, discussed in Section 2, to distinguish trends from the structural (μ_t) and airline plus noise (μ_t^A) models. We draw this latter distinction due to what we view there as a definitional difference of the components, and so view part of the difference between, say, $\hat{\mu}_t^{Hom}$ and $\hat{\mu}_t^{A,Hom}$ as definitional. However, any difference between $\hat{\mu}_t^{Hom}$ and $\hat{\mu}_t^{Het}$ we view as due to either parameter estimation error (if $\sigma_{\varepsilon,s}^2 = 0$ but $\hat{\sigma}_{\varepsilon,s}^2 > 0$) or to model misspecification with the homoskedastic model (if $\sigma_{\varepsilon,s}^2 > 0$). In fact, an interesting question we shall explore in Section 4 is how much difference we find between homoskedastic and heteroskedastic model signal extraction estimates for a given model form (comparing $\hat{\mu}_t^{Hom}$ with $\hat{\mu}_t^{Het}$ or $\hat{\mu}_t^{A,Hom}$ with $\hat{\mu}_t^{A,Het}$), versus how much difference we find between estimates from the structural versus the airline plus noise model (comparing $\hat{\mu}_t^{Hom}$ with $\hat{\mu}_t^{A,Hom}$ or $\hat{\mu}_t^{Het}$ with $\hat{\mu}_t^{A,Het}$). We will similarly examine differences between seasonal component estimates (substitute γ for μ in the previous sentence), and signal extraction error variances, etc.

Another question noted earlier arises as to whether, for the seasonally heteroskedastic models, ε_t^s should be removed as part of seasonal adjustment or relegated to the irregular? If the decision is not to remove an estimate of ε_t^s as part of seasonal adjustment, then the seasonally adjusted series from the heteroskedastic model is given by

$$y_t - \hat{\gamma}_t^{Het} = \hat{\mu}_t^{Het} + (\hat{\varepsilon}_t^n + \hat{\varepsilon}_t^s).$$

If we instead decide to remove $\hat{\varepsilon}_t^s$ as part of seasonal adjustment, then the resulting estimate is

$$y_t - \hat{\gamma}_t^{Het} - \hat{\varepsilon}_t^s = \hat{\mu}_t^{Het} + \hat{\varepsilon}_t^n$$

which could be called “seasonal noise adjusted.” Section 4 will illustrate differences between these two options for seasonal adjustment.

3.2 Filters

A component estimate at a given time point can be expressed as the result of applying a linear filter to the surrounding observations. For instance, for the trend component estimate from a

homoskedastic structural model, we can write

$$\hat{\mu}_t^{Hom} = \sum_k w_{\mu,k}^{Hom} y_{t+k}. \quad (11)$$

where k is a positive integer representing the lead or lag of the observation being weighted, relative to the signal estimate time point. Examining the set of filter weights, $\{w_{\mu,k}^{Hom}, k = 0, \pm 1, \pm 2, \dots\}$ is informative about exactly how the trend estimate is produced. It is useful to consider the bi-infinite case, $k = 0, \pm 1, \pm 2, \dots$, which abstracts from the effects of finite series length and gives a good approximation to signal extraction away from the end-points of a long series.

In the homoskedastic case the underlying models are time-invariant, a property shared by their optimal bi-infinite filters. In the seasonally heteroskedastic case, however, the optimal filters are time-dependent, varying seasonally. In particular,

$$\hat{\mu}_t^{Het} = \sum_k w_{\mu,k}^{Het}(j) y_{t+k} \quad (12)$$

where the notation $w_{\mu,k}^{Het}(j)$ indicates that the filter weights sequence depends on the season $j(t)$ corresponding to time t . The pattern of the bi-infinite weights repeats seasonally because, for any time point corresponding to season j , the pattern of high versus low variances in the surrounding observations – their timing relative to season j – will be the same.

The w_k^{Hom} and $w_k^{Het}(j)$ can be computed indirectly, with Koopman and Harvey's (2002) modification of the state space recursions, or directly, with the formulas given by McElroy (2008). (These two strategies apply to the finite sample case. The bi-infinite weighting function may be handled as in McElroy and Trimbur (2015) by embedding the different monthly series in a vector time series, an idea originally proposed by Gladyshev (1961)). Relative to weights for a filter from a homoskedastic model, the filter weights for a corresponding heteroskedastic model are generally reduced for the observations in high-variance seasons, and increased for the observations in low-variance seasons. This will be seen for the empirical example given in Section 4.

Expressions analogous to (11) and (12) obviously hold for the airline plus seasonal noise model. We denote the corresponding filter weights as $w_{\mu,k}^{A,Hom}$ and $w_{\mu,k}^{A,Het}(j)$. These can also be computed as just discussed.

4 Application of the alternative models to the monthly time series of Midwest housing starts

In this section we present an application of the models discussed above to a monthly time series of housing starts, which signal imminent activity in the construction sector of the macroeconomy. As noted in Section 2.3, there is strong evidence of seasonal heteroskedasticity in housing starts for the NE and MW regions due to variations in winter weather around the typical patterns. Here, for the MW housing starts series, we address the objective of examining signal extraction estimation of series components for models with and without heteroskedasticity, and we show how an indicator series may be adjusted to account for seasonal noise.

Subsection 4.1 describes the dataset and form of heteroskedasticity used in the modelling, and then reviews evidence on seasonal heteroskedasticity from Trimbur and Bell (2012). Subsections 4.2 to 4.5 then compare signal extraction results for estimating the time series components – seasonal, trend, irregular, and seasonally adjusted series – using the four alternative models: the homoskedastic and seasonally heteroskedastic STSM and airline models.

4.1 Data and model estimation

We model monthly estimates of the number of total housing starts for the MW region of the U.S. from January, 1959 to December, 2015. The data are from the U.S. Census Bureau’s Survey of Construction; the most recent data and additional information are available at https://www.census.gov/construction/nrc/historical_data/index.html. Note that the series is published as both seasonally adjusted figures (an indirect adjustment is used) and unadjusted figures. Here we use the unadjusted data since we aim to model the seasonality in levels along with the seasonal heteroskedasticity. Prior to applying the four alternative models, the series was logged. Figure 1 shows the original logged series and corresponding signal extraction estimates of the trend for the homoskedastic STSM. The estimated trend tracks the major changes in level over the sample period.

