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Background and Perspectives for ARIMA Model-Based Seasonal Adjustment

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Abstract

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1. Introduction

Methodology and implementation details of ARIMA model-based seasonal adjustment are presented, with key features illustrated with the simplest formulas that provide concrete representations of the methd devoloped by Tiao and Hillmer (1978) and Hillmer and Tiao (1982), with important implementation contributions by Burman (1980), Gómez and Maravall (1996) and others. We use the abbreviation SA for seasonal adjustment and AMBSA for this ARIMA-Model-Based SA method. Typically this refers to the decomposition of a span of Seasonal-ARIMA-modeled time series data into component time series for the same time span, usually seasonal, trend and irregular component series, with the irregular obtained in such a way that the decomposition is "canonical". This term means that it conforms to Tiao and Hillmer's attractive way of specifying a unique decomposition by requiring the white noise irregular component to have maximal variance (possibly zero). Most commonly, the ARIMA model's differencing operator has a seasonal sum factor and a trend differencing factor. When only the latter is present, the decomposition has trend and irregular components but no seasonal component.

AMBSA is applied to time series which, in the course of obtaining the ARIMA (sometimes ARMA) model, have been preadjusted for identified effects of outliers, holidays, and other calendar effects. *The model obtained is treated as correct in all calculations.* In particular, autocovariances (and spectral densities) calculated for components from the model, after any model-specified differencing, ignore uncertainties in the ARIMA model's selection and estimation procedures. Similarly, coefficients of the component estimates and the mean square errors calculated for component estimates are treated as non-stochastic.

After preadjustment, if only the seasonal adjustment is wanted, the task and methods can be viewed as those of basic *statistical signal processing* applied to the preadjusted series in order to statistically suppress the *seasonal* "noise" to better reveal the *nonseasonal* "signal," (or the non-trend to better reveal the trend if it is the signal, etc.)

For AMBSA in such two-component decomposition situations, under weak assumptions, mean square optimal, i.e. minimum mean square error linear unobserved component estimation can be formulated as a linear regression problem with refinements to accommodate ARIMA differencing operations, as displayed in the easily programmed matrix formulas of McElroy (2008) shown in Subsection 9.3.

Calculation of these estimates and their mean square error variance matrix via the matrix formulas is an option in some AMBSA software. However, the matrix calculations are not as numerically efficient and stable as those of the two traditional, less elementary calculation methods, the "Wiener-Kolmogorov" method presented in Section 13, and the state space method presented in Durbin and Koopman (2012).

AMBSA software refers to the software currently in wide use by National Statistical Institutes and Central Banks. This includes JDemetra+, (see Eurostat 2015), TRAMO-SEATS, see Gómez and Maravall (1996) and X-13ARIMA-SEATS, see U.S. Census Bureau (2017).

AMBSA should not be a black box procedure to its users because default software procedures are sometimes seriously inadequate. Also user decisions regarding software options and model choice can strongly impact the results obtained, for better or worse. A seasonal adjuster who understands the basic facets of the method and some of its diagnostics, as outlined and then detailed in this document, will have a greater capacity to obtain successful adjustments. Maravall (2016) has alternative treatments of some of the topics we consider and treatments of further important topics relevant to the AMBSA methods described in Gómez and Maravall (1996) and adopted in the software of the U.S. Bureau of the Census and Eurostat.

The reader is assumed to be familiar with ARIMA time series models. For an understanding of the nature of the component estimates, only a basic background in linear regression sufficient for the review of regression in Section 4 is needed. The regression formulas are first illustratively applied in Section 6 to a stationary case, to obtain the canonical decomposition of a span of data from an $AR(1)_r$, a first-order seasonal autoregressive model with seasonal period r (r observations per year). In this case, they yield simple revealing time-varying filter formulas for the signal and noise component estimates and for their error variance matrix.

Thereafter technical background for nonstationary ARIMA data with multiple unobserved components is developed, moving step by step through the simplest formulas that provide concrete representations of key features of AMBSA.

The reader might start by perusing the Sections whose titles have a *, and later reading for details as their content becomes more directly relevant.

2. Conceptual Overview* (* indicates especially fundamental material.)

Estimating Two-Component Decompositions. Two-component decompositions can illustrate the main concepts of AMBSA. We begin with a span $Z_1, \ldots, Z_n$ of n stationary zero-mean data, in vector form $Z = (Z_1, \ldots, Z_n)'$, that has an ARMA model and consequently an autocovariance matrix $\Sigma_{ZZ} = EZZ'$ that is positive definite[1].

[1] See Wikipedia Contributors (2017b)

Suppose Z_t is considered to be the sum of two unobserved, mutually uncorrelated, stationary component series.

$$Z_t = S_t + N_t. \tag{2.1}$$

Thus there is an autocovariance decomposition,

$$\Sigma_{ZZ} = \Sigma_{SS} + \Sigma_{NN}. \tag{2.2}$$

In the cases we consider, the unobserved components can be usually be estimated with the aid of appropriate properties specified or inferred for Σ_{SS} and Σ_{NN}, as we first illustrated simply with (2.4).

Henceforth, I denotes the identity matrix of order n. From (2.2), the linear regression formulas of Section 4 provide the $n \times n$ coefficient matrices $\beta_S = [\beta_S (j,k)]$, $1 \leq j,k \leq n$ and $\beta_N = I - \beta_S$ of the *minimum mean square error* (MMSE) linear estimates, $\beta_S Z = \hat{S} = \left(\hat{S}_1, \ldots, \hat{S}_n\right)'$ of $S = (S_1, \ldots, S_n)'$, and $\beta_N Z = \hat{N} = I - \hat{S} = \left(\hat{N}_1, \ldots, \hat{N}_n\right)'$ of $N = (N_1, \ldots, N_n)'$ in (2.3). The j-th row of β_S shows the data coefficients of the linear estimate $\hat{S}_j = \Sigma_{k=1}^n \beta_S (j,k) Z_k$ of the decomposition of Z_j and correspondingly for $\hat{N}_j$ in

$$Z = \hat{S} + \hat{N}. \tag{2.3}$$

With a two-component decomposition, the component of greater interest can be labeled *signal* S_t and the other labeled *noise* N_t. The simplest case is that of *white noise*, uncorrelated and constant variance N_t, resulting in $\Sigma_{NN} = \sigma^2 I$. A specification of $\sigma^2 > 0$ small enough that $\Sigma_{SS} = \Sigma_{ZZ} - \sigma^2 I$ is a covariance matrix, i.e. positive semi-definite (all eigenvalues nonnegative, one or more positive) will provide a decomposition (2.2) that yields estimates (2.3).

$$\Sigma_{ZZ} = \Sigma_{SS} + \sigma^2 I. \tag{2.4}$$

A Limited Elementary Stationary-Case Specification of σ^2. Here is a possible specification of σ^2 in (2.4) that only uses standard matrix concepts: Specify σ^2 as the maximal white noise variance compatible with (2.4). From $\Sigma_{SS} = \Sigma_{ZZ} - \sigma^2 I$ one sees that this σ^2 is the smallest eigenvalue $\alpha_{\min}$ of Σ_{ZZ}. It is positive because every ARMA covariance matrix Σ_{ZZ} is positive definite[2]. The resulting $\hat{N}$ and $\hat{S} = Z - \hat{N}$ from the regression formulas (4.2) of Section 4 are the MMSE linear estimates of the white noise $\hat{N}$ and the signal $\hat{S}$ for the specified decomposition (2.4). (The estimated $\hat{N}_t$ are not white noise, see (4.5)).

A choice of σ^2 different from (smaller than) $\alpha_{\min}$ produces a different decomposition (2.4) with different estimated components. So the choice $\alpha_{\min}$ is just one of many possibilities for stationary data.

[2] Every positive definite matrix has such a $\sigma^2 = \alpha_{\min}$ decomposition. No model connection is required.

For ARMA Z_t, AMBSA software uses the less elementary σ^2 specification of Hillmer and Tiao (1982) described in the next paragraph. Differently from $\sigma^2 = \alpha_{\min}$, its definition of σ^2 admits a generalization for ARIMA Z_t, the usual case for AMBSA. So it provides some useful conceptual consistency between stationary and nonstationary cases. See Section 7.

The Canonical Two-Component Stationary-Case Decomposition. The generalizable definition of σ^2 requires the ARMA model for Z_t to be invertible, which is the typical case in practice. This means that the *spectral density* $g_Z(\lambda)$, defined by (5.13) and abbreviated sd (or Sd, plural sds or Sds) has a positive minimum value, $\min_\lambda g_Z(\lambda) > 0$. For ARMA Z_t, the *canonical decomposition* (Tiao and Hillmer, 1978) specifies $\sigma^2 = \min_\lambda g_Z(\lambda)$ in (2.4).

Sd functions are autocovariance generating functions, see (5.5). The matrix decomposition **(2.4)** is generated for all series lengths n by the sd decomposition $g_Z(\lambda) = g_S(\lambda) + \sigma^2 = g_S(\lambda) + g_N(\lambda)$ with $g_S(\lambda) = g_Z(\lambda) - \sigma^2$. Invertibility of the ARMA model guarantees that $g_S(\lambda) = g_Z(\lambda) - \sigma^2 \geq 0$. It follows that $g_S(\lambda)$ specifies a stationary component S_t having a non-invertible ARMA model. such that the variance σ^2 of the white noise component N_t is maximal. Such maximality is the defining property of the *canonical decomposition*, also when there are several non-white-noise components.

Nonstationary Case. For invertible ARIMA Z_t, σ^2 is specified as the minimum of the *pseudo-spectral density* (p-sd) of the model, defined by (7.1). The trend plus irregular decomposition (8.6) of Subsection 8.1, with trend component denoted p_t, illustrates this.

The sd and p-sd formulas (5.13) and (7.1) are sources of the great versatility of AMBSA. They reflect the model structure in ways that facilitate the specification of decompositions with appropriate components, see Section 8 and Subsection 21.

If decomposition calculations provide an sd or a p-sd for each component, and a nonnegative value for the constant calculated to be the maximal variance, then the decomposition is *admissible* (or *acceptable*). If this constant is negative, the decomposition is *nonadmissible*. Section 8 provides two examples of fundamental ARIMA models with admissible decompositions for all model parameter values and one ARIMA model whose decomposition is admissible only for a subinterval of model coefficient values.

3. Fundamental Examples*

Section 10 offers graphs of central and concurrent (latest observation time) MMSE filters from a standard seasonal model, and also graphs of the associated time-varying error variances of their AMBSA seasonal adjustments. Subsections 11.2 and 12.1 illustrate how error variances and covariances can be used to obtain probability intervals for an estimated growth rate and also for its revised value that will be obtained by reestimating with additional later data.

Sections 10–12 illustrate, in various ways, how greater AMBSA smoothing is associated with greater instability of the AMBSA estimates, a theme relevant for Section 17.

The seasonal random walk is the nonstationary analogue of the seasonal $\mathrm{AR}(1)_r$. Its $r=2$ biannual model is the only nonstationary model for which we show exact formulas for seasonal, trend and irregular filters, symmetric and asymmetric. The symmetric filters of this model are derived in Section 15. The asymmetric filters are obtained using MMSE forecasts and backcasts of data required by the symmetric filters but not available.

Reflecting the complexity of most economic indicators that are seasonally adjusted, the trend component has often been called the *trend-cycle component* and regarded as consisting of a long-term trend component plus one or more higher frequency cyclical components. An approach to providing such a decomposition with one cycle component, available in some AMBSA software, is described in Subsection 21.2.

4. Linear Regression Applied for Signal Extraction*

As in Section 2, we start from zero mean data $Z_1,\ldots,Z_n$ whose $n\times n$ autocovariance matrix Σ_{ZZ}, is positive definite and focus on two-unobserved-component decompositions $Z_t=S_t+N_t$ with uncorrelated components, $ES_jN_k=0,\ j,k=1,\ldots n$. With $Z=(Z_1,\ldots,Z_n)'$, the task is to use a specified autocovariance decomposition (2.2) to obtain $n\times n$ coefficient matrices β_S, β_N of linear estimates $\hat{S}=\beta_S Z,\ \hat{N}=\beta_N Z$ of a decomposition (2.3) that have have minimum mean square error. To accomplish this, setting $Z=(Z_1,\ldots,Z_n)'$ and $S=(S_1,\ldots,S_n)'$, we seek the $n\times n$ coefficient matrix β_S such that the error $e=\hat{S}-S$ of the linear estimate $\hat{S}=\beta_S Z$ is uncorrelated with Z, $Ee'Z=0$, which then also holds for the error $\hat{N}-N=-e$. This property characterizes linear estimates βZ whose mean square error $Ee'e=\Sigma_{j=1}^n e_j^2$ is minimal, as Section 4.1 of Whittle (1963) shows[3]. Other MMSE characterizations include: For any positive definite $n\times n$ matrix Q, the MMSE error e_t minimizes $\Sigma_{j,k=1,\ldots,n}\ E\{e_jQ_{j,k}e_k\}$. With Gaussian Z, the MMSE $\hat{S}$ and $\hat{N}$ are the conditional expectations of S and N given Z.

Let 0_n denote the zero matrix of order n. Because $\Sigma_{SN}=ESN'=0_n$ yields $\Sigma_{SZ}=\Sigma_{SS}$, we have, with $\Longleftrightarrow$ denoting equivalence,

$$E\left(S-\beta_S Z\right)Z'=0_n \Longleftrightarrow \Sigma_{SZ}-\beta_S\Sigma_{ZZ}=0_n \Longleftrightarrow \Sigma_{SS}=\beta_S\Sigma_{ZZ}, \tag{4.1}$$

resulting in $\hat{S}=\beta_S Z$ with $\beta_S=\Sigma_{SS}\Sigma_{ZZ}^{-1}$. Analogously, $\hat{N}=\beta_N Z$ with $\beta_N=\Sigma_{NN}\Sigma_{ZZ}^{-1}$, whence $\beta_S+\beta_N=I$.

In summary,

$$\hat{S}=\beta_S Z,\quad \beta_S=\Sigma_{SS}\Sigma_{ZZ}^{-1},\quad \hat{N}=\beta_N Z,\ \beta_N=\Sigma_{NN}\Sigma_{ZZ}^{-1},\quad \beta_S+\beta_N=I. \tag{4.2}$$

It follows that covariance matrices of the estimates have the formulas

$$\Sigma_{\hat{S}\hat{S}}=\Sigma_{SS}\Sigma_{ZZ}^{-1}\Sigma_{SS},\ \Sigma_{\hat{N}\hat{N}}=\Sigma_{NN}\Sigma_{ZZ}^{-1}\Sigma_{NN}. \tag{4.3}$$

[3] Wikipedia Contributors (2013) derives and applies the analogous MMSE estimate identifying property for simpler non-time series contexts.

The last formula in (4.2) shows that the estimates provide a decomposition (2.3). For a specified decomposition (2.2), the estimate $\hat{S} = Z - \hat{N}$ can be regarded as an optimally "de-noised" version of the data for revealing the signal S.

For $1 \leq t \leq n$, the t-th row of β_S provides the *coefficients* of the MMSE linear estimate $\hat{S}_t = \Sigma_{j=1}^{n} \beta_{S,t,j} Z_j$ and correspondingly for β_N and $\hat{N}_t$, see examples in Subsection 6.1.

From (2.1) and (2.3), the estimation error $e = S - \hat{S}$ is equal to $\hat{N} - N$, so both estimates have the same *error variance matrix*,

$$\begin{aligned} \Sigma_{ee} &= E(S-\hat{S})(S-\hat{S})' = E(N-\hat{N})(N-\hat{N})' \\ &= \Sigma_{NN}\Sigma_{ZZ}^{-1}\Sigma_{SS} = \Sigma_{SS}\Sigma_{ZZ}^{-1}\Sigma_{NN} = \hat{\Sigma}_{\hat{N}\hat{S}} = \Sigma_{\hat{S}\hat{N}}. \end{aligned} \quad (4.4)$$

Change of Scale Results. It follows from the preceding formulas that if Z and its components are multiplied by scalar $\alpha \neq 0$, then Σ_{ee} and other covariance matrices become multiplied by α^2 but the filter coefficient vectors β_S and β_N are unchanged, they are *scale invariant*.

4.1. Basic Examples of Covariance Properties Not Inherited by Estimates

The final formulas for Σ_{ee} show that, whereas $\Sigma_{SN} = \Sigma_{NS} = 0_n$, the estimates are cross-correlated, $\Sigma_{\hat{S}\hat{N}} = \Sigma_{\hat{N}\hat{S}} = \Sigma_{ee}$, a positive definite matrix if both Σ_{SS} and Σ_{NN} have this property, otherwise positive semidefinite. This generalizes to AMBSA. Estimates of uncorrelated components are cross-correlated because all are linear functions of the data Z (after differencing to stationarity in nonstationary cases). Also, a component estimate has covariance properties different from the component. Most basically, for a white noise component N of non-white-noise Z, the estimate $\hat{N}$, is not white noise. From (4.3),

$$\Sigma_{\hat{N}\hat{N}} = \sigma^4 \Sigma_{ZZ}^{-1} = \sigma^4 \left(\Sigma_{SS} + \sigma^2 I\right)^{-1}. \quad (4.5)$$

5. Spectral Densities of Stationary Series*

Optional Review of Complex Numbers. Spectral densities can be defined without using complex numbers as we show, but then formulas and important seasonal decomposition calculations lose simplicity. We use standard notation, $z = a + ib$ with a and b real and $i^2 = -1$. The number a is the real part of z, $a = \operatorname{Re}(z)$, and b the is imaginary part, $b = \operatorname{Im}(z)$. $\bar{z} = a - ib$ is the complex conjugate of z. Its properties are $z + \bar{z} = 2\operatorname{Re}(z)$, $z - \bar{z} = 2i\operatorname{Im}(z)$ and $\sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$, which is the magnitude of z, denoted $|z|$ (the distance from (a, b) to $(0, 0)$ in the coordinate plane). Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ for real θ shows that $e^{-i\theta}$ is the complex conjugate of $e^{i\theta}$ and that $\left|e^{i\theta}\right|^2 = \cos^2\theta + \sin^2\theta = 1$. The *polar* representation of z is $|z|\, e^{i\theta}$, with phase θ. Especially relevant are calculations like

$$\left|1 \pm \theta e^{i\lambda}\right|^2 = \left(1 \pm \theta e^{i\lambda}\right)\left(1 \pm \theta e^{-i\lambda}\right) = 1 + \theta^2 \ \pm \theta\left(e^{i2\pi\lambda} + e^{-i2\pi\lambda}\right) = 1 + \theta^2 \pm 2\theta\cos\lambda. \quad (5.1)$$

For more information, see Wikipedia Contributors (2012).

Notational Convention: Hereafter, w_t denotes a covariance stationary series, sometimes ARMA, possibly the stationary transform $w_t = \delta(B) Z_t$ of an ARIMA Z_t with differencing operator $\delta(B)$. When Z_t is stationary, then $w_t = Z_t$.

With $\gamma_j = E w_t w_{t-j}, j = 0, \pm 1, \ldots$, the *spectral density* of w_t is the function defined for $-1/2 \leq \lambda \leq 1/2$ by

$$g_w(\lambda) = \sum_{j=-\infty}^{\infty} \gamma_j e^{i2\pi j\lambda}. \tag{5.2}$$

Because $\gamma_{-j} = \gamma_j$,

$$g_w(\lambda) = \gamma_0 + \sum_{j=1}^{\infty} \gamma_j \left(e^{i2\pi j\lambda} + e^{-i2\pi j\lambda}\right), \tag{5.3}$$

$$= \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j \cos 2\pi j\lambda. \tag{5.4}$$

$g_w(\lambda)$ is also called the *autocovariance generating function* of w_t due to

$$\gamma_j = \int_{-1/2}^{1/2} e^{-i2\pi j\lambda} g_w(\lambda)\, d\lambda = 2\int_{0}^{1/2} \cos 2\pi j\lambda g_w(\lambda), \quad j = 0, \pm 1, \ldots . \tag{5.5}$$

Thus $g_w(\lambda)$ is a frequency domain re-expression of the autocovariance properties of w_t. It is an even function, $g_w(-\lambda) = g_w(\lambda), -1/2 \leq \lambda \leq 1/2$, which is nonnegative, a property expressed in the ARMA spectral density formula (5.13). Any integrable function with these properties is the spectral density of a stationary time series, see Brockwell and Davis (1991).

White noise, $w_t = a_t$, with $\sigma_a^2 = E a_t^2$ has the simplest sd. From (5.3), its spectral density is a constant,

$$g_a(\lambda) = \sigma_a^2, \ -1/2 \leq \lambda \leq 1/2. \tag{5.6}$$

Conversely, if (5.6) holds, it follows from (5.5) that the autocovariances of a_t are zero at nonzero lags, i.e. a_t is white noise. To illustrate (5.2), an MA(1), $w_t = (1 - \theta B) a_t$, has autocovariances $\gamma_0 = \sigma_a^2\left(1 + \theta^2\right)$, $\gamma_{\pm 1} = -\sigma_a^2 \theta$ and $\gamma_j = 0$ for $|j| \geq 2$, so from (5.3) and (5.1),

$$g_w(\lambda) = \sigma_a^2\left(1 + \theta^2\right) - \sigma_a^2 \theta \left(e^{i2\pi j\lambda} + e^{-i2\pi j\lambda}\right) = \sigma_a^2 \left|1 - \theta e^{i2\pi\lambda}\right|^2 . \tag{5.7}$$

The final formula in (5.7) is an instance of the general ARMA sd formula (5.13).

5.1. Transfer Functions and the ARMA Spectral Density Formula*

5.1.1. ARMA Conventions

For a stationary ARMA(p,q) series w_t, by definition

$$\phi(B) w_t = \theta(B) a_t, \tag{5.8}$$

with white noise a_t. Unlike some AMBSA software, we use the sign convention of Box and Jenkins (1976), with the basic ARMA(1,1) expressed as $(1-\phi_1 B)\, w_t = (1-\theta_1 B)\, a_t$. Stationarity requires $\phi(z) = 1 - \phi_1 z + \cdots - \phi_p z^p$ to satisfy

$$\phi(z) \neq 0, \ |z| \leq 1, \tag{5.9}$$

($|\phi_1| < 1$ if $p = 1$). With no loss of generality under Gaussian assumptions, $\theta(z)$ is assumed to satisfy

$$\theta(z) \neq 0, \ |z| < 1, \tag{5.10}$$

a condition that is always imposed by the ARIMA coefficient estimation routines of AMBSA software. If also $\theta(z) \neq 0$ whenever $|z| = 1$, then w_t and its model are said to be *invertible*. Both are *noninvertible* if $\theta(z) = 0$ for a z with $|z| = 1$.

When w_t is a *seasonal* ARMA process, $\phi(B)\,\Phi(B^r)\, w_t = \theta(B)\,\Theta(B^r)\, a_t$ with seasonal period $r \geq 2$, then (5.9) and (5.10) apply to the *total* AR and MA polynomials[4] $\varphi(z) = \phi(z)\,\Phi(z^r)$ and $\vartheta(z) = \theta(z)\,\Theta(z^r)$.

For simplicity, we usually use this total notation for all ARMA and ARIMA models,

$$\varphi(B)\, Z_t = \vartheta(B)\, a_t, \tag{5.11}$$

seasonal or nonseasonal, referring to $\vartheta(z)$ as the MA polynomial and $\varphi(z)$ as the AR polynomial, which can include the differencing polynomial in the nonstationary case.

The formula (5.13) of the spectral density of such a w_t follows from (5.8) via a fundamental fact: When a stationary series y_t is the output of a *linear filter* $\beta(B) = \sum_j \beta_j B^j$, i.e. $y_t = \sum_j \beta_j x_{t-j}$ for some stationary x_t, then the spectral densities of the input series x_t and the output series y_t are related by

$$g_y(\lambda) = \left|\beta\left(e^{i2\pi\lambda}\right)\right|^2 g_x(\lambda), \tag{5.12}$$

see Theorems 4.4.1 and 4.10.1 of Brockwell and Davis (1991). The function $\beta\left(e^{-i2\pi\lambda}\right) = \sum_j \beta_j e^{-i2\pi j\lambda}$ is the *transfer function* of the filter and $\left|\beta\left(e^{i2\pi\lambda}\right)\right|^2$ is the *squared gain* of the filter. The filter is *symmetric*, $\beta_j = \beta_{-j}$ for $j \neq 0$, when $\beta\left(e^{-i2\pi\lambda}\right) = \beta\left(e^{i2\pi\lambda}\right)$ for all λ.

5.1.2. The Spectral Density Formula

Using (5.12), it follows from (5.6) and (5.8) that $\left|\varphi\left(e^{i2\pi\lambda}\right)\right|^2 g_w(\lambda) = \left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2 \sigma_a^2$ and therefore that an ARMA sd has the form

$$g_w(\lambda) = \sigma_a^2 \frac{\left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2}{\left|\varphi\left(e^{i2\pi\lambda}\right)\right|^2}, \quad -1/2 \leq \lambda \leq 1/2. \tag{5.13}$$

[4] ϑ is script θ "theta" and φ is script ϕ "phi"

Thus an ARMA(1,1) w_t has sd

$$w(\lambda) = \sigma_a^2 \frac{\left|1-\theta e^{i\lambda}\right|^2}{\left|1-\phi e^{i\lambda}\right|}.$$

Invertibility is equivalent to the sd having a positive minimum, $\sigma^2 = \min_\lambda g_w(\lambda) > 0$.

5.2. Spectral Density Sums and Uncorrelated Decompositions*

If stationary times series x_t and $\tilde{x}_t$ are uncorrelated, then

$$E(x_t + \tilde{x}_t)(x_{t-j} + \tilde{x}_{t-j}) = Ex_t x_{t-j} + E\tilde{x}_t \tilde{x}_{t-j} \quad j = 0, \pm 1, \dots \ .$$

It follows that the sd of the sum series $w_t = x_t + \tilde{x}_t$ is the sum of the component sds,

$$g_w(\lambda) = g_x(\lambda) + g_{\tilde{x}}(\lambda). \tag{5.14}$$

Conversely, if the spectral density $g_w(\lambda)$ of a stationary series is found to have a decomposition (5.14), then as regards its autocovariance properties, one one can treat w_t as admitting a decomposition $w_t = x_t + \tilde{x}_t$ with uncorrelated components having spectral densities $g_x(\lambda)$ and $g_{\tilde{x}}(\lambda)$. (The possible correlated decompositions yielding (5.14) lack practical value, see Findley (2012).)

As a fundamental example of (5.14), in the invertible case, $\sigma^2 = \min_\lambda g_w(\lambda) > 0$, then with $g_s(\lambda) = g_w(\lambda) - \sigma^2$, the decomposition

$$g_w(\lambda) = \left\{g_w(\lambda) - \sigma^2\right\} + \sigma^2 \equiv g_S(\lambda) + g_N(\lambda) \tag{5.15}$$

specifies a canonical two-component decomposition, $w_t = S_t + N_t$, with white noise N_t.

5.2.1. The Canonical Sd Decomposition of an Invertible MA(1)

For an MA(1) w_t with $|\theta| < 1$, from (5.1), $\sigma^2 = \min_\lambda \sigma_a^2\left(1+\theta^2 - 2\theta\cos\lambda\right) = \sigma_a^2\left(1+\theta^2 - 2|\theta|\right)$. Therefore

$$g_S(\lambda) = g_w(\lambda) - \sigma^2 = \sigma_a^2\left(2|\theta| - 2\theta\cos\lambda\right).$$

6. Canonical Decomposition of a First-Order Seasonal Autoregression*

The time-varying filters and MMSE errors for (5.15) can given in detail for data from the first-order seasonal autoregressive model AR$(1)_r$, with seasonal period $r \geq 2$,

$$w_t = \Phi w_{t-r} + a_t, -1 < \Phi < 1. \tag{6.1}$$

From Box and Jenkins (1976, p. 329),

$$\gamma_j = Ew_{t+j}w_t = \sigma_a^2 \begin{cases} \left(1-\Phi^2\right)^{-1}\Phi^k, & |j| = kr, \quad k = 0, 1, \ldots \\ 0, & \text{otherwise.} \end{cases} . \tag{6.2}$$

From (5.13) and (5.15), we obtain $g_w(\lambda)$ and its canonical decomposition components:

$$g_w(\lambda) = \sigma_a^2 \left|1 - \Phi e^{i2\pi r\lambda}\right|^{-2}, \tag{6.3}$$

$$g_N(\lambda) = \sigma^2 = \min_{\lambda} g_w(\lambda) = g_w(0) = \sigma_a^2 \left(1 + |\Phi|\right)^{-2}. \tag{6.4}$$

Formula (19) of Findley, Lytras and Maravall (2015) shows that for $\Phi > 0$, $g_S(\lambda) = g_w(\lambda) - g_N(\lambda)$ can be expressed as

$$g_S(\lambda) = \sigma_a^2 \Phi \left(1+\Phi\right)^{-2} \frac{\left|1 + e^{i2\pi\lambda}\right|^2}{\left|1 - \Phi e^{i2\pi\lambda}\right|^2}.$$

For simplicity, we only consider $\Phi > 0$. Then the minimum in (6.4) occurs at the frequencies in $-1/2 \leq \lambda \leq 1/2$ where $\cos 2\pi r\lambda = -1$, such as $\lambda = \pm (2r)^{-1}$.