We consider the two heteroskedastic models for which months are classified into two groups with different variances. For the “high variance months”, taken here as December, January, and February, $\sigma_{\varepsilon,s}^2(j) = \sigma_{\varepsilon,s}^2 > 0$, where $\sigma_{\varepsilon,s}^2$ (written without functional dependence on the

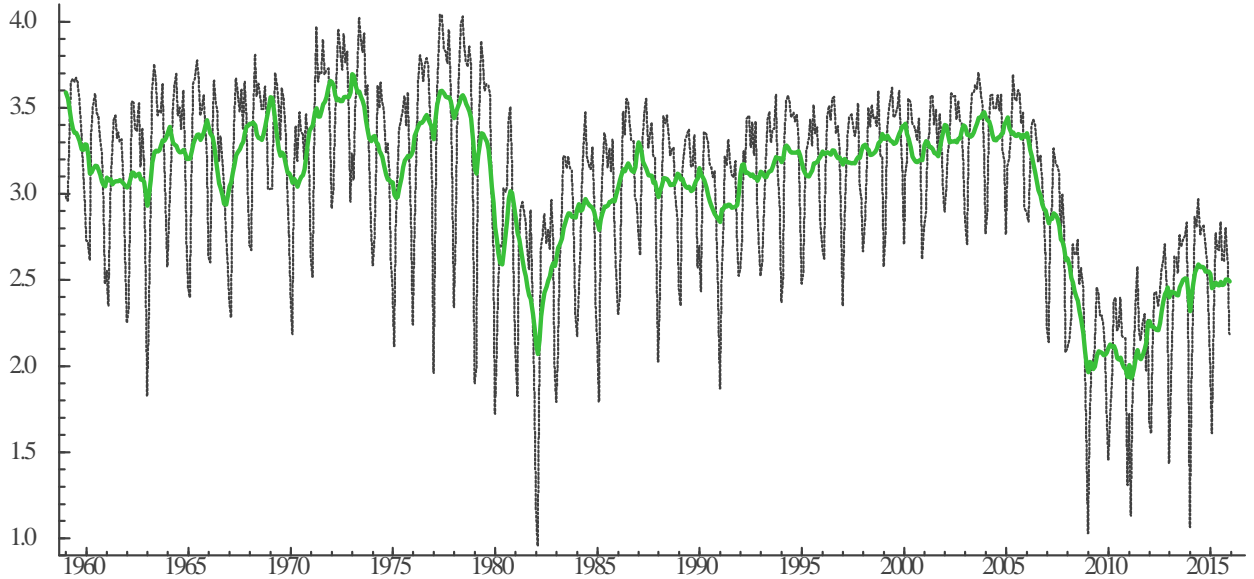


Figure 1: Estimated trend for homoskedastic STSM (solid line), along with the observations (dotted line) on total MW housing starts (logged). The sample period is 1959:1 to 2015:12.

season j) is a constant parameter that represents the extra variability present in each winter month. The remaining months are “low variance months” with $\sigma_{\varepsilon,s}^2(j) = 0$ so that only the baseline irregular variation is present.

The month grouping algorithm and LR test results presented by Trimbur and Bell (2012) support the assumption of a winter effect on variability for the NE and MW regions, particularly for the MW. These results were based on a prior sample consisting of roughly the first half of the series used here. When we applied the grouping algorithm to MW starts over the full sample period used here it also selected December to February as the high variance months for both the heteroskedastic STSM and the airline plus seasonal noise models.

Model parameters were estimated via Maximum Likelihood. The prediction error decomposition was used within a State Space framework to evaluate the log-likelihood at each candidate parameter vector, and the log-likelihood was then optimized numerically. Computations were performed using the Ox programming language of Doornik (2013) and the SsfPack set of State Space routines, described in Koopman, Shephard, and Doornik (2008). The models used here may also be handled with the REGCMPNT software introduced in Bell (2011).

4.2 Comparing results for the trend component

The trend estimates for the series are significantly affected by intense seasonal noise, as is evident from the numerous instances in Figure 1 where the STSM trend estimate is dragged down by sharp dips in the series. Figure 2 plots a close-up over the recent sub-sample from 2008 to 2015, showing the data and comparing the trend estimates from the homoskedastic airline and STS models. The two trends generally match each other well in their overall evolution, though their differences are readily apparent, as the canonical airline trend is visibly smoother. Note, for instance, the values enclosed in the red rectangular region, which show the trend from the airline model passing through a short-run oscillation in the STSM trend. Both trends show excessive short-term reactions to observations affected by seasonal noise, e.g. the unusually high values around the winter season centered on January 2012 lead to a temporary rise in the trends' rate of ascent going into January that is rapidly reversed (into a slow decline) until mid-2012, at which point the trend appears to ascend at roughly the same rate as before. Likewise for the surprisingly negative seasonal noise that impacts the January 2014 observation, both trends undergo a short-run dip. Such temporary fluctuations do not fit with our intuition of trend and indicate spill-over of temporary seasonal noise into the component that is supposed to reflect long-run movements.

Figure 3 shows a comparison of the two estimated STSM trends. Though it is usually close to the homoskedastic trend, the heteroskedastic trend is clearly more robust to large-magnitude irregulars around the turn of the year. The heteroskedastic trend tends to pass through temporary episodes that are unusual relative to seasonal norms and the present level of the trend, such as the observation in January 2014. Whereas, beginning in late 2011 the homoskedastic trend has a short-run upturn and responds excessively to the unusually high observations, the heteroskedastic trend continues to increase gradually and at a steadier rate during this time. Overall, a comparison of the trends makes it clear that insufficient noise is being eliminated around winter in the homoskedastic case; note the gyrations around the turn of the year in the circled regions in Figure 3. The heteroskedastic trend provides an effective remedy to this problem.

The weights placed on the observations when extracting the trend with the STSM are displayed in Figure 4 as a function of the time separation between the time point where the

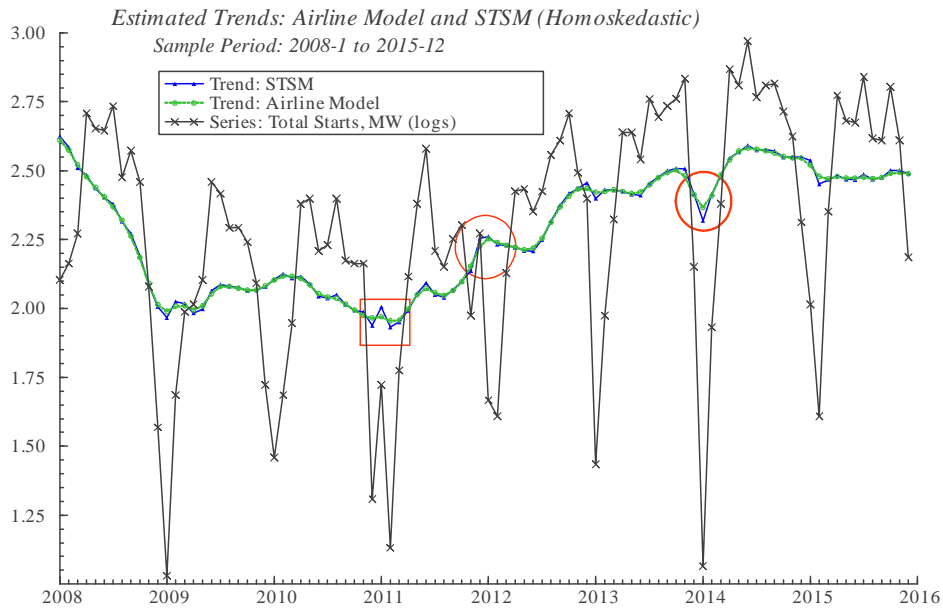


Figure 2: Estimated trends for homoskedastic models, with the STSM trend shown as the solid line (with triangles) and the airline model trend shown as the dotted line (with circles), along with the observations on MW total housing starts (logged). The values are shown for the period 2008:1 to 2015:12.