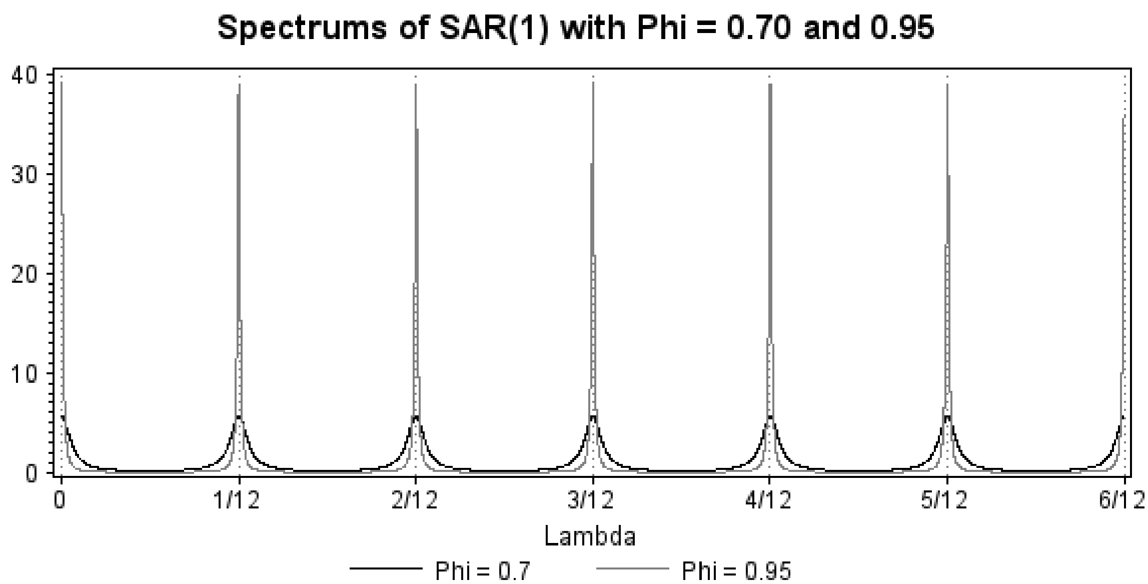

Figure 1: Two $r = 12$ SAR(1) sds for $0 \leq \lambda \leq 1/2$, for $\Phi = 0.70$ (darker line) and $\Phi = 0.95$, with $\sigma_a^2 = 1 - \Phi^2$ to have $\gamma_0 = 1$, hence area 1/2 below each graph.

The peaks in Figure 1 are at $\lambda = 0$ and at each seasonal frequency, $k/12$ cycles per year, $1 \leq k \leq 6$, always with amplitude $\sigma_a^2 (1-\Phi)^{-2} = (1+\Phi)(1-\Phi)^{-1}$. The peaks for $\Phi = 0.70$

are broader and much lower than those for $\Phi = 0.95$. The minimum value $(1-\Phi)(1+\Phi)^{-1}$ occurs midway between each pair of peaks.

The canonical sd decomposition $g_w(\lambda) = \left(g_w(\lambda) - \sigma_a^2 (1+\Phi)^{-2}\right) + \sigma_a^2 (1+\Phi)^{-2}$ identifies the matrix decomposition

$$\Sigma_{ww} = \left(\Sigma_{ww} - \sigma_a^2 (1+\Phi)^{-2} I\right) + \sigma_a^2 (1+\Phi)^{-2} I \equiv \Sigma_{SS} + \Sigma_{NN}. \tag{6.5}$$

Substitution from (6.2) into the regression formulas (4.2) yields the estimates $\hat{S}_t$ and $\hat{N}_t$ of the canonical decomposition, as we illustrate.

For the seasonal AR$(1)_r$ model, the entries of the inverse matrix Σ_{ww}^{-1} have known, relatively simple formulas, see Wise (1955) and Zinde-Walsh (1988). For example, when $r = 2$, $n = 7$,

$$\Sigma_{ww}^{-1} = \sigma_a^{-2} \begin{bmatrix} 1 & 0 & -\Phi & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\Phi & 0 & 0 & 0 \\ -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 & 0 \\ 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 \\ 0 & 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi \\ 0 & 0 & 0 & -\Phi & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\Phi & 0 & 1 \end{bmatrix}. \tag{6.6}$$

For $r \geq 2$ and all $n \geq 2r+1$, as (6.6) indicates, Σ_{ww}^{-1} has a tridiagonal symmetric form, with nonzero values only on the main diagonal and the r-th diagonals above and below. The sub- and superdiagonals have the entries $-\Phi\sigma_a^{-2}$. The first and last r entries of the main diagonal are σ_a^{-2} and the rest are $\sigma_a^{-2}\left(1+\Phi^2\right)$.

For $\beta_N = \Sigma_{NN}\Sigma_{ww}^{-1} = \sigma_N^2 \Sigma_{ww}^{-1} = (1+\Phi)^{-2}\sigma_a^2 \Sigma_{ww}^{-1}$, one has, when $r = 2$, $n = 7$,

$$\beta_N = (1+\Phi)^{-2} \begin{bmatrix} 1 & 0 & -\Phi & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\Phi & 0 & 0 & 0 \\ -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 & 0 \\ 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 \\ 0 & 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi \\ 0 & 0 & 0 & -\Phi & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\Phi & 0 & 1 \end{bmatrix}, \tag{6.7}$$

Further, from $\beta_S = I - \beta_N$,

$$\beta_S = \Phi \left(1+\Phi\right)^{-2} \begin{bmatrix} (2+\Phi) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & (2+\Phi) & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & (2+\Phi) & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & (2+\Phi) \end{bmatrix}. \tag{6.8}$$

6.1. Signal and Noise Filters of the Initial, Intermediate, and Final Years

For general $r \geq 2$ and $n \geq 2r+1$, the Σ_{ww}^{-1} formula of Wise (1955) yields the filter formulas for $\hat{N}$ and $\hat{S} = w - \hat{N}$ shown in (6.9)–(6.13) and (6.14)–(6.16). For the *intermediate times* $r+1 \leq t \leq n-r$, the noise component estimate $\hat{N}_t$ is given by a symmetric filter (6.9) with equal negative initial and final coefficients smaller in magnitude than the positive central coefficient.

$$\hat{N}_t = \frac{1}{\left(1+\Phi\right)^2} \left(-\Phi w_{t-r} + \left(1+\Phi^2\right) w_t - \Phi w_{t+r}\right). \tag{6.9}$$

The filters for the initial and final years are asymmetric. For the *initial* year $1 \leq t \leq r$,

$$\hat{N}_t = \frac{1}{\left(1+\Phi\right)^2} \left(w_t - \Phi w_{t+r}\right) \tag{6.10}$$

$$= \frac{1}{\left(1+\Phi\right)^2} \left(-\Phi \left\{\Phi w_t\right\} + \left(1+\Phi^2\right) w_t - \Phi w_{t+r}\right). \tag{6.11}$$

The filter for the *final year* $n-r+1 \leq t \leq n$ is the time-reverse of the initial year filter,

$$\hat{N}_t = \left(1+\Phi\right)^{-2} \left(-\Phi w_{t-r} + w_t\right) \tag{6.12}$$

$$= \frac{1}{\left(1+\Phi\right)^2} \left(-\Phi w_{t-r} + \left(1+\Phi^2\right) w_t - \Phi \left\{\Phi w_t\right\}\right). \tag{6.13}$$

In comparison with (6.9), the value $\{\Phi w_t\}$ in the re-expression (6.11) appears as the MMSE $\mathrm{AR}(1)_r$ *backcast* of the missing w_{t-r} and in (6.13) as the MMSE $\mathrm{AR}(1)_r$ *forecast* of the missing w_{t+r}.

Also for $\hat{S}_t$ at intermediate times $r+1 \leq t \leq n-r$ the filter formula is symmetric,

$$\hat{S}_t = \frac{\Phi}{\left(1+\Phi\right)^2} \left(w_{t-r} + 2w_t + w_{t+r}\right) = \frac{4\Phi}{\left(1+\Phi\right)^2} \left(\frac{1}{4} w_{t-r} + \frac{1}{2} w_t + \frac{1}{4} w_{t+r}\right). \tag{6.14}$$

As with $\hat{N}_t$, for the initial and final years, the $\hat{S}_t$ filters are asymmetric. For $1 \leq t \leq r$,

$$\hat{S}_t = \frac{\Phi}{\left(1+\Phi\right)^2} \left(\left(\Phi+2\right) w_t + w_{t+r}\right) = \frac{4\Phi}{\left(1+\Phi\right)^2} \left(\frac{1}{4}\left\{\Phi w_t\right\} + \frac{1}{2} w_t + \frac{1}{4} w_{t+r}\right), \tag{6.15}$$

and for $n-r+1 \leq t \leq n$, the filter is the time reverse of the initial year filter,

$$\hat{S}_t = \frac{\Phi}{(1+\Phi)^2}\left(w_{t-r} + (\Phi+2)\,w_t\right) = \frac{4\Phi}{(1+\Phi)^2}\left(\frac{1}{4}w_{t-r} + \frac{1}{2}w_t + \frac{1}{4}\{\Phi w_t\}\right). \tag{6.16}$$

The role of $\{\Phi w_t\}$ in (6.15) and (6.16) is as in (6.11) and (6.13).

The classical nonregression approach, illustrated in Subsection 14.1, quickly yields the symmetric filters but explicitly needs forecasts and backcasts to obtain the asymmetric filters.

Figure 2 shows the extracted signal $\hat{S}_t$ from an $n = 144$ simulated $r = 12$ (monthly) Z_t with $\Phi = 0.95$, after MMSE suppression of Z_t's white noise component. The $\hat{S}_t$ track all but the most rapid movements of the Z_t series, but with fewer changes of direction over the 12 years. Findley et al. (2016) gives a formal sense in which the calendar month series of the $\hat{S}_t$ are smoother than those of the Z_t.

6.2. The Error Variance Matrix of the Estimates

From (4.4), for $q = 2$ and $n = 7$, the error variance matrix has the formula

$$\Sigma_{ee} = \sigma_a^2 \frac{\Phi}{(1+\Phi)^4} \begin{bmatrix} 2+\Phi & 0 & \Phi & 0 & 0 & 0 & 0 \\ 0 & 2+\Phi & 0 & \Phi & 0 & 0 & 0 \\ \Phi & 0 & 2 & 0 & \Phi & 0 & 0 \\ 0 & \Phi & 0 & 2 & 0 & \Phi & 0 \\ 0 & 0 & \Phi & 0 & 2 & 0 & \Phi \\ 0 & 0 & 0 & \Phi & 0 & 2+\Phi & 0 \\ 0 & 0 & 0 & 0 & \Phi & 0 & 2+\Phi \end{bmatrix}. \tag{6.17}$$

The error variances of the initial and final years are larger than the error variance $2\sigma_a^2(1+\Phi)^{-4}\Phi$ at intermediate times by the amount $\sigma_a^2\Phi^2(1+\Phi)^{-4}$, which is the mean square error[5] of using $\Phi(1+\Phi)^{-2}\{\Phi w_t\}$ to forecast/backcast $\Phi(1+\Phi)^{-2} w_{t\pm q}$ in (6.10) and (6.13), since from (6.2) we have

$$E\left(w_{t\pm q} - \Phi w_t\right)^2 = \left(1+\Phi^2\right)\gamma_0 - 2\Phi\gamma_q = \left(1-\Phi^2\right)\gamma_0 = \sigma_a^2. \tag{6.18}$$

The fact that the intermediate-time mean square error has the same positive value for all $n \geq 5$ shows that the mean square error does not become negligible with large n. *Unobserved components can be estimated only to limited precision.*

[5] With model-based estimates from more general models for Z_t, more forecasts and backcasts are needed and their error cross-covariances occur in the mean square error formulas, which are less simple.

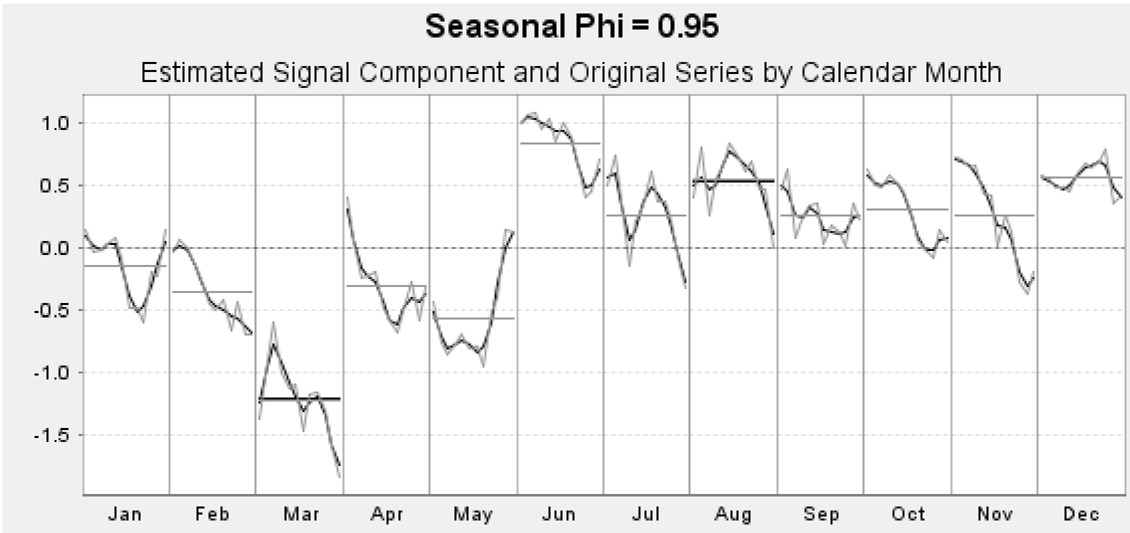

Figure 2: The 12 calendar-month subseries, their averages (horizontal lines), and canonical $\hat{S}_t$ (darker line) of a length 144 simulated $\Phi = 0.95$ SAR(1) Z_t.

7. Pseudo-Spectral Densities of ARIMA Models*

With nonstationary ARIMA Z_t, the *pseudo-spectral density* (p-sd) takes over the role of the sd in decomposition calculations. Its *partial fraction decomposition* (e.g.,Wikipedia Contributors (2011)) is often the starting place for deriving the canonical decomposition as is shown with examples below.

Let Z_t denote a nonstationary ARIMA time series with differencing operator $\delta(B) = 1 + \sum_{j=1}^{d} \delta_j B^j$ for $d \geq 1$ such that $w_t = \delta(B) Z_t$ has the model (5.8). We always assume that $\delta(B)$ and $\vartheta(B)$ have no common factors: there is no overdifferencing.

With $g_w(\lambda)$ as in (5.13), the *pseudo-spectral density* (p-sd, plural p-sds) of Z_t is defined by

$$g_Z(\lambda) = \frac{g_w(\lambda)}{\left|\delta\left(e^{i2\pi\lambda}\right)\right|^2} = \sigma_a^2 \frac{\left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2}{\left|\delta\left(e^{i2\pi\lambda}\right)\right|^2 \left|\varphi\left(e^{i2\pi\lambda}\right)\right|^2}, \ -1/2 \leq \lambda \leq 1/2. \tag{7.1}$$

The p-sd is a non-integrable function, $\int_{-1/2}^{1/2} g_Z(\lambda)\, d\lambda = \infty$, as each zero of $\delta\left(e^{i2\pi\lambda}\right)$ occurs where $g_w(\lambda) > 0$.

The most basic p-sd is that of the (0,1,0) or *random walk*,

$$(1-B)Z_t = a_t \iff Z_t = Z_{t-1} + a_t, \tag{7.2}$$

where a_t is white noise with variance σ_a^2. From (7.1),

$$g_Z(\lambda) = \frac{\sigma_a^2}{\left|1 - e^{i2\pi\lambda}\right|^2}. \tag{7.3}$$

The model (7.2) is the special case $\theta = 0$ of the *invertible IMA(1,1) trend model*,

$$(1 - B)Z_t = (1 - \theta B)a_t, \quad -1 < \theta < 1, \tag{7.4}$$

which, by (5.7) and (7.1), has the p-sd

$$g_Z(\lambda) = \sigma_a^2 \left|\frac{1 - \theta e^{i2\pi\lambda}}{1 - e^{i2\pi\lambda}}\right|^2, -1 < \theta < 1. \tag{7.5}$$

Especially informative will be the $r \geq 2$ *seasonal (1,1)$_r$* generalization of (7.4),

$$(1 - B^r)Z_t = (1 - \Theta B^r)\, a_t, -1 < \Theta < 1, \tag{7.6}$$

considered in Subsection 8.2. Its p-sd is

$$g_Z(\lambda) = \sigma_a^2 \left|\frac{1 - \Theta e^{i2\pi r\lambda}}{1 - e^{i2\pi r\lambda}}\right|^2, -1 < \Theta < 1. \tag{7.7}$$

8. Canonical Pseudo-Spectral Density Decompositions*

8.1. The Canonical Trend-Irregular Decomposition of the IMA(1,1)

The differencing operator $1 - B$ of (7.4), the IMA(1,1), is a trend differencing, so the "signal" of its p-sd decomposition is a trend. We use p_t for trend (later trend-cycle) and N_t for the white noise irregular of the canonical two-component decomposition, as in Section 6,

$$Z_t = p_t + N_t. \tag{8.1}$$

The minimum value of the p-sd $g_Z(\lambda)$ in (7.5) is not obvious, so we calculate the partial fraction decomposition,

$$\sigma_a^{-2} g_Z(\lambda) = \frac{\left|1 - \theta e^{i2\pi\lambda}\right|^2}{\left|1 - e^{i2\pi\lambda}\right|^2} = \frac{a}{\left|1 - e^{i2\pi\lambda}\right|^2} + b. \tag{8.2}$$

The constants a, b are obtained by multiplying (8.2) by $\left|1 - e^{i2\pi\lambda}\right|^2$ and expanding the resulting expressions,

$$\begin{aligned} \left|1 - \theta e^{i2\pi\lambda}\right|^2 &= a + b\left|1 - e^{i2\pi\lambda}\right|^2 \\ \left(1 + \theta^2\right) - \theta\left(e^{i2\pi\lambda} + e^{-i2\pi\lambda}\right) &= a + b\left(2 - \left(e^{i2\pi\lambda} + e^{-i2\pi\lambda}\right)\right) \\ &= (a + 2b) - b\left(e^{i2\pi\lambda} + e^{-i2\pi\lambda}\right). \end{aligned}$$

Equating constants and coefficients of $\left(e^{i2\pi\lambda}+e^{-i2\pi\lambda}\right)$ on both sides yields $a+2b=1+\theta^2$, $b=\theta$. Thus $a=(1-\theta)^2$ and we have

$$\sigma_a^{-2} g_Z(\lambda)=\frac{(1-\theta)^2}{\left|1-e^{i2\pi\lambda}\right|^2}+\theta, \tag{8.3}$$

revealing that the variance of the canonical white noise irregular component is

$$\sigma_a^{-2}\sigma_N^2=\min_\lambda \sigma_a^{-2} g_Z(\lambda)=\frac{(1-\theta)^2}{4}+\theta=\frac{(1+\theta)^2}{4} \tag{8.4}$$

in units of σ_a^2. Therefore, for the p-sd of the trend, from (7.5),

$$\begin{aligned}
\sigma_a^{-2} g_p(\lambda) &= \sigma_a^{-2}\left(g_Z(\lambda)-\sigma_N^2\right)=\frac{(1-\theta)^2}{\left|1-e^{i2\pi\lambda}\right|^2}-\frac{(1-\theta)^2}{4} \\
&= \frac{1}{4}(1-\theta)^2 \frac{4-\left|1-e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2}=\frac{1}{4}(1-\theta)^2 \frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2}.
\end{aligned} \tag{8.5}$$

In summary, from (8.4) and (8.5), the canonical IMA(1,1) p-sd decomposition

$$g_Z(\lambda)=g_p(\lambda)+g_N(\lambda)$$

has

$$g_p(\lambda)=\frac{1}{4}(1-\theta)^2 \frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2}\sigma_a^2, \quad g_N(\lambda)=\frac{1}{4}(1+\theta)^2\sigma_a^2, \tag{8.6}$$

and is admissible for all $-1 \leq \theta < 1$.

The canonical trend's model is the noninvertible IMA(1,1),

$$(1-B)\,p_t=(1+B)\,b_t, \tag{8.7}$$

with white noise b_t having variance $\sigma_b^2=\sigma_a^2(1-\theta)^2/4$.

For $0<\theta<1$, the *noncanonical,* admissible, IMA(1,1) trend-irregular *Structural Model* decomposition is derived in Section 18.

8.2. A Sometimes Nonadmissible 3-Component Canonical Decomposition*

For three-component *seasonal-trend-irregular* decompositions, we follow the notation of Maravall (2016) in using s_t, p_t and u_t for the respective components,

$$Z_t=s_t+p_t+u_t. \tag{8.8}$$

We start with our most revealing example, the canonical (8.8) decomposition of (7.6) for $r=2$ (biannual data). Partial fraction calculations for the p-sd (7.7) like those used for (8.3) yield

$$\sigma_a^{-2} g_Z(\lambda)=\frac{(1-\Theta)^2}{4\left|1+e^{i2\pi\lambda}\right|^2}+\frac{(1-\Theta)^2}{4\left|1-e^{i2\pi\lambda}\right|^2}+\Theta. \tag{8.9}$$

If Θ is nonnegative, this is an admissible p-sd decomposition, but not the canonical decomposition: the condition $\Theta \geq 0$ is too restrictive.

To obtain the candidate canonical p-sd decomposition, the positive (equal) minimum values of the nonconstant p-sds in (8.9),

$$\min_{\lambda} \frac{(1/4)(1-\Theta)^2}{\left|1+e^{i2\pi\lambda}\right|^2} = \min_{\lambda} \frac{(1/4)(1-\Theta)^2}{\left|1-e^{i2\pi\lambda}\right|^2} = (1-\Theta)^2/16, \tag{8.10}$$

are shifted to the constant term Θ in (8.9) to yield the larger constant,

$$\Theta + 2(1-\Theta)^2/16 = \frac{(1-\Theta)^2+8\Theta}{8} = \frac{1+6\Theta+\Theta^2}{8},$$

and the noninvertible seasonal and trend p-sds

$$\begin{aligned}
\frac{(1/4)(1-\Theta)^2}{\left|1+e^{i2\pi\lambda}\right|^2} - (1-\Theta)^2/16 &= \frac{(1-\Theta)^2}{16}\frac{4-\left|1+e^{i2\pi\lambda}\right|^2}{\left|1+e^{i2\pi\lambda}\right|^2} = \frac{(1-\Theta)^2}{16}\frac{\left|1-e^{i2\pi\lambda}\right|^2}{\left|1+e^{i2\pi\lambda}\right|^2},\\
\frac{(1/4)(1-\Theta)^2}{\left|1-e^{i2\pi\lambda}\right|^2} - (1-\Theta)^2/16 &= \frac{(1-\Theta)^2}{16}\frac{4-\left|1-e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2} = \frac{(1-\Theta)^2}{16}\frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2}.
\end{aligned}$$

These result in the new decomposition

$$\sigma_a^{-2} g_Z(\lambda) = \frac{(1-\Theta)^2}{16}\frac{\left|1-e^{i2\pi\lambda}\right|^2}{\left|1+e^{i2\pi\lambda}\right|^2} + \frac{(1-\Theta)^2}{16}\frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2} + \frac{1+6\Theta+\Theta^2}{8}. \tag{8.11}$$

This is the canonical p-sd decomposition when the admissibility condition $1+6\Theta+\Theta^2 \geq 0$ holds, which is equivalent to $\Theta \geq -3+2\sqrt{2} \doteq -0.1716$. For such Θ, the canonical p-sd decomposition for (8.8) has

$$g_s(\lambda) = \sigma_a^2 \frac{(1-\Theta)^2}{16}\frac{\left|1-e^{i2\pi\lambda}\right|^2}{\left|1+e^{i2\pi\lambda}\right|^2},\ g_p(\lambda) = \sigma_a^2 \frac{(1-\Theta)^2}{16}\frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2},\ g_u(\lambda) = \sigma_u^2, \tag{8.12}$$

with the maximal irregular component variance

$$\sigma_u^2 = \frac{\left(\Theta^2+6\Theta+1\right)}{8}\sigma_a^2. \tag{8.13}$$

Thus, for these Θ, we have a canonical decomposition (8.8) with white noise u_t having variance (8.13) and with ARIMA s_t and p_t having the noninvertible biannual seasonal and trend models,

$$\begin{aligned}
(1+B)\,s_t &= (1-B)\,c_t,\\
(1-B)\,p_t &= (1+B)\,b_t,
\end{aligned}$$

respectively, with

$$\sigma_b^2 = \sigma_c^2 = (1-\Theta)^2\sigma_a^2/16. \tag{8.14}$$

For output tables of AMBSA software, the white noise variance σ_a^2 of Z_t's ARIMA model is often set equal to 1.0 in the calculation of seasonal decomposition components' white noise variances like σ_b^2 and σ_c^2. These are then identified as being in "units of $var(a)$".

For the subinterval $-1 < \Theta < -3 + 2\sqrt{2}$, (8.13) yields a negative σ_u^2. For these Θ, the p-sd decomposition is *nonadmissible* and there is no AMBSA decomposition from the estimated model of Z_t. The automatic nonadmissible decomposition model replacement option[6] of some AMBSA software replaces Θ in this one-parameter case with the closest Θ from an admissible decomposition, $\Theta \doteq -.1716$ for the model (7.6) with $r = 2$. Then $\sigma_u^2 = 0$ so there there is no irregular component.

Hillmer and Tiao (1982, pp. 66-67) also provide results for $r > 2$ and for more general seasonal models, but only for $r = 2$ (biannual data) do they obtain simple formulas. In practice, estimated Θ are usually positive for an Airline i.e., $(0,1,1)(0,1,1)_r$ model, and $\Theta \geq 0$ results in admissibility for this and some similar seasonal models, as Hillmer and Tiao show. Our final canonical p-sd decomposition example will provide a revealing collection of filter formulas for component estimation.

8.3. 3-Component Pseudo-Spectral Density Decomposition of the Biannual Seasonal Random Walk

Setting $\Theta = 0$ and $r = 2$ in (7.6) yields the biannual Seasonal Random Walk, the $(0,1,0)_2$ or SRW_2,

$$(1 - B^2)Z_t = a_t, \tag{8.15}$$

whose p-sd is given by

$$\sigma_a^{-2} g_Z(\lambda) = \frac{1}{|1 - e^{i2\pi 2\lambda}|^2} = \frac{1}{|1 + e^{i2\pi\lambda}|^2} \frac{1}{|1 - e^{i2\pi\lambda}|^2}. \tag{8.16}$$

Setting $\Theta = 0$ in (8.12) and (8.13) yields the p-sd formulas for its 3-component decomposition (8.8),

$$g_s(\lambda) = \frac{\sigma_a^2}{16} \frac{|1 - e^{i2\pi\lambda}|^2}{|1 + e^{i2\pi\lambda}|^2}, \quad g_p(\lambda) = \frac{\sigma_a^2}{16} \frac{|1 + e^{i2\pi\lambda}|^2}{|1 - e^{i2\pi\lambda}|^2}, \quad g_u(\lambda) = \frac{\sigma_a^2}{8}. \tag{8.17}$$

Thus s_t and p_t are nonstationary, with respective differencing polynomials $\delta_s(B) = 1 + B$ and $\delta_p(B) = 1 - B$, with $\delta_p(B)$ also the differencing polynomial of the seasonal adjustment $sa_t = p_t + u_t$. Section 15 provides the symmetric filter formulas, also obtained by Maravall and Pierce (1987), whose tutorial study of (8.15) displays the infinitely many decompositions

[6] One should check that the software's replacement model has acceptable goodness-of-fit diagnostics. If it does not, the user could explore other models and/or shorter data spans to try to obtain an estimated model with an admissible decomposition having acceptable diagnostics. Nonadmissibility is generally associated with data whose graph shows quite erratic movements or strong trend movements.

different from the canonical that can result from apportioning p-sd minima like (8.10) differently between two or more component p-sds (or sds).

We now consider methods for estimating components of nonstationary decompositions and their filters' properties.

9. Matrix Formulation of Signal Extraction: Difference-Stationary Case

For data with a known ARIMA model, three approaches for calculating MMSE estimates (providing identical estimates) are available among the main AMBSA programs. The only elementary approach, also the easiest to program for two-component decompositions, is presented in Subsection 9.3. To best reveal to the reader how the nonstationary case differs from the stationary case, we start with the need for an assumption that provides MMSE optimality of forecasts, backcasts and component estimates.

9.1. Assumption A and Random Walk Forecasts and Backcasts*

Purely autocovariance-based formulas like those above, e.g. in (4.2), are not applicable to the inherently more complex case of nonstationary ARIMA Z_t because variances and autocovariances cannot be estimated for nonstationary variates. We illustrate why this is so with the simplest ARIMA process, the random walk (7.2). Its Z_t, $t \geq 2$, are generated recursively from Z_1 and future white noise $a_t, t \geq 2$,

$$Z_t = Z_{t-1} + a_t = Z_{t-2} + a_{t-1} + a_t = \cdots = Z_1 + \sum_{j=2}^{t} a_j, t \geq 2. \tag{9.1}$$

Assumption A of Bell (1984) for (7.2) is that the initial value Z_1 of the data generating formula (9.1) is uncorrelated with all a_t, including the $a_t, t \leq 1$ which generate earlier Z_t via
$Z_{t-1} - Z_t = -a_t, t \leq 1$.