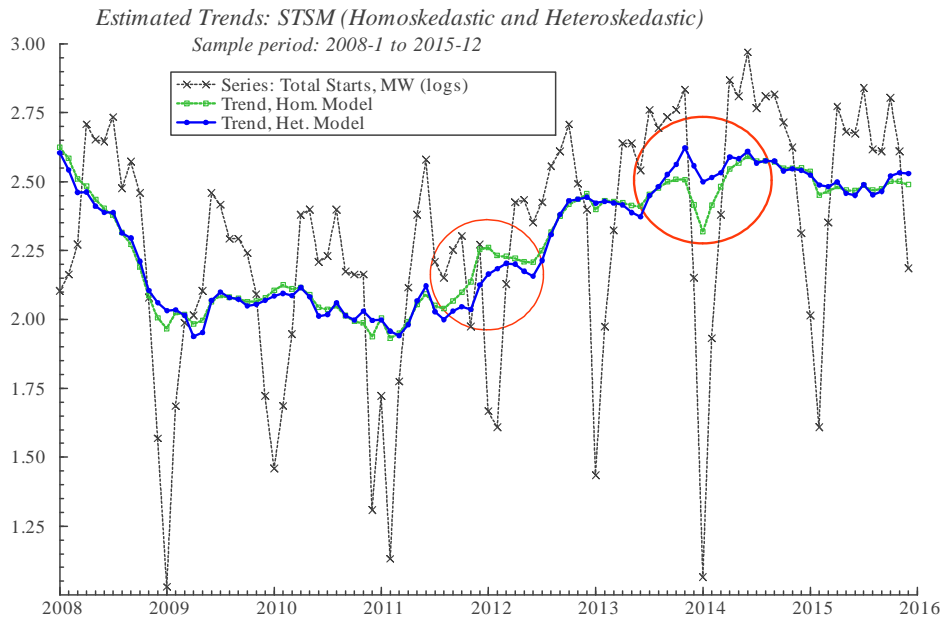


Figure 3: Estimated trends over time shown for the homoskedastic (dotted line, open boxes) and heteroskedastic (solid line, dots) STSM, along with the observations on total MW housing starts (logged). The values are shown for the period 2008:1 to 2015:12.

trend is estimated and the time point of the observation being weighted. The weight function in the upper left panel applies to the homoskedastic case; it roughly follows the shape of a double exponential, with the current observation emphasized most heavily and with weights decaying away from the time point of the estimate. The weights for the heteroskedastic model depend on the month when the signal is computed. The upper-right panel pertains to estimation of trend at a January time point, and it reveals how the December to February data values are substantially down-weighted relative to the homoskedastic weight function. Hence, the greater robustness to seasonal noise is reflected by this strongly modified pattern in the weights for estimating the trend. The pattern here is symmetric since each adjacent month is high variance, the second nearest months are both low variance, and so forth. An examination of the weight function in the bottom left panel, which pertains to estimates for a February time point, reveals an asymmetric weight function. Relative to the homoskedastic case, the weights for the two left observations and the current one are substantially reduced, with the weight transferred to the nearest low-variance observations. Likewise, the March function has a specific form of asymmetry based on emphasizing lower variance observations. The trend produced by such time-varying weights is attractive as it better reflects the general level and the ongoing trajectory of the time series, and the weight function for the observed data shows exactly how this is achieved.

The trend estimates for the airline and airline-seasonal noise models are plotted in Figure 5. These tell the same basic story as those for the STS models. While the trend from the homoskedastic airline model reacts to individual winter observations that are unusually low or high relative to the ongoing trend-seasonal, the trend for the airline-seasonal noise model is more robust and more likely to pass through such events. The canonical airline trends in Figure 5 are also smoother than the STSM trends in Figure 3, part of the “definitional difference” discussed earlier. This may be an attractive property depending on the application.

For estimating the trend in January, since the nearest observations are associated with a higher irregular variance, the degree of uncertainty in the estimate is higher. Likewise, in December and February the current observation and those located 1-month and 2-months away on one side are less informative, so that trends in December and February are also measured with less precision than is implied by the homoskedastic model. Hence, as shown in Figure 6,

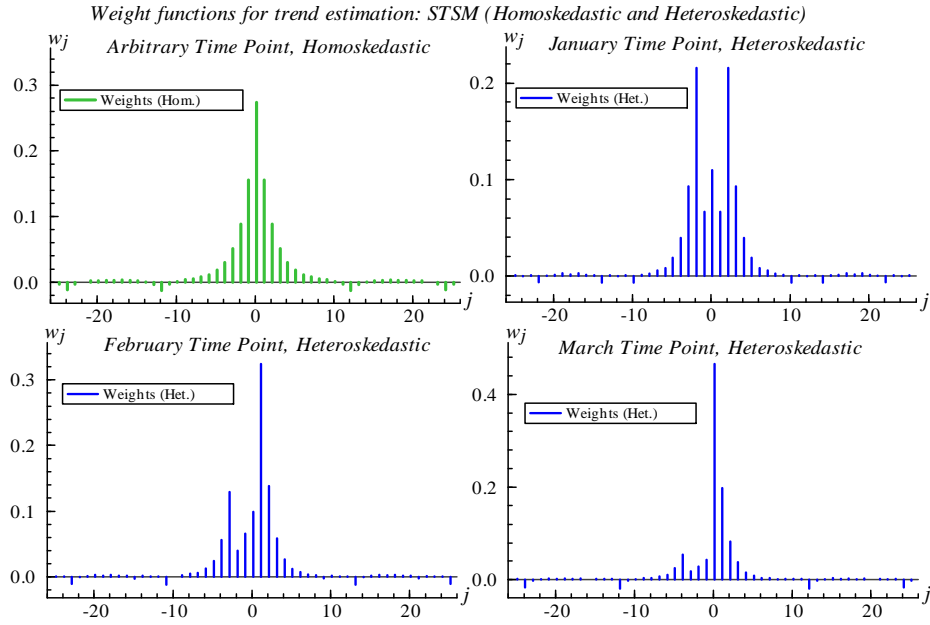


Figure 4: Estimated weight functions on observations, for estimating the trend, shown for the homoskedastic (green) and heteroskedastic (blue) STSM. The trend estimate location is away from the series end-points.

which focuses on a recent sub-sample, the signal extraction root mean squared error (RMSE) for the heteroskedastic STSM expands around the turn of the year beyond the RMSE computed for the homoskedastic STSM, while the heteroskedastic STSM’s RMSE falls substantially below that for the homoskedastic STSM the rest of the year. The RMSE also widens sharply for the last few months of the series (which ends in December 2015) in both cases. On average, the RMSE is a good deal lower for the heteroskedastic model, and the increase during the winter months gives an accurate reflection of the higher uncertainty around this time. Similar results (not shown for brevity) were obtained for the airline and airline plus seasonal noise models.

4.3 Comparing results for the seasonal component

Figure 7 shows the estimated seasonal component for the homoskedastic STSM; the sub-sample from January 1975 to December 2004 is shown for visibility purposes. We see it exhibits a sharp winter decrease each calendar year, followed by regular increases in spring and summer. The detrended series, equivalent to seasonal plus irregular, is indicated by the black \times ’s and dotted line. The seasonal pattern varies noticeably over the sample period; in particular, the

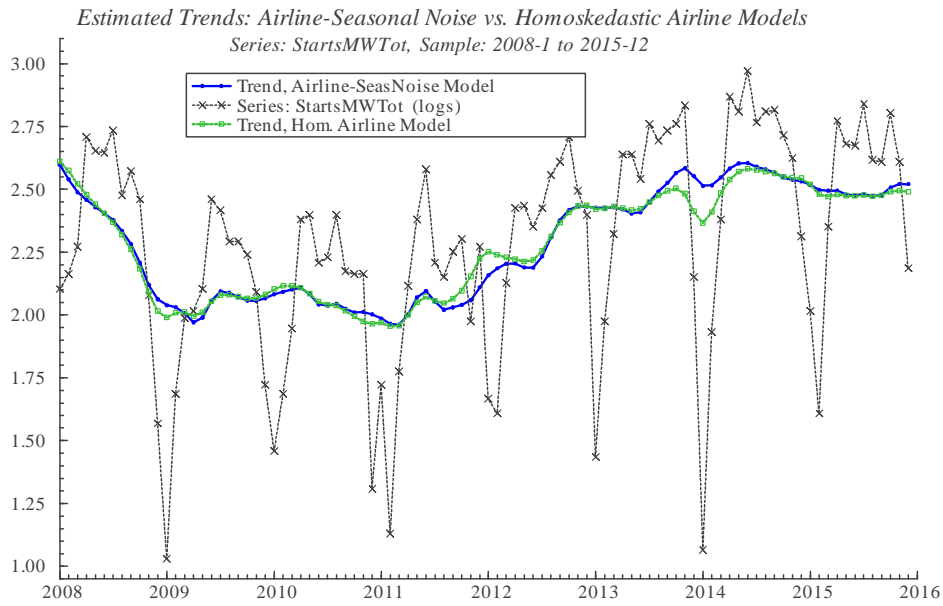


Figure 5: Estimated trends over time shown for the airline (dotted line, open boxes) and airline-seasonal noise (solid line, dots) models, along with the observations on total MW housing starts (logged). The values are shown for the period 2008:1 to 2015:12.

depth of the winter trough becomes more intense from the mid 1970s to the early 1980s, a time period corresponding to several irregular values that are unusually negative around the turns of the years, as indicated by the first region enclosed in a red rectangle. While the irregulars are reflecting these temporarily low observations, the estimated seasonal is also being significantly influenced by these observations that are affected by weather or other transient factors. Similarly, there is a cluster of surprisingly high irregulars (that is, less negative rather than lower magnitude) enclosed by the red ellipse; both of these episodes have clear effects on the estimated winter seasonal trough. There is also a second period of several highly negative irregulars in the part of the sample not shown from around 2009 to 2014, which leads to another deepening in the estimated seasonal trough.

Figure 8 shows the estimated seasonal component and the seasonal plus irregular for the heteroskedastic STSM over the same time frame used for Figure 7. The deep troughs in the series around winter months in the late 1970s and early 1980s show up more clearly in the seasonal plus irregular than is the case in Figure 7 (note the different scale of the vertical axis from that of Figure 7), as now the more pronounced temporary swings are absorbed by the

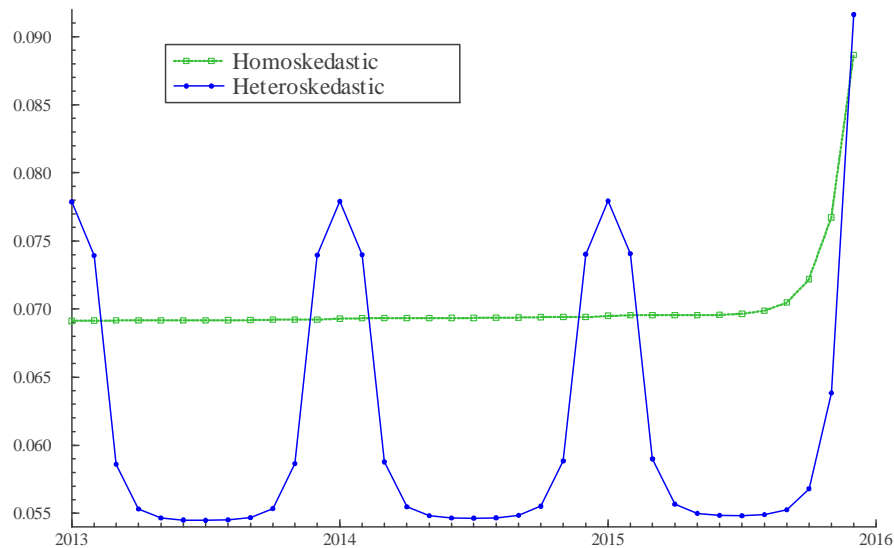


Figure 6: Root-Mean Squared Error of trend estimate over time shown for the homoskedastic (open boxes) and heteroskedastic (solid dots) STSM. The sample period is 1959:1 to 2015:12, and the values are shown for the period 2013:1 to 2015:12.

seasonal noise part of the total irregular. A more gradual and plausible evolution in seasonal pattern is also transparent in Figure 8, as the estimated seasonal component shows very little reaction to the bursts of seasonal noise, such as in the regions enclosed in the red rectangle and ellipse. (There is a similar robustness in the estimated seasonal from the period around 2009 to 2014 of intensely negative irregulars.)

Figure 9 shows the seasonal pattern in the form of a seasonal component by month plot, which shows the estimated seasonal component values, plotted separately for each given calendar month, over all the years in the sample. Results are shown for the homoskedastic versions of both the airline and STS models. The span of the seasonal estimates is nearly 0.3, or 30 percent, for both January and February over the sample period. The evolution is non-monotonic and indicates a sizeable expansion, followed by a period where the trough contracts in size, and then finally by continued expansion in the last several years. While a contraction of the seasonal amplitude would fit with intuition about technological improvements making it more feasible to initiate construction on new residences despite unusually cold temperatures or high precipitation, the expansions of the trough seen in the early and late parts of the sample period have no such intuitive explanation and are due to bursts of temporary seasonal noise. The

estimated seasonal components for the airline and STS models are generally very close, with that for the airline model showing slightly larger movements over time.