EZ_1^2 cannot be estimated consistently from one datum Z_1. Nor can the covariance $EZ_1 a_t$. So Assumption A can neither be verified nor contradicted. It can be replaced by other assumptions but doing so leads to more complex formulas with no advantages, see Bell (1984), where it is also indicated why Assumption A justifies the standard ARIMA forecast formulas by guaranteeing that they provide MMSE forecasts. We establish this for the random walk.

For (7.2), the "well known" (if rarely fully derived) result is that *for all $h \geq 1$, the MMSE h-step forecast of Z_{t+h} from data $Z_1, \ldots, Z_t$ is the latest datum, $\widehat{Z}_{t+h} = Z_t$*. To establish this result, note from (9.1) that the forecast error $Z_{t+h} - Z_t = a_{t+1} + a_{t+2} + \cdots + a_{t+h}$ is uncorrelated with Z_1 by Assumption A and with $a_2, \ldots, a_t$ by the white noise property. Consequently, the error of forecast $\widehat{Z}_{t+h} = Z_t$ is uncorrelated with the data, which is the MMSE characterizing property, see Section 4. Analogous calculations show that Z_1 *is the MMSE backcast of $Z_1, \ldots, Z_t$ for all $h \geq 1$.*

9.2. Required Properties of the Stationarized Data, Signal, and Noise

When signal and noise components are ARIMA , there are three differencing operators $\delta(B) = 1 + \delta_1 B + \cdots + \delta_d B^d$ for the series Z_t, $\delta_S(B) = 1 + \delta_1^S B + \cdots + \delta_{d_S}^S B^{d_S}$ for S_t, and $\delta_N(B) = 1 + \delta_1^N B + \cdots + \delta_{d_N}^N B^{d_N}$ for N_t. If either S_t or N_t is a stationary component, such as an irregular, transitory, or cycle component (see Subsection 21.2) one sets $\delta_S(B) = 1$ or $\delta_N(B) = 1$. For MMSE component estimates from series of length $n > d$, the formulas of McElroy(2008) below and most of those of Section 13 require:

1. $\delta(B) = \delta_S(B)\,\delta_N(B)$
2. $\delta_S(B)$ and $\delta_N(B)$ have no common zeroes
3. The stationary processes

$$U_t = \delta_S(B)\,S_t, \quad V_t = \delta_N(B)\,N_t, \tag{9.2}$$

   are uncorrelated[7]: $EU_sV_t = 0$, for $s = d_s + 1, \ldots, n$, $t = d_N + 1, \ldots, n$.

4. The $d = \deg \delta(B)$ initial values $Z_1, \ldots, Z_d$ are uncorrelated with the series U_t and V_t. (Assumption A of Bell (1984)).

We refer to these as Requirements 1–3 and Assumption A. Regarding 1 and 2, when $\delta(B) = (1-B)^d\left(1-B^{12}\right)$, as is common with monthly data Z_t, with S_t the seasonal and $N_t = Z_t - S_t$ the nonseasonal component, these Requirements are met by $\delta_S(B) = 1 + B + \cdots + B^{11}$ and $\delta_N(B) = (1-B)^{d+1}$.

9.3. The Two-Component Estimation Formulas

For $U_t = \delta_S(B)\,S_t$, define $U = (U_{d_S+1}, \ldots, U_n)'$ and let Δ_S be the $(n - d_S) \times n$ matrix implementing $\delta_S(B)$, i.e. such that

$$U = \Delta_S S.$$

Thus, when $\delta_S(B) = 1 + \sum_{j=1}^{d_S} \delta_j^S B^j$, the matrix Δ_S has the form

$$\Delta_S = \begin{bmatrix} \delta_{d_S}^S & \delta_{d_{S-1}}^S & \cdots & 1 & 0 & \cdots & 0 \\ 0 & \delta_{d_S}^S & \delta_{d_{S-1}}^S & \cdots & 1 & \cdots & 0 \\ \vdots & & & & & & \vdots \\ & & & & & & \\ 0 & & & & & 1 & 0 \\ 0 & 0 & \cdots & & \cdots & \delta_1^S & 1 \end{bmatrix}.$$

[7] Findley (2012) shows how this requirement can be weakened.

Set $\Sigma_U = EUU'$ and let Σ, Σ_V and Δ_N be the analogous matrices for $\delta(B) Z_t$, $V_t = \delta_N(B) N_t$, and $\delta_N(B)$. With the autocovariance matrices Σ_U and Σ_V and differencing matrices Δ_S and Δ_N, McElroy (2008) shows, under Requirements 1–4, that the MMSE linear estimate

$$\hat{S} = \beta_S Z \tag{9.3}$$

is obtained with

$$\beta_S = \left(\Delta_S' \Sigma_U^{-1} \Delta_S + \Delta_N' \Sigma_V^{-1} \Delta_N\right)^{-1} \Delta_N' \Sigma_V^{-1} \Delta_N, \tag{9.4}$$

and the variance matrix Σ_{ee} of the signal extraction error $e = S - \hat{S}$ has the formula

$$\Sigma_{ee} = \left(\Delta_S' \Sigma_U^{-1} \Delta_S + \Delta_N' \Sigma_V^{-1} \Delta_N\right)^{-1}. \tag{9.5}$$

For each $1 \leq t \leq n$, the t-th row of β_S consists of the filter coefficients $\beta_{S,t,j}$ used to obtain $\hat{S}_t = \sum_{j=1}^{n} \beta_{S,t,j} Z_j$. The *change of scale results* below formulas (4.4) also apply in the ARIMA case, e.g., the filters do not depend on σ_a^2. McElroy (2008) establishes various properties of the filters, including their *reverse symmetry*: the coefficients for $t = n$ are those of $t = 1$ in reverse order, and similarly for $t = n - 1$ and $t = 2$, etc. If n is odd, $n = 2m + 1$, then the filter for the midpoint $t = m + 1$ is *symmetric*, $\beta_{S,m+1,j} = \beta_{S,m+1,n-j}$ for $1 \leq j \leq m$. Otherwise, the AMBSA seasonal adjustment filters are asymmetric. As in the stationary case, $\hat{N} = \beta_N Z$ with $\beta_N = I - \beta_S$.

A conspicuous feature of the formula for β_S is the noise differencing operator Δ_N on the right: When N_t is nonstationary, signal extraction starts by stationarizing the noise component.

Extensions of the matrix formulas are developed in McElroy (2006) and McElroy and Holan (2012) for the long-term trend and cycle estimates of Subsection 21.2.

9.4. Filter and Error Variance Properties of The Canonical Decomposition*

Remark 1 of McElroy (2008) describes how the formulas (9.4) and (9.5) simplify when one component is stationary. For the canonical decomposition case, with $\Sigma_{NN} = \sigma^2 I, \sigma^2 > 0$, and $W = \Delta Z$, the result is

$$\begin{aligned} \beta_S &= I - \sigma^2 \Delta' \Sigma_W^{-1} \Delta, \\ \Sigma_{ee} &= \sigma^2 \beta_S. \end{aligned}$$

This Σ_{ee} formula is a generalization of (4.4) to nonstationary Z_t and S_t. When, for example, S_t is the trend component of the IMA(1,1) model considered in Subsection 8.1, then $\sigma^2 = (1 + \theta)^2 / 4$ in units of σ_a^2, see (8.4).

The formulas reveal that, with white noise N, $\beta_S = \sigma^{-2} \Sigma_{ee}$ is a positive definite matrix, since Σ_{ee} is. It follows that the filter coefficient $\beta_{S,t,t}$ of Z_t in $\hat{S}_t = \Sigma_{j=1}^{n} \beta_{S,t,j} Z_j$ is positive for each $1 \leq t \leq n$. Further, the largest magnitude coefficient in β_S is on the main diagonal, i.e., is the coefficient $\beta_{S,t,t}$ of Z_t in $\hat{S}_t$ for some t, and thus positive, see Theorem 12.4 of Noble (1969). Also, $\beta_N = 1 - \beta_S = \sigma^2 \Delta' \Sigma_W^{-1} \Delta$ has these properties.

The largest magnitude coefficient property does not generalize to 3-component decompositions: In Subsection 15.2, the seasonal component filter (15.6) of the canonical three-component decomposition of a seasonal random walk is displayed. Its largest magnitude coefficient is negative and not on the main diagonal.

Section 5.2 of McElroy (2008) shows a way, which is implemented in some AMBSA software to, use matrix formulas to obtain AMBSA estimates for decompositions with more components. But usually the state space method or the W-K filter-based calculation method described in Burman (1980) is used. Each is available in widely used AMBSA software and can handle any number of components. Both have important computational efficiency advantages over the matrix formulas. The different methods produce the same finite-sample estimates and filter coefficients (up to rounding error). Only the matrix and state space calculations produce finite-sample variances and covariances rather than infinite-sample-based approximations.

10. Illustrative Seasonal Adjustment Filter and Standard Error Graphs*

For a monthly time series of length $n = 131$ from the $\theta = 0$ Airline model,

$$(1 - B)\left(1 - B^{12}\right) \log Z_t = \left(1 - \Theta B^{12}\right) a_t, \tag{10.1}$$

with $\Theta = 0.3$, Figure 3 shows the filter coefficients for the symmetric midpoint SA filter ($t = 66$, solid lines) and for the one-sided concurrent SA filter ($t = 131$, dashed lines). Focusing on the larger coefficients, both filters have effective lengths of about two years, with large coefficients of different signs between time t and the adjacent same-calendar-month times, also for other t not shown. As a consequence, their SA estimates can be adaptive to short-term changes in the features of the series, potentially providing considerable smoothing. But with future data, the SA, especially its concurrent value, can have large *revisions*, i.e., large changes in value from its initial estimate, in comparison to revisions from Figure 4's $\Theta = 0.9$ filters, as the standard errors in Figure 5 indicate.

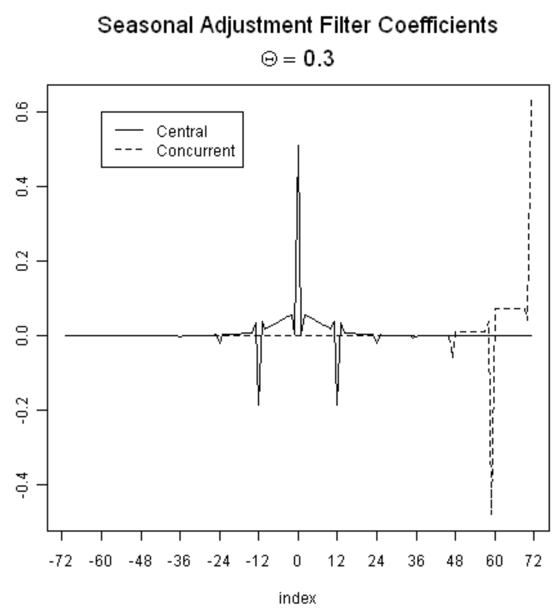

Figure 3: Symmetric central ($t = 66$) and one-sided concurrent ($t = 131$) seasonal adjustment filter coefficients for $n = 131$ from the canonical decomposition of the $(0,1,0)(0,1,1)_{12}$ model with $\Theta = 0.3$.

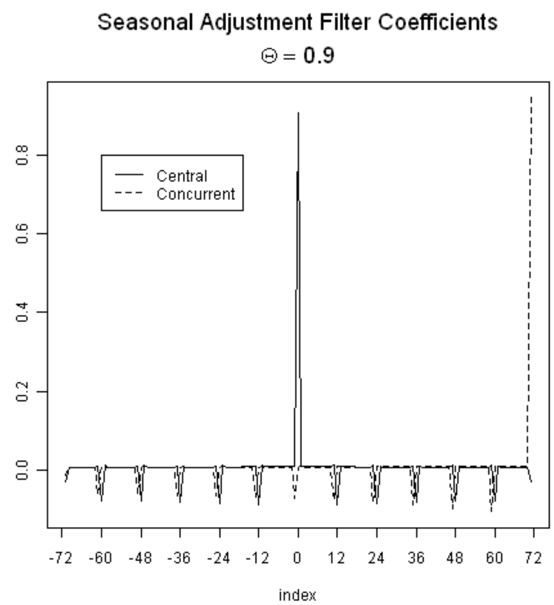

Figure 4: The $\Theta = 0.9$ analogue of the preceding Figure. The coefficients decrease slowly, so the filters are not very responsive to short-term data fluctuations.

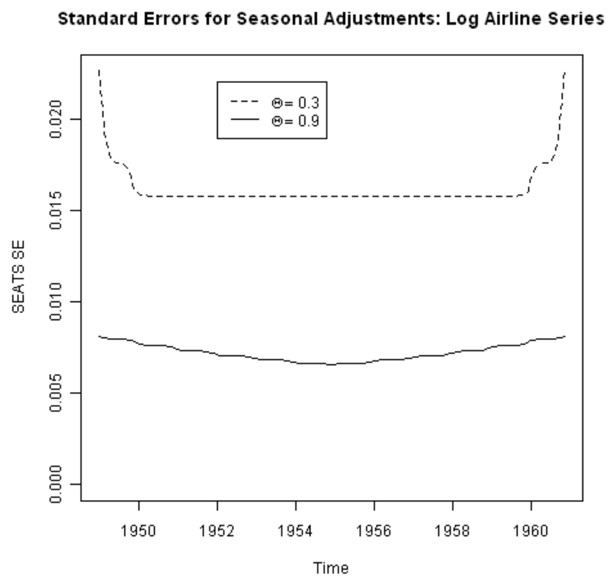

Figure 5: The finite-sample standard errors of seasonal adjustments from the models of Figures 3 and 4, increase with the asymmetry of the filters as more forecasts and backcasts are used.

Figure 5 shows that additional smoothing from the $\Theta = 0.3$ model results in more than twice the standard error of the less smooth seasonal adjustment from the $\Theta = 0.9$ model. Note too how the standard errors increase with the distance from the center of the series. The same σ_a^2 is used in (10.1) for both models.