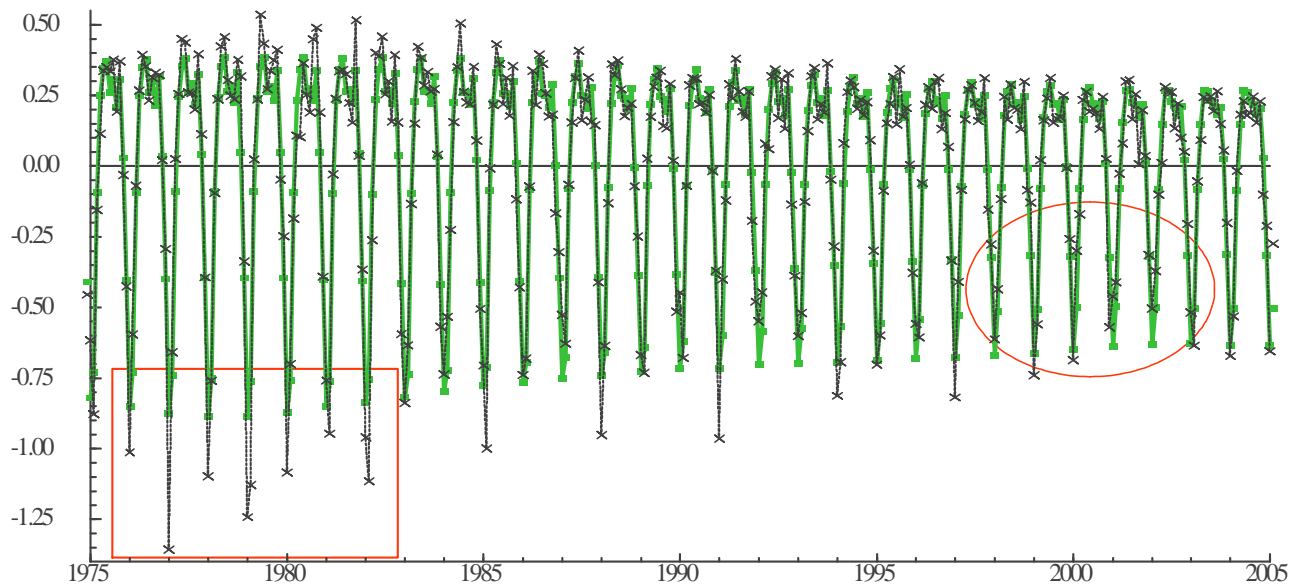


Figure 7: Estimated seasonal component shown over time (solid line, closed boxes), along with the detrended (seasonal plus irregular) series (dotted line, x's), for the homoskedastic STSM. The values are shown for 1975:1 to 2004:12.

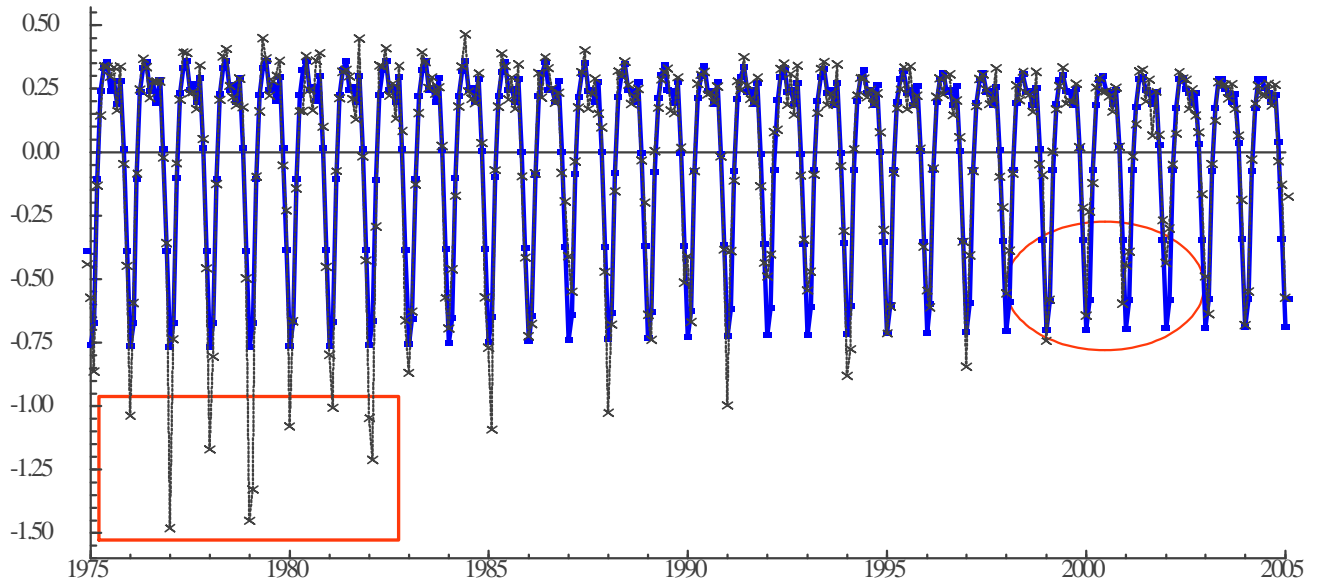


Figure 8: Estimated seasonal component shown over time (solid line, closed boxes), along with the detrended (seasonal plus total irregular) series (dotted line, x's), for the heteroskedastic STSM. The values are shown for 1975:1 to 2004:12.

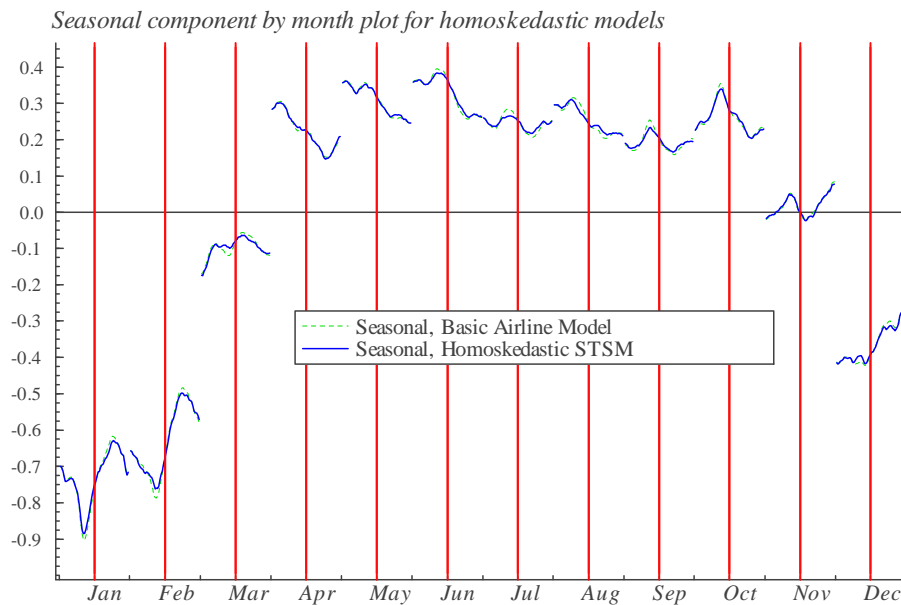


Figure 9: Estimated seasonal component shown by month for the homoskedastic STSM (dashed line, green) and the airline model (solid line, blue). The sample period is 1959:1 to 2015:12