11. Standard Errors of Change for Additive Estimates

11.1. Change from the Preceding Estimate

Error variances and covariances from Σ_{ee} can be used to describe the uncertainty in measures of change. Let S_t denote the seasonal adjustment component of interest (e.g., the seasonal adjustment or the trend). With an additive decomposition, the error of the one-month change estimate $\hat{S}_t - \hat{S}_{t-1}$ is the difference of the errors of the two estimates,

$$(S_t - S_{t-1}) - \left(\hat{S}_t - \hat{S}_{t-1}\right) = \left(S_t - \hat{S}_t\right) - \left(S_{t-1} - \hat{S}_{t-1}\right) = e_t - e_{t-1},$$

so the error variance is

$$E\left(e_t - e_{t-1}\right)^2 = \left(\Sigma_{ee}\right)_{t,t} + \left(\Sigma_{ee}\right)_{t-1,t-1} - 2\left(\Sigma_{ee}\right)_{t,t-1}. \tag{11.1}$$

Hence the standard error of $\hat{S}_t - \hat{S}_{t-1}$ is given by

$$\sqrt{(\Sigma_{ee})_{t,t} + (\Sigma_{ee})_{t-1,t-1} - 2\,(\Sigma_{ee})_{t,t-1}}, \tag{11.2}$$

from which probability intervals for $S_t - S_{t-1}$ can be calculated under standard assumptions.

We illustrate such an interval for the more complex case of future revisions of an estimate.

11.2. Revisions with Future Data*

The $n \times n$ matrix Σ_{ee} depends only on the models for S_t and N_t, not on Z_t data, see (9.5). So it can be calculated for future series lengths $n+h$ for any $h = 1, 2, \ldots$ when only n observations are available. Let $\hat{S}_{t|_1^n}$ and $\hat{S}_{t|_1^{n+h}}$ denote the estimates of S_t from these two series lengths, resulting in the revision $\hat{S}_{t|_1^{n+h}} - \hat{S}_{t|_1^n}$, and let $\Sigma_{ee}^{(n)}$ and $\Sigma_{ee}^{(n+h)}$ denote the corresponding error variance matrices. The main result is that $R_t\,(h) = E\left(\hat{S}_{t|_1^{n+h}} - \hat{S}_{t|_1^n}\right)^2$, called the *revision variance* or the *mean square revision,* is given by the nonnegative quantity

$$R_t\,(h) = \left(\Sigma_{ee}^{(n)}\right)_{t,t} - \left(\Sigma_{ee}^{(n+h)}\right)_{t,t}. \tag{11.3}$$

Hence, assuming normality, a 95 percent probability interval for the revised estimate $\hat{S}_{t|_1^{n+h}}$ with additional data $Z_{n+1}, \ldots, Z_{n+h}$ is given by $\hat{S}_{t|_1^n} - 1.96\sqrt{R_t\,(h)} \leq \hat{S}_{t|_1^{n+h}} \leq \hat{S}_{t|_1^n} + 1.96\sqrt{R_t\,(h)}$.

With $R_t\,(\infty) = \lim_{h\to\infty} R_t\,(h)$, McElroy and Gagnon (2008) show how to calculate the analogue $1 - \sqrt{1 - R_t\,(h)\,/R_t\,(\infty)}$ of a revision measure produced by AMBSA software for revisions of estimates from the infinite past $\hat{S}_{t|_{-\infty}^n} - \hat{S}_{t|_{-\infty}^{n+h}}, 1 \leq h < \infty$.

McElroy and Gagnon further show how this is a measure of the proportion of the total root mean square revision $\sqrt{R_t\,(\infty)}$ of $\hat{S}_{t|_1^n}$ that is obtained after h months. Their numerical comparisons indicate that the software-default infinite-data-based measures, which are computationally less expensive (especially for large n), differ little from $1 - \sqrt{1 - R_t\,(h)\,/R_t\,(\infty)}$ except when n is small.

12. Multiplicative Decomposition Estimates from Logs*

Most often, it is the logs of positive economic data z_t, $Z_t = \log z_t$, that can be modeled with an ARIMA model. In this case, the conceptual two-component signal and noise decomposition model is a multiplicative decomposition, $z_t = s_t n_t$, estimated from the additive decomposition $Z_t = S_t + N_t$, with $S_t = \log s_t$ and $N_t = \log n_t$.

If lognormality (e.g., Wikipedia Contributors (2017a)) is assumed, the MMSE estimate of $s_t = \exp S_t$ is

$$\exp\left(\hat{S}_t + \frac{1}{2}\,(\Sigma_{ee})_{t,t}\right) = \left(\sqrt{\exp\,(\Sigma_{ee})_{t,t}}\right)\exp \hat{S}_t,$$

whose mean square error is $\exp 2\hat{S}_t \left(\exp\left(2\left(\Sigma_{ee}\right)_{t,t}\right) - \exp\left(\left(\Sigma_{ee}\right)_{t,t}\right)\right)$, and analogously for $n_t = \exp N_t$. The product of the optimal estimates is thus not z_t but $z_t\left(\Sigma_{ee}\right)_{t,t}$. In AMBSA practice however, $\hat{s}_t = \exp \hat{S}_t$ and $\hat{n}_t = \exp \hat{N}_t$ are taken as the estimates, giving up idealized mean square optimality for the practical goal of having a multiplicative decomposition $z_t = \hat{s}_t \hat{n}_t$.

Another effect of estimating components by exponentiating estimates made from log transformed data is that the resulting *level estimates are downwardly biased*, as a consequence of the geometric-arithmetic mean inequality: $\exp\left(n^{-1}\Sigma_{t=1}^{n} \log x_t\right) < n^{-1}\Sigma_{t=1}^{n} x_j$ for $n \geq 2$ unless all x_j have the same value. Thus trend estimates obtained by exponentiating the trend estimates of the log transformed data are downwardly biased. A simple often effective procedure of Maravall to reduce this bias is implemented in all AMBSA software. See Proietti and Riani (2017) for a more encompassing analysis and discussion of transformations.

12.1. Standard Errors for Growth Rates*

Growth rates from multiplicative decompositions are calculated as
$\left(s_t - s_{t-1}\right)/s_{t-1}$ in the one-period case. When they are reasonably small, e.g. < 0.10, then

$$\hat{S}_t - \hat{S}_{t-1} = \log\left(\frac{\hat{s}_t}{\hat{s}_{t-1}}\right) = \log\left(1 + \frac{\hat{s}_t - \hat{s}_{t-1}}{\hat{s}_{t-1}}\right) \doteq \frac{\hat{s}_t - \hat{s}_{t-1}}{\hat{s}_{t-1}}, \tag{12.1}$$

and the error variance of $\left(\hat{s}_t - \hat{s}_{t-1}\right)/\hat{s}_{t-1}$ can be estimated by the error variance of $\hat{S}_t - \hat{S}_{t-1}$, i.e. by (11.1), and its standard error by (11.2). The standard errors of $\hat{S}_t - \hat{S}_{t-1}$ shown in Figure 6 for the models of Figures 3 and 4 reveal that those for $\Theta = 0.3$ are substantially larger and increase much more sharply near the ends of the series than those for $\theta = 0.9$.

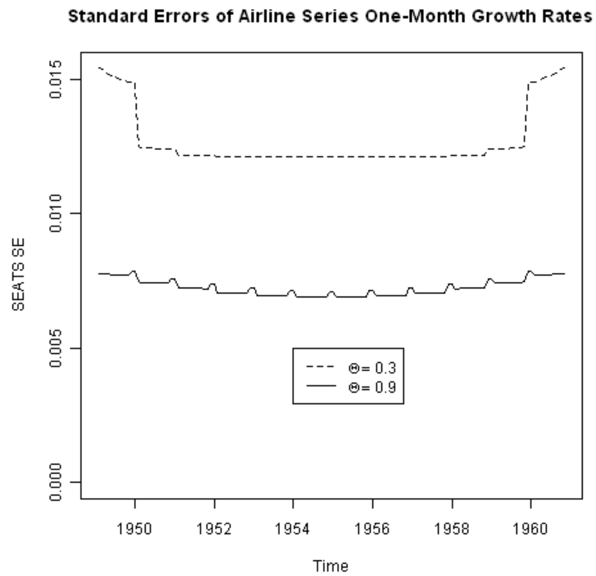

Figure 6: Finite-sample ($n = 131$) standard errors of the approximate one-month growth rates (12.1) from (11.2) for the canonical decompositions of the (0,1,0)(0,1,1) models with $\Theta = 0.3$ and $\Theta = 0.9$. The value used for the white noise variance σ_a^2 is the same as for Figure 3 and the interpretation of the graphs is analogous.

13. ARMA and ARIMA Wiener-Kolmogorov Filters

MMSE component estimation for the case of bi-infinite data Z_m, $-\infty < m < \infty$ is the conceptual starting point for the approach of Hillmer and Tiao (1982) approach to MMSE finite-sample component estimation. For signal plus noise decompositions of stationary series, the symmetric MMSE filter formulas (13.1) and associated transfer function formulas below were independently published by A. N. Kolmogorov (1939) and N. Wiener (1949). Bell (1984) provided conceptual foundations for the ARIMA generalization. As Subsection 6.1 illustrated with finite symmetric filters for stationary data and Subsection 15 illustrates for nonstationary data, forecasts and backcasts replace the unavailable data required by the symmetric filter, thereby defining a time-varying asymmetric filter. The resulting estimates are referred to as Wiener-Kolmogorov or W-K estimates.

Starting from the always symmetric, usually bi-infinite filters introduced here, AMBSA software that does not apply state space methods or matrix formulas will apply W-K filters and the algorithm of Tunnicliffe-Wilson published in the Appendix of Burman (1980). The algorithm exploits the fact that, because recursion relations from the model can be used, the desired finite-sample MMSE estimates can be obtained from rapidly from a relatively small number of forecasts and backcasts from the finite available ARMA or ARIMA data.

We first consider estimation of a two-component decomposition, $Z_t = S_t + N_t$ for Z_t with an invertible ARIMA model.

To obtain the MMSE estimates $\hat{S}_t$ and $\hat{N}_t$, $-\infty < t < \infty$, from bi-infinite data[8] Z_m, $-\infty < m < \infty$, the Requirements 1 and 2 and Assumption A of Subsection 9.2 are retained. Requirement 3 is reformulated for $U_t = \delta_S(B) S_t$ and $V_t = \delta_N(B) N_t$ as $EU_tV_{t-j} = 0$ for all $j = 0, \pm 1, \pm 2, \ldots$.

Under these conditions, the symmetric filters

$$\beta_S(B) = \beta_{S,0} + \sum_{j=1}^{\infty} \beta_{S,j}\left(B^{-j} + B^j\right), \quad \beta_N(B) = \beta_{N,0} + \sum_{j=1}^{\infty} \beta_{N,j}\left(B^{-j} + B^j\right)$$

of the bi-infinite data estimates,

$$\hat{S}_t = \beta_S(B) Z_t = \beta_{S,0} Z_t + \sum_{j=1}^{\infty} \beta_{S,j}\left(Z_{t+j} + Z_{t-j}\right)$$

and $\hat{N}_t = \beta_N(B) Z_t$, with $\beta_N(B) = 1 - \beta_S(B)$, have W-K transfer functions defined by ratios (13.1) of p-sds of S_t, N_t and Z_t. It is now helpful to use the fact that the defining formulas (5.13) of sds and (7.1) of p-sds are each functions of $e^{i2\pi\lambda}$. We will write $g_Z\left(e^{i2\pi\lambda}\right)$ instead of $g_Z(\lambda)$ and similarly for all other p-sds. The W-K formulas are

$$\beta_S\left(e^{i2\pi\lambda}\right) = \frac{g_S\left(e^{i2\pi\lambda}\right)}{g_Z\left(e^{i2\pi\lambda}\right)}, \quad \beta_N\left(e^{i2\pi\lambda}\right) = \frac{g_N\left(e^{i2\pi\lambda}\right)}{g_Z\left(e^{i2\pi\lambda}\right)}. \tag{13.1}$$

With stationary Z_t, sds replace p-sds in these formulas and the S_t series is assumed to be uncorrelated with the N_t series. Appendix A of Findley, Lytras and Maravall (2015) provides an elaboration of the derivation of Whittle (1963) for the stationary case formula $\beta_S\left(e^{i2\pi\lambda}\right) = g_S\left(e^{i2\pi\lambda}\right)/g_Z\left(e^{i2\pi\lambda}\right)$ and of the estimation error spectral density formula: in the original notation $g_e(\lambda) = g_S(\lambda)\, g_N(\lambda)/g_Z(\lambda)$, an analogue of the formula $\Sigma_{ee} = \Sigma_{SS}\Sigma_{ZZ}^{-1}\Sigma_{NN}$ of (4.4).

The ps-d generalization of (5.12) shows that multiplying the squares of the transfer functions (13.1) by the p-sd or sd of Z_t yields the p-sds or sds of the component estimates. From these the estimates' ARIMA or ARMA models are revealed, see Findley et al. (2016) for examples. These models provide seasonal adjustment quality diagnostics, see Maravall (1987), Findley, McElroy and Wills (2005) and their references.

[8] Bell (1984) shows how the infinite past Z_m, $m \leq 0$ can be generated recursively from the degree d differencing polynomial $\delta(B)$ of Z_t, the starting values $Z_1, \ldots, Z_d$, and the stationary process $w_t = \delta(B) Z_t, -\infty < t < \infty$. This was illustrated with the random walk, which has $d = 1$ and $\delta(B) = (1 - B)$, in Subsection 9.1.

Analogous W-K formulas apply for decompositions with more components, e.g., with seasonal s_t, trend p_t, and irregular u_t,

$$\beta_s\left(e^{i2\pi\lambda}\right)=\frac{g_s\left(e^{i2\pi\lambda}\right)}{g_Z\left(e^{i2\pi\lambda}\right)},\quad \beta_p\left(e^{i2\pi\lambda}\right)=\frac{g_p\left(e^{i2\pi\lambda}\right)}{g_Z\left(e^{i2\pi\lambda}\right)},\quad \beta_u\left(e^{i2\pi\lambda}\right)=\frac{g_u\left(e^{i2\pi\lambda}\right)}{g_Z\left(e^{i2\pi\lambda}\right)}. \tag{13.2}$$

14. W-K Filter Formula Examples

14.1. Rederiving the AR(1)$_r$ Symmetric Filters

From (13.1), we can quickly re-obtain (6.9): First, from (6.4) and (6.3),

$$\beta_N\left(e^{i2\pi\lambda}\right)=\frac{g_N\left(e^{i2\pi\lambda}\right)}{g_w\left(e^{i2\pi\lambda}\right)}=\frac{(1+\Phi)^{-2}}{\left|1-\Phi e^{i2\pi r\lambda}\right|^{-2}}=(1+\Phi)^{-2}\left|1-\Phi e^{i2\pi r\lambda}\right|^2. \tag{14.1}$$

Next, for translations from transfer functions of the form $\left|\Sigma_j\alpha_j e^{i2\pi r\lambda}\right|^2$ to filters, we can adopt a device of Maravall and Pierce (1987), replacing $e^{\pm i2\pi j\lambda}$ by $B^{\pm j}$ to obtain symmetric filter formulas,

$$\left|\Sigma_j\alpha_j B^j\right|^2=\left(\Sigma_j\alpha_j B^j\right)\left(\Sigma_j\alpha_j B^{-j}\right). \tag{14.2}$$

From (14.1) and (14.2),

$$\beta_N(B)=(1+\Phi)^{-2}\left(1-\Phi B^r\right)\left(1-\Phi B^{-r}\right)=(1+\Phi)^{-2}\left(-\Phi B^r+\left(1+\Phi^2\right)-\Phi B^{-r}\right).$$

This is the filter that produces (6.9).