Figure 10 shows the seasonal-by-month plot for the homoskedastic and heteroskedastic STS models. As was noted from Figure 9, for the homoskedastic model (shown in green), during January and February the winter trough first deepens until the mid 1980s portion of the sample, then moderates until around 2010, and finally expands over the last few years of the sample. For the heteroskedastic model (shown in blue) in contrast, there is a modest reduction in seasonal amplitude – mostly due to a slow, monotonic rise in the trough over the sample period – which could be explained, for instance, by advances in technology serving to facilitate winter construction. Overall, the heteroskedastic seasonal is far less responsive to bursts of seasonal noise over the sample period and roughly passes through the swings in the homoskedastic seasonal. For the airline versus airline-seasonal noise models, the seasonals show the same kind of divergence, albeit to a lesser degree than do those for the STSM's.

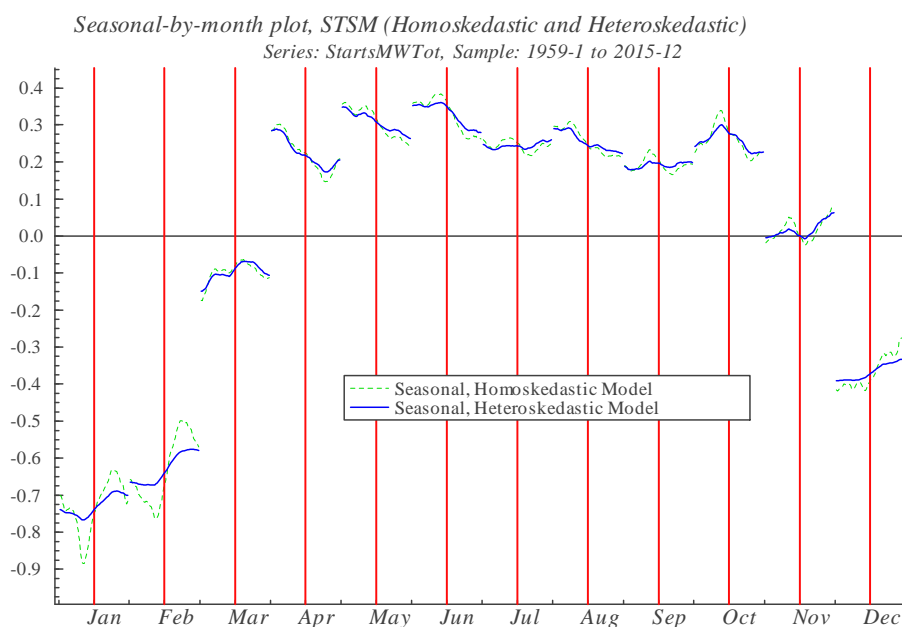


Figure 10: Estimated seasonal component shown by month for the homoskedastic (dashed line, green) and heteroskedastic (solid line, blue) STSM. The sample period is 1959:1 to 2015:12.

Figure 11 shows the seasonal-by-month plot for the airline-seasonal noise and heteroskedastic STS models. There is more difference between the two seasonals than was seen for the homoskedastic models in Figure 9, particularly in December, January, and February, where the seasonal component varies somewhat more over time for the airline-seasonal noise model.

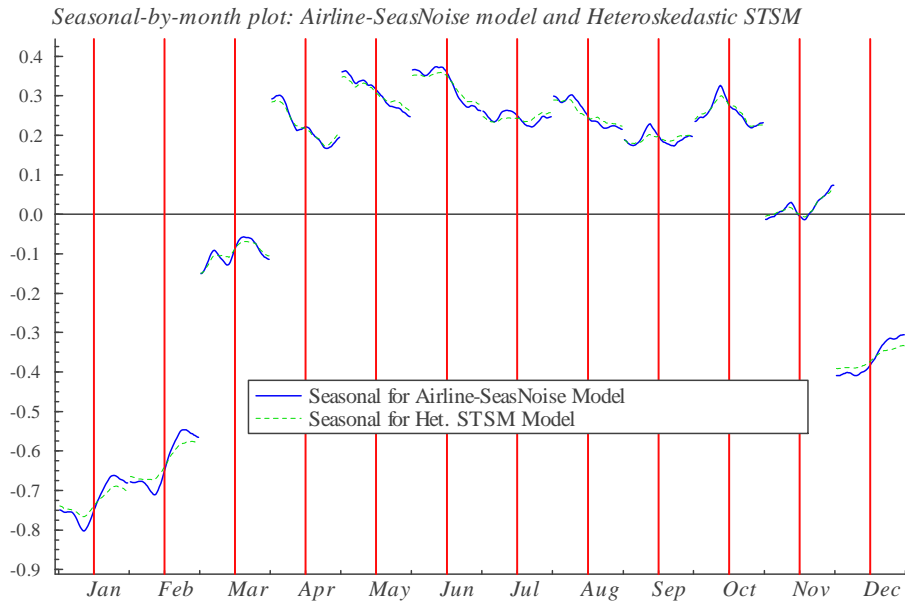


Figure 11: Estimated seasonal component shown by month for the heteroskedastic STSM (dashed line, green) and the airline-seasonal noise model (solid line, blue). The sample period is 1959:1 to 2015:12.

4.4 Comparing results for the irregular component

Figure 12 shows the differences in the estimated total irregular from explicitly modeling the seasonal noise in the STSM versus assuming the homoskedastic version of the model. This total irregular-by-month plot, analogous to the seasonal-by-month plot, separately plots the estimated total irregulars for each calendar month over successive years in the sample. For clarity here, the irregulars are shown for just the last eight years in the sample, that is, from 2008 onward. The empirical standard deviations of the estimated irregulars for each month are indicated by the solid horizontal bars (green for the homoskedastic STSM and blue for the heteroskedastic STSM) and pertain to the full sample period. The standard deviation bands for the homoskedastic case show greater variability around the winter months. This is also true for the heteroskedastic model, but there the span of this winter month variability is noticeably wider, particularly for January and February. Therefore, the heteroskedastic model is assigning more of these temporary aberrations to the irregular via the explicit allowance for seasonal heteroskedasticity. Three examples of divergent irregular estimates are as follows. First, the lowest value occurs in January 2014 and corresponds to unusually severe weather

for that month. There the estimated irregular is about 20 percentage points lower (recall that the logged-series is used) for the heteroskedastic model. As a second example, November 2013 is associated with an unusually high observation relative to the trend-seasonal baseline, and the heteroskedastic model attributes less of this movement to noise and more to trend, given that November represents a low-variance month. Lastly, the highest value of the irregular over this sub-sample occurs in December 2011, when unusually warm weather led to construction exceeding the seasonal norm; the heteroskedastic STSM assigns about 0.2 more to the irregular than does the homoskedastic STSM. This increment to the irregular is directly reflected as a smaller shock to the trend. The resulting improvement in the robustness of the trend to temporary seasonal noise that is achieved using the heteroskedastic model is clearly an attractive feature.