14.2. Infinite W-K Filters

A W-K filter is infinite if and only if the model for Z_t has a moving average component.

14.2.1. A Stationary Case: The Invertible Seasonal MA(1)$_r$

Suppose $Z_t=(1-\theta B^r)\,a_t$, with $r\geq 2$, $\sigma_a^2=1$, $0<|\theta|<1$. Then
$g_Z(\lambda)=\left|1-\theta e^{i2\pi r\lambda}\right|^2$ and the white noise sd $g_N(\lambda)=\sigma^2$ has

$$\sigma^2=\min_{\lambda}\left|1-\theta e^{i2\pi r\lambda}\right|^2=(1-|\theta|)^2.$$

Thus its W-K transfer function is

$$\beta_N\left(e^{i2\pi\lambda}\right)=\frac{\sigma^2}{g_Z\left(e^{i2\pi\lambda}\right)}=(1-|\theta|)^2\left|1-\theta e^{i2\pi r\lambda}\right|^{-2}.$$

From (5.13), this has the form of the sd of a seasonal AR(1)$_r$ with AR coefficient θ, white noise variance $(1-|\theta|)^2$, and thus variance $(1-|\theta|)^2\,(1-\theta^2)^{-1}=(1-|\theta|)\,(1+|\theta|)^{-1}_{.}$ Hence, from

(6.2) and (5.3), the filters are symmetric and infinite:

$$\begin{aligned}
\beta_N(B) &= \frac{1-|\theta|}{1+|\theta|}\left(1+\sum_{j=1}^{\infty}\theta^j\left(B^{jr}+B^{-jr}\right)\right),\\
\beta_S(B) &= 1-\beta_N(B)=\frac{2|\theta|}{1+|\theta|}+\sum_{j=1}^{\infty}\left\{-\frac{1-|\theta|}{1+|\theta|}\theta^j\right\}\left(B^{jr}+B^{-jr}\right).
\end{aligned}$$

Therefore

$$\hat{N}_t = \beta_N(B)Z_t=\frac{1-|\theta|}{1+|\theta|}\left(Z_t+\sum_{j=1}^{\infty}\theta^j\left(Z_{t-jr}+Z_{t+jr}\right)\right), \tag{14.3}$$

$$\hat{S}_t = \beta_S(B)Z_t=\frac{2|\theta|}{1+|\theta|}Z_t+\sum_{j=1}^{\infty}\left\{-\frac{1-|\theta|}{1+|\theta|}\theta^j\right\}\left(Z_{t-jr}+Z_{t+jr}\right). \tag{14.4}$$

The filter coefficients are nonzero only at lag zero and seasonal lags. Lag zero has the largest magnitude coefficient. The magnitudes at seasonal lags decrease exponentially with increasing seasonal lag. For $\theta > 0$, the seasonal lag $jr \geq 1$ coefficients are positive for the white noise estimate $\hat{N}_t$ and negative for the signal estimate $\hat{S}_t$. The coefficients decay exponentially at the rate $|\theta|^j$ starting at $j = 1$.

Next we consider the nonstationary case, starting with a general result.

14.2.2. Transfer Function Form and Coefficient Decay Rate of ARIMA W-K Filters

All W-K transfer functions like those in (13.2) have the form of the sd of an ARMA model because the differencing operator factor of $g_Z\left(e^{i2\pi\lambda}\right)$, e.g., $\left|\delta_S\left(e^{i2\pi\lambda}\right)\right|^{-2}\left|g_N\left(e^{i2\pi\lambda}\right)\right|^{-2}$ in the general two-component case, becomes the numerator factor $\left|\delta_S\left(e^{i2\pi\lambda}\right)\right|^{2}\left|g_N\left(e^{i2\pi\lambda}\right)\right|^{2}$ in $g_Z\left(e^{i2\pi\lambda}\right)^{-1}$, cancelling any differencing factors of $g_S\left(e^{i2\pi\lambda}\right)$ and $g_N\left(e^{i2\pi\lambda}\right)$. It follows that the coefficients of the W-K filter $\beta(B)$ coincide with the autocovariances γ_j of this ARMA model,

$$\beta(B)=\gamma_0+\sum_{j=1}^{\infty}\gamma_j\left(B^{-j}+B^{j}\right). \tag{14.5}$$

This formula reveals the important property that, as in Figures 3 and 4, the weight γ_0 given by W-K filters to the contemporaneous datum Z_t in the estimate $\hat{\beta}_t = \beta(B)Z_t$ is positive and greater in magnitude than all other coefficients, $\gamma_0 > \left|\gamma_j\right|$ for all $j > 1$, because $\left|\gamma_j/\gamma_0\right| < 1$ holds for ARMA autocorrelations. These properties were established for the filters of every canonical two-component finite-sample decomposition in Subsection 9.4.

For specific examples of an ARIMA model's component filters, we return to the fundamental model of Subsection 8.1.

14.2.3. Nonstationary Case: IMA(1,1) Trend-Irregular Decomposition Filters

For the trend-irregular decomposition (8.1) of the model $(1-B)\,Z_t = (1-\theta B)\,a_t, -1<\theta<1$, the sd and p-sd functions of the irregular and trend estimates, $g_u(\lambda)$ and $g_p(\lambda) = 1-\beta_u(\lambda)$, were obtained in (8.6). Here we derive the formulas of the components' estimation filters and note some features that generalize. From the formulas of (8.6) for $g_u(\lambda)$ and of (7.5) for $g_Z(\lambda)$, the transfer function (13.2) for the bi-infinite data estimate $\hat{u}_t = \beta_u(B)\,Z_t$ of the white noise irregular component is

$$\beta_u(\lambda) = \sigma_u^2 g_Z(\lambda)^{-1} = \frac{1}{4}(1+\theta)^2 \frac{\left|1-e^{i2\pi\lambda}\right|^2}{\left|1-\theta e^{i2\pi\lambda}\right|^2}. \tag{14.6}$$

From (8.4), this is the sd of an ARMA(1,1) with AR coefficient θ, MA coefficient 1, and white noise variance $(1+\theta)^2/4$. The transfer function of the trend component,

$$\beta_p(\lambda) = g_p(\lambda)\,g_Z(\lambda)^{-1} = \frac{1}{4}(1-\theta)^2 \frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-\theta e^{i2\pi\lambda}\right|^2}, \tag{14.7}$$

is the sd of an ARMA(1,1) with AR coefficient θ, MA coefficient -1 and innovation variance $(1-\theta)^2/4$. By (14.5), each has as filter coefficients the autocovariances of its sd's model. We calculate the formula for $\beta_p(B)$ in this way and then obtain $\beta_u(B)$ as $1-\beta_p(B)$. Applying the recursions (3.4.7) of Box and Jenkins (1976) for ARMA(1,1) autocovariances to (14.7), we obtain

$$\begin{aligned}
\beta_p(B) &= \frac{2}{1-\theta}\left\{1+\frac{1}{2}\left(B+B^{-1}\right)+\frac{1}{2}\sum_{j=2}^{\infty}\theta^{j-1}\left(B^j+B^{-j}\right)\right\}. \\
\beta_u(B) &= 1-\beta_p(\theta) = \frac{2}{1-\theta}\left\{\frac{1+\theta}{2}-\frac{1}{2}\left(B+B^{-1}\right)-\frac{1}{2}\sum_{j=2}^{\infty}\theta^{\,j-1}\left(B^{\,j}+B^{-\,j}\right)\right\}.
\end{aligned}$$

The coefficients decay exponentially at the rate $|\theta|^j$ starting[9] at $j=2$. For $\theta>0$, the midpoint $(j=0)$ coefficients, $2(1-\theta)^{-1}$ for $\beta_p(B)$ and $(1+\theta)(1-\theta)^{-1}$ for $\beta_u(B)$, are the largest in magnitude. Also, for the trend estimate, all coefficients are positive. For the irregular estimate, except at the midpoint, all are negative.

15. Biannual Seasonal Random Walk Filters*

It is useful to consider detailed results for the 3-component decomposition of (8.15) with $r=2$. The results illustrate AMBSA for the rather rare case of a model with no MA component. Here,

[9] In general, when the model of Z_t has a total AR polynomial $\varphi(B)$ (including any differencing operator) and an MA polynomial with no zeroes of order greater than one, it can be shown that decay rate τ^j begins at $j=\deg\varphi(B)+1$ with τ equal to the maximum magnitude of the reciprocals of the zeroes of the MA polynomial $\Theta(z)$. For example, in the MA(1) case, $\Theta(z)=1-\theta z$ is zero for $z=\theta^{-1}$, whose reciprocal is θ.

unless the data interval is short, symmetric filters can be applied in an interior interval and asymmetric filters are required only near the two ends of the series. So one wants to be aware of differences in the properties of the two kinds of filters.

From (8.16),

$$g_Z(\lambda)^{-1} = \sigma_a^{-2}\left|1-e^{i2\pi 2\lambda}\right|^2 = \sigma_a^{-2}\left|1+e^{i2\pi\lambda}\right|^2\left|1-e^{i2\pi\lambda}\right|^2. \tag{15.1}$$

Multiplication into (8.17) provides the filter transfer functions of the canonical decomposition,

$$\beta_s\left(e^{i2\pi\lambda}\right) = \frac{1}{16}\left|1-e^{i2\pi\lambda}\right|^4,\ \ \beta_p\left(e^{i2\pi\lambda}\right) = \frac{1}{16}\left|1+e^{i2\pi\lambda}\right|^4,\ \ \beta_u\left(e^{i2\pi\lambda}\right) = \frac{1}{8}\left|1-e^{i2\pi 2\lambda}\right|^2, \tag{15.2}$$

with $\beta_{sa}\left(e^{i2\pi\lambda}\right) = 1-\beta_s\left(e^{i2\pi\lambda}\right)$ for the seasonal adjustment filter.

15.1. The Symmetric Filters

For the symmetric filters, (15.2) yields the length 5 formulas

$$\beta_s(B) = \frac{1}{16}|1-B|^4 = \frac{1}{16}\left(B^2-4B+6-4B^{-1}+B^{-2}\right). \tag{15.3}$$

$$\beta_p(B) = \frac{1}{16}|1+B|^4 = \frac{1}{16}\left(B^2+4B+6+4B^{-1}+B^{-2}\right). \tag{15.4}$$

$$\beta_{sa}(B) = 1-\beta_s(B) = \frac{1}{16}\left(-B^2+4B+10+4B^{-1}-B^{-2}\right).$$

$$\beta_u(B) = \frac{1}{8}\left(1-B^2\right)\left(1-B^{-2}\right) = \frac{1}{8}\left(-B^2+2-B^{-2}\right). \tag{15.5}$$

For $n \geq 6$, these filters produce estimates for times $3 \leq t \leq n-2$.

15.2. The Asymmetric Filters

Forecasts of the series values at times $n+1$ and $n+2$ are needed for component estimates at $t = n-1, n$. So are backcasts for $t = 1, 2$. We illustrate with odd n, $n = 2m+1$, for which $\hat{Z}_{2m+2} = Z_{2m}$, $\hat{Z}_{2m+3} = Z_{2m+1}$ and $\hat{Z}_{-1} = Z_1$, $\hat{Z}_0 = Z_2$ are needed for component estimates. The resulting filters are asymmetric. Subsection 8.2 of Findley et al. (2016) derives these results and the asymmetric filter formulas for the initial year and final year. Only final year filters are displayed below. We identify forecasts $\hat{Z}$ only for $\hat{u}_{2m}$, $\hat{u}_{2m+1}$ and $\hat{s}_{2m+1}$.

$$\begin{aligned}
\hat{u}_{2m} &= \frac{1}{8}\left\{-Z_{2m-2}+2Z_{2m}-\hat{Z}_{2m+2}\right\} = \frac{1}{8}\left\{-Z_{2m-2}+Z_{2m}\right\} = \frac{1}{8}\left(1-B^2\right)Z_{2m}. \\
\hat{u}_{2m+1} &= \frac{1}{8}\left\{-Z_{2m-1}+2Z_{2m+1}-\hat{Z}_{2m+3}\right\} = \frac{1}{8}\left\{-Z_{2m-1}+Z_{2m+1}\right\} = \frac{1}{8}\left(1-B^2\right)Z_{2m+1}.
\end{aligned}$$

$$\begin{aligned}
\hat{s}_{2m+1} &= \frac{1}{16}\left\{Z_{2m-1}-4Z_{2m}+6Z_{2m+1}-4\hat{Z}_{2m+2}+2\hat{Z}_{2m+3}\right\} \\
&= \frac{1}{16}\left\{B^2-8B+7\right\}Z_{2m+1} = \frac{1}{16}(7-B)(1-B)Z_{2m+1}.
\end{aligned} \tag{15.6}$$

$$\begin{aligned}
\hat{s}_{2m} &= \frac{1}{16}\left\{B^2-4B+7-4B^{-1}\right\}Z_{2m}=\frac{1}{16}\left(-B+3-4B^{-1}\right)(1-B)\,Z_{2m}.\\
\widehat{sa}_{2m} &= \frac{1}{16}\left\{-B^2+4B+9+4B^{-1}\right\}Z_{2m}=\frac{1}{16}\left(-B+5+4B^{-1}\right)(1+B)\,Z_{2m},\\
\widehat{sa}_{2m+1} &= \frac{1}{16}\{-B^2+8B+9\}Z_{2m+1}=\frac{1}{16}(9-B)(1+B)\,Z_{2m+1}.\\
\hat{p}_{2m} &= \frac{1}{16}\left\{B^2+4B+7+4B^{-1}\right\}Z_{2m}=\frac{1}{16}\left(B+3+4B^{-1}\right)(1+B)\,Z_{2m}.\\
\hat{p}_{2m+1} &= \frac{1}{16}\left\{B^2+8B+7\right\}Z_{2m+1}=\frac{1}{16}(7+B)(1+B)\,Z_{2m+1}. \qquad (15.7)
\end{aligned}$$

Since $\left(1-B^2\right)=(1-B)(1+B)$, the factored formulas on the right for both types of filters show that the trend and seasonal differencing operator factors of an asymmetric filter are of lower degree than those of the symmetric filters. For example, whereas the factorization $\beta_s(B)=B^{-2}(1-B)^4$of (15.3) shows that $\beta_s(B)$ can annihilate a cubic trend to estimate the seasonal s_t, the concurrent filters of $\hat{s}_{2m}$ and $\hat{s}_{2m+1}$ involving forecasts and backcasts can only annihilate a constant mean.