Seasonal noise estimates for the STSM, which are present only for the months of January, February, and December, are shown in Figure 13. While the empirical standard deviation of these estimates is the highest for January, and higher for February than December, it appears that the simple binary grouping of these three months as high variance and the others as low variance enables the model to capture a great deal of the temporary noise around winter, which fits our aim of the heteroskedastic model generalization.

4.5 Comparing results for the seasonally adjusted series

Figure 14 compares seasonal adjustments from the homoskedastic airline and airline plus seasonal noise models when only the estimated seasonal components, $\hat{\gamma}_t^{A,Hom}$ and $\hat{\gamma}_t^{A,Het}$, are removed. The two seasonally adjusted series are rather close, reflecting the generally small difference between $\hat{\gamma}_t^{A,Hom}$ and $\hat{\gamma}_t^{A,Het}$ over this sub-sample. This closeness of the seasonally adjusted series continues to hold throughout the full sample period, and a similar closeness of the seasonally adjusted series (graph not shown) also occurs for the homoskedastic and heteroskedastic STS models.

Figure 15 shows the consequences of removing seasonal noise as part of seasonal adjustment, plotting the SA series from the heteroskedastic STSM ($y_t - \hat{\gamma}_t^{Het}$) along with the seasonal-noise adjusted (SNA) series from the same model ($y_t - \hat{\gamma}_t^{Het} - \hat{\varepsilon}_t^s$). The seasonal noise preserved in

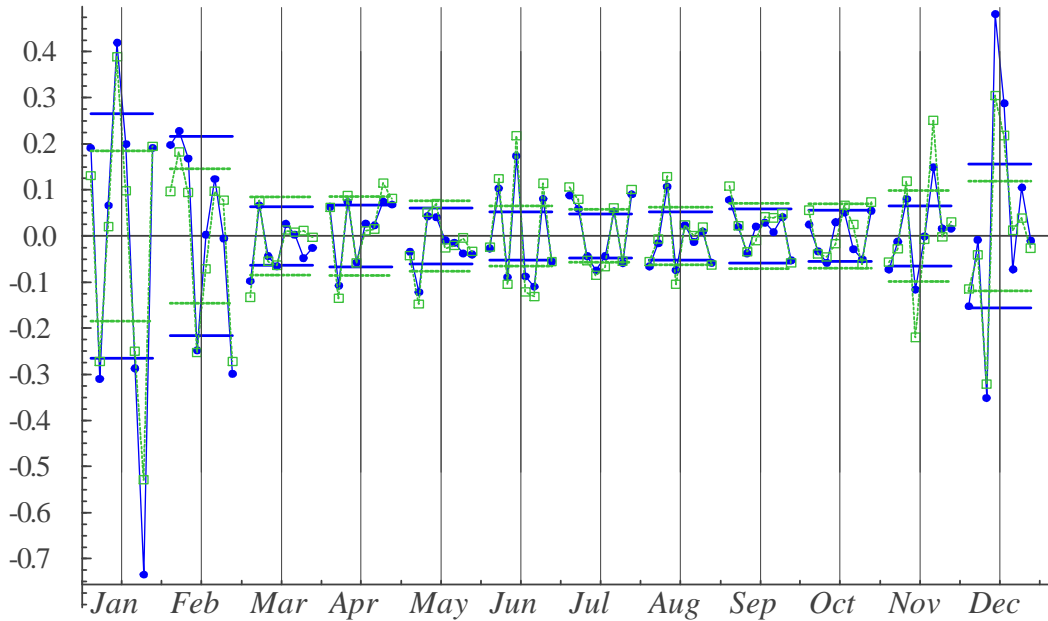


Figure 12: Estimated total irregular, shown by month for the STSM, with the homoskedastic case as open boxes and the heteroskedastic case as dots. The values are shown for the period 2008:1 to 2015:12. Horizontal lines indicating plus and minus one empirical standard deviation of the estimates from each month (computed over the full sample) are also shown for the homoskedastic (green dotted line) and heteroskedastic (blue solid line) models.

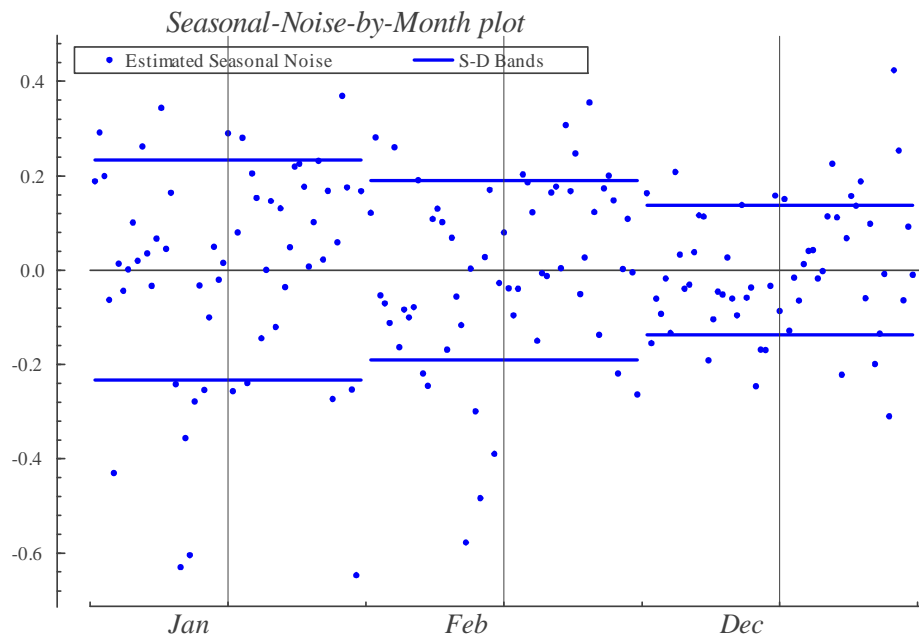


Figure 13: Estimated seasonal noise, shown by month for the heteroskedastic STSM. The sample period is 1959:1 to 2015:12. The horizontal lines indicate plus and minus one empirical standard deviation of the estimates from each month.

the SA series shows up very clearly on the graph for the months of December to February. (Differences between the SA and SNA series are difficult to discern in the other months.) The SNA series removes this noise in December to February, and includes just the baseline degree of irregular movements. Certain SA values, such as the massive dip at the beginning of 2014, could lead to increased attention on the part of policymakers, and perhaps strengthen the argument for intervention. However, once the effect of the seasonal noise is removed, it is seen that, relative to the course of events in a typical calendar year, the severity of the dip is grossly overstated by the homoskedastic model. SNA series from the heteroskedastic models effectively remove large shocks associated with seasonal noise to produce more stable trend and seasonal components and adjusted series. On the other hand, if one wants the adjusted series to show these shocks (something certainly not wanted for the trend estimates), then removal of just the seasonal component would be appropriate.

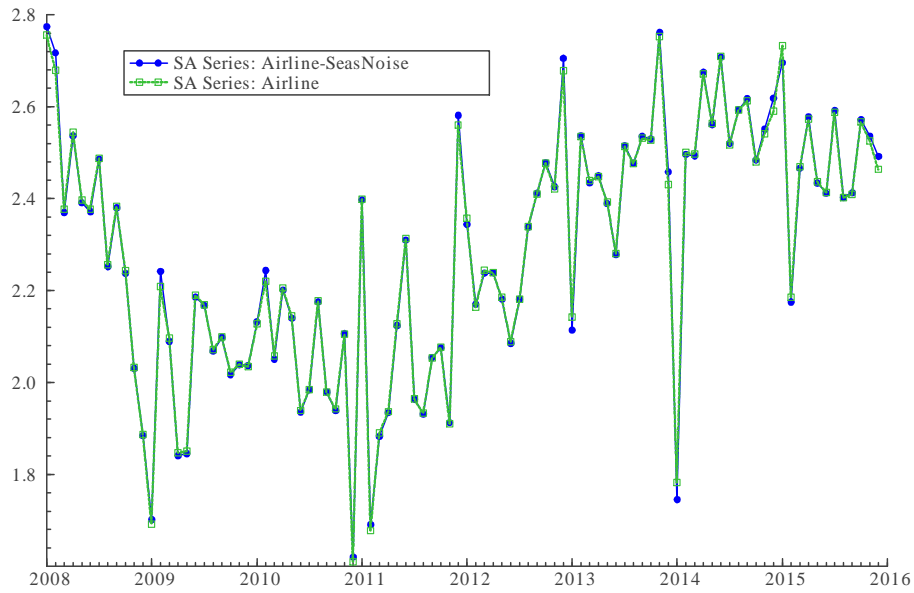


Figure 14: Estimated SA series over time shown for the airline (open boxes) and airline-seasonal noise (solid dots), along with the observations on total MW housing starts (logged). The sample period is 1959:1 to 2015:12, and the values are shown for the period 2008:1 to 2015:12.

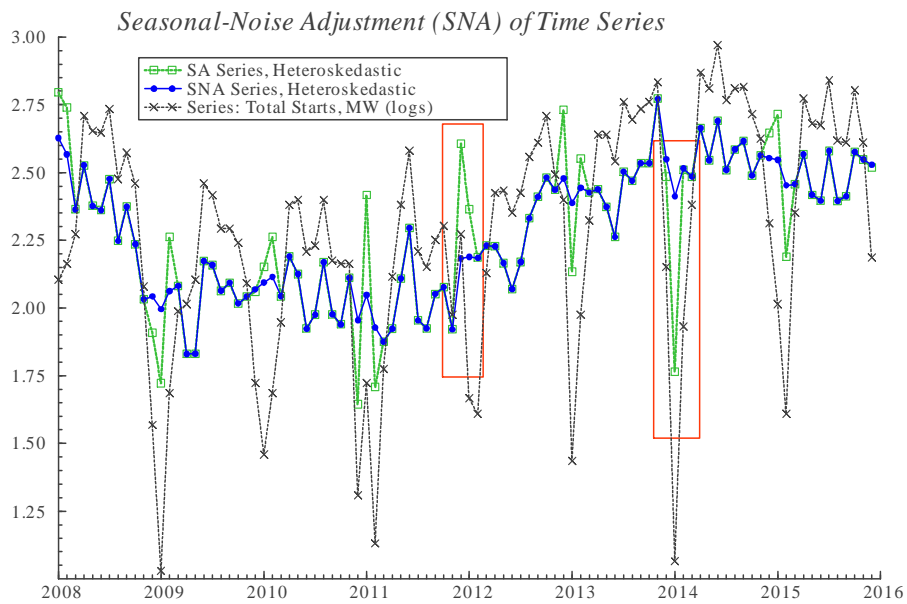


Figure 15: Estimated SA (open boxes) and SNA (solid dots) series shown for the heteroskedastic STSM, along with the observations on total MW housing starts (logged). The values are shown for the period 2008:1 to 2015:12.

5 Conclusions

In this paper, we set out the framework for signal extraction in the presence of seasonal heteroskedasticity. The presence of seasonal changes in variability may affect the essential assessment of an indicator’s signal, as illustrated here for regional housing starts, where the practical implications are easy to see in the estimated components. Our empirical results for the MW housing starts series show that models that account for seasonal heteroskedasticity produce trend estimates that are relatively unaffected by the large-magnitude realizations of seasonal noise present in winter. This contrasts with the responsiveness of trend estimates from the homoskedastic models, for which part of the increased variation in winter leaks into the trend estimates. Seasonal adjustments done in the standard way – just removing seasonality in levels – show some effects from the seasonal noise, though of much less importance. Examining filter weights for the signal extraction estimates shows that the high variance months are down-weighted by the heteroskedastic model relative to the homoskedastic model.

The empirical results also show that differences in signal extraction results due to choice of a heteroskedastic versus homoskedastic model form are far greater than are those due to choice between the two general model forms considered (structural ARIMA components models versus the airline model with the canonical decomposition). Finally, removing estimated seasonal noise along with the estimated seasonal as part of the adjustment (seasonal noise adjusted data) removes the large shocks that can be produced by seasonal noise, to produce much more stable adjusted estimates.

An alternative approach to that of this paper for dealing with periodic fluctuations in seasonal variation is to augment models with regression covariates that attempt to model this variation explicitly. Goodman (1987), Cammorata (1989), Coulson and Richard (1996), and Fergus (1999) report on various analyses relating national and regional monthly housing starts to weather variables reflecting seasonally unusual precipitation and temperature. The weather variables were constructed using weighted averages of precipitation and temperature data from weather stations spread around the regions. While results varied somewhat with the different analyses performed and the different data sets used, there was also some agreement in finding more weather related variation in the winter months for the NE and MW (formerly called “North Central”) regions. More recently, Boldin and Wright (2015) modeled effects of unusual

weather on national employment data. An interesting question not addressed in these references is whether seasonal heteroskedasticity may remain present after the regression modeling of weather effects. Though we did not explore regression modeling of weather effects here, we intend to do so in future research.

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