16. Differencing Operators of General ARIMA Filters

In the nonstationary case, with $\delta(B)$ denoting the differencing operator of the ARIMA model of Z_t, the denominator factors $\left|\delta\left(e^{i2\pi\lambda}\right)\right|^2$ in (7.1) give rise to factors of $\delta(B)\delta(B^{-1})$ in the filters (13.2). For example, if $\delta(B)=(1-B)(1+B)=\delta_p(B)\delta_s(B)$ as in the SRW_2 , then $\beta_s(B)$ and $\beta_u(B)$ will contain the factor $(1-B)\left(1-B^{-1}\right)=B^{-2}(1-B)^2$, which will annihilate a linear trend $a+bt$ because differencing lowers the degree of a polynomial by one,
e.g., $(1-B)\,t^2=t^2-(t-1)^2=2t-1$. Consequently, the trend filter $\beta_p(B)=1-\beta_s(B)-\beta_u(B)$ will preserve such a trend without change.

The SRW_2 asymmetric filter only has a single $1-B$ factor, as the filter formulas of Section 15 show. Therefore only a constant level term is annihilated by $\beta_s(B)$ and $\beta_u(B)$ and preserved by $\beta_p(B)$. Similarly, in the asymmetric case, only the first power of $(1+B)$ occurs in $\beta_p(B)$. The filter $\beta_u(B)$ annihilates a period $r=2$ deterministic seasonal component $a(-1)^t$ which is preserved by $\beta_s(B)$.

16.1. What Seasonal Decomposition Filters Annihilate or Preserve

The tables of Bell (2012, 2015) cover the annihilation and preservation properties of more general differencing operators and also several generations of symmetric and asymmetric X-11 filters. Table 2 of Bell (2012) summarizes the main results for the practical case of asymmetric filters.

17. Canonical Decomposition and Smoothing Trade-Offs*

White noise has neither seasonal features nor the smooth properties expected of a trend or any other component of interest for cyclical analysis. Therefore specifying all components other than the irregular in a way that makes them white noise free (after differencing if needed), as the canonical decomposition does, is appropriate. Hillmer and Tiao (1982) shows that the ARIMA models of the canonical seasonal and the canonical trend have smaller one-step-ahead forecast error variances than all noncanonical models for these components. Thus the seasonal and trend components are more predictable one period ahead, a kind of increased smoothness – something that always has a cost. Corollary 3 of Maravall (1986) shows that each canonical non-irregular component has a larger revision variance than a noncanonical specification for the component. Mostly summarizing Maravall, this is the price paid for cleaning the signal of white noise. It is an instance of an important phenomenon mentioned in Section 10: greater smoothing is usually associated with some kind of increased statistical instability. Another example: the AMBSA trend is a smoothed version of the AMBSA seasonal adjustment, see Findley et al. (2016), and concurrent trend estimates generally have statistically larger revisions than concurrent seasonal adjustments. Maravall and Planas (1999) analyze features and costs of an interesting variety of conceptually less simple alternatives to the canonical p-sd specification.

18. Structural Models: A Trend Estimation Example

The models known as *Structural Time Series Models* offer a different approach to component estimation by directly specifying models of a simple form for the components and jointly estimating the parameters of their sum, which is the implied model for the data. See Harvey and Koopman (2000). This approach requires less modeling background and modeling effort than ARIMA modeling, possibly at the cost of reduced forecast performance and goodness of fit compared to what can be obtained from direct ARIMA modeling of the data. However, unless parameter estimation fails, this approach always yields an admissible p-sd decomposition.

We illustrate with the simplest structural trend model for nonstationary Z_t. This prescribes a random walk trend p_t^* and a white noise irregular N_t^*,

$$\begin{aligned} Z_t &= p_t^* + N_t^* \\ (1-B)\,p_t^* &= b_t^*, \end{aligned} \tag{18.1}$$

with mutually uncorrelated white noise N_t^* and b_t^*. The parameters to be estimated are σ_b^2 and $q = \sigma_N^2/\sigma_b^2$. For $w_t = (1-B)\,Z_t$, the prescriptions result in $w_t = b_t^* + (1-B)\,N_t^*$ being a stationary process with $\gamma_0 = \sigma_{b^*}^2 + 2\sigma_{N^*}^2$, $\gamma_{\pm 1} = -\sigma_{N^*}^2$ and $\gamma_k = 0$, for $|k| > 1$. Thus w_t is an MA(1) process, $w_t = a_t - \theta a_{t-1}$, with a negative lag one autocorrelation $\rho_1 = \gamma_1/\gamma_0 = -\sigma_N^2\left(2\sigma_{N^*}^2 + \sigma_a^2\right)^{-1}$. Since also $\rho_1 = -\theta/\left(1+\theta^2\right)$, necessarily $\theta > 0$. Thus a differently parameterized constrained IMA(1,1) model with the constraint $\theta > 0$ is prescribed. Its trend model (18.1) differs from that of the canonical decomposition (8.7).

This ARIMA model representation is the *reduced form* of the Structural Model. Structural Models always have a parameter-constrained ARIMA reduced form whose parameters can be determined from the autocovariances. See Section 19 and, for this example, also Subsection 19.1. The reduced form is not needed to derive the likelihood function and is not usually of interest.

Unfortunately, Structural Models tend to have parameter estimation problems, see Bell (1993) for example. Eurostat does not recommend the use of Structural Models for seasonal adjustment.

19. Spectral Factorization

A nonzero function of the form

$$Q(\lambda) = \gamma_0 + \sum_{j=1}^{q} \gamma_j \left(e^{i2\pi j\lambda} + e^{-i2\pi j\lambda} \right), \ q \geq 1 \tag{19.1}$$

that is nonnegative for all λ is the spectral density of an MA(q) process with autocovariances γ_j, $0 \leq j \leq q$. More specifically,

$$Q(\lambda) = \sigma^2 \left| 1 - \sum_{j=1}^{q} \theta_j e^{i2\pi j\lambda} \right|^2, \tag{19.2}$$

with MA(q) polynomial $\theta(z) = 1 - \sum_{j=1}^{q} \theta_j z^j$ such that $\theta(z) \neq 0$ for $|z| < 1$, a property that uniquely determines the coefficients θ_j and σ^2, see Findley (2012). The determination of $\theta(z)$ from (19.1) is known as the *spectral factorization of* $Q(\lambda)$. It can be accomplished by constructing a polynomial of degree q, scaled to have $\theta(0) = 1$ at $z = 0$, whose zeroes are the zeroes of $\gamma_0 + \sum_{j=1}^{q} \gamma_j \left(z^j + z^{-j} \right)$ having $|z| \geq 1$ (usually found by numerical methods if $q > 1$). Some AMBSA software does not use spectral factorization to calculate component estimates, but uses it for other purposes, e.g., to express the ARIMA or ARMA models of the bi-infinite seasonal, trend and irregular estimators in a familiar form for diagnostic purposes. The efficient algorithm derived in Appendix A of Maravall and Mathis (1994) is used. (In formula (A.4) of this reference, z_j should be be z_j^{-1}.)

19.1. The MA(1) Case

When $q = 1$, the quadratic formula provides (19.2): For an MA(1) $Z_t = a_t - \theta a_{t-1}$ with $|\theta| \leq 1$, the coefficient θ and the first order autocorrelation $\rho_1 = \gamma_1/\gamma_0 = -\theta/\left(1 + \theta^2\right)$ are connected by the properties that $\theta = 0$ if and only if $\rho_1 = 0$ and, for $\rho_1 \neq 0$, $\theta^2 + \rho_1^{-1}\theta + 1 = 0$. Thus θ satisfies

$$\theta = \frac{-\rho_1^{-1} \pm \sqrt{\rho_1^{-2} - 4}}{2}.$$

For $\theta(z) = 1 - \theta z$, the requirement that $\theta(z) \neq 0$ for $|z| < 1$ is equivalent to $|\theta| \leq 1$, which determines the $\pm$ choice. Finally $\gamma_0 = \left(1 + \theta^2\right)\sigma_a^2$ yields $\sigma_a^2 = \left(1 + \theta^2\right)^{-1}\gamma_0$.

20. Regarding AMBSA Software Model Choices and Decomposition Components

Terminology. An ARIMA model is *balanced* if the numerator and denominator functions on the right in (7.1) have the same degree, ($\deg \vartheta = \deg \delta + \deg \varphi$), *bottom heavy* if the denominator has larger degree, ($\deg \delta + \deg \varphi > \deg \vartheta$), and top heavy if the numerator has larger degree ($\deg \vartheta > \deg \delta + \deg \varphi$).

The automatic ARMA or ARIMA modeling procedures of AMBSA software are biased toward balanced models because these more often have p-sds with admissible decompositions. With top heavy p-sds, the partial fraction decomposition used to obtain their components always yields an additive moving average component in addition to a balanced component, see Wikipedia Contributors (2011). Among the p-sd examples of Section 7, (7.3) is bottom heavy, and (7.5) and (7.7) are balanced. Model (6.3) of Hillmer and Tiao (1982) is a top heavy model whose canonical p-sd decomposition is shown by the authors to be admissible for a range of parameter values. With balanced and bottom heavy models, the nature of the p-sd decomposition is determined by a factorization of the total autoregressive polynomial $\varphi(B) = \delta(B)\,\phi(B)\,\Phi(B^r)$ and a corresponding decomposition of the p-sd $\varphi(z)$.

21. Additional Components of Some AMBSA Software Decompositions

For ARIMA seasonal time series, the differencing operator usually has the form

$$\delta(B) = (1 - B)^d (1 - B^r) = (1 - B)^{d+1} U(B), d \geq 0,$$

with even-length seasonal period $r \geq 2$ and $U(z) = 1 + \sum_{j=1}^{r-1} z^j$. Assuming $d + D > 0$, the zero $z = 1$ of $(1 - z)^{d+D}$ is the *trend unit root*. The $r - 1$ zeroes of $U(z)$ are the *seasonal unit roots*. These always include $z_{r/2} = -1$ and for $r > 2$, also $z_k = e^{i2\pi k/r}$, $k = \pm 1, \ldots, \pm(r/2 - 1), r/2$.

The associated functions $z_k(\lambda) = e^{i2\pi\lambda(k/r)}$, $k = 0, \pm, 1 \ldots, r/2$ are periodic, repeating $|k|$ times a year.

21.1. Stationary Components

When the ARIMA model has a stationary autoregressive polynomial $\varphi(z) = \phi(z)\Phi(z^r)$ with zeroes close to the seasonal unit roots and/or to the trend root, and/or certain other unit roots, it is decomposed as $\varphi(z) = \phi_{seas}(z)\phi_{trend}(z)\phi_{other}(z)$, with any factor set equal to 1 when there are no zeroes that qualify for inclusion of the factor. The magnitude of "close" is defined by a software default, often a user-changeable magnitude. With these AR factors, the p-sd decomposition can include a seasonal component p-sd with denominator $\left|\varphi_{seas}\left(e^{i2\pi\lambda}\right)\right|^2$, where $\varphi_{seas}\left(e^{i2\pi\lambda}\right) = U(e^{i2\pi\lambda})\phi_{seas}\left(e^{i2\pi\lambda}\right)$, a trend component p-sd with denominator $\left|\varphi_{trend}\left(^{i2\pi\lambda}\right)\right|^2$, where $\varphi_{trend}\left(e^{i2\pi\lambda}\right) = \left(1 - e^{i2\pi\lambda}\right)^{d+1}\phi_{trend}\left(e^{i2\pi\lambda}\right)$, and possibly a spectral density for a stationary component not connected to trend or seasonal, with denominator $\left|\phi_{other}\left(e^{i2\pi\lambda}\right)\right|^2$, which is generally called a *transitory component*, with $\varphi_{trans}\left(e^{i2\pi\lambda}\right) = \phi_{other}\left(e^{i2\pi\lambda}\right)$ in the total AR notation. However, in the monthly case, if $\phi_{other}(z)$ is of degree 2 and has complex zeroes with arguments λ close in magnitude to the main trading frequency 0.348 cycles/month, i.e. $|\lambda| \doteq \pm 0.348$, then it is called a stochastic trading day component, with $\varphi_{td}\left(e^{i2\pi\lambda}\right) = \phi_{trans}\left(e^{i2\pi\lambda}\right)$. If $\phi_{other}(z)$ is of degree 3 and contains such a degree 2 factor, then it is factored as $\phi_{other}(z) = \varphi_{td}(z)\varphi_{trans}(z)$ resulting in denominator factors $\left|\varphi_{td}\left(e^{i2\pi\lambda}\right)\right|^2$ and $\left|\varphi_{trans}\left(e^{i2\pi\lambda}\right)\right|^2$. Then, with the irregular, there are five possible stationary decomposition components.

21.2. A Trend-Cycle Decomposition Option for Long Series

If a monthly ARIMA series has an estimated nonstationary trend component $\hat{p}_t$ of length at least least ten years, then by (changeable) default, some AMBSA software automatically applies a Hodrick-Prescott (HP) filter (see Wikipedia (2017c)), here denoted $H(B)$, to the trend estimate $\hat{p}_t$ extended by forecasts and backcasts. The result is an estimate $\hat{C}_t = H(B)\hat{p}_t$ of a stationary cycle C_t and the estimate $\hat{T}_t = \hat{p}_t - \hat{C}_t$ of the nonstationary long-term trend T_t. Expressed in terms of transfer functions, the trend filter transfer function $g_p\left(e^{i2\pi\lambda}\right)$ is decomposed as the sum of transfer functions $g_C\left(e^{i2\pi\lambda}\right) = \left|H\left(e^{i2\pi\lambda}\right)\right|^2 g_p\left(e^{i2\pi\lambda}\right)$ and $g_T\left(e^{i2\pi\lambda}\right) = g_p\left(e^{i2\pi\lambda}\right) - g_C\left(e^{i2\pi\lambda}\right)$.

Kaiser and Maravall (2001) and Maravall (2005) give further background, as do McElroy (2006) and McElroy and Holan (2012), who provide matrix formulas for MMSE finite-sample estimates and their mean square errors, and also results of simulation experiments. Trend-Cycle decompositions are also available with other observation frequencies, e.g. quarterly data, from some AMBSA software for sufficiently long series. Wikipedia (2017c) discusses limitations of HP filters for cycle extraction, but doesn't consider the situation in which, as here, an ARIMA model for the $\hat{p}_t$ can be derived from which any needed forecasts and backcasts can be be obtained.

Many users of seasonally adjusted data are interested in detecting cyclical movements in the adjusted data. But validation and interpretation of cycle estimates is not part of the discipline of seasonal adjustment. It belongs to a less developed discipline requiring substantial data knowledge and practical training as well as technical knowledge. It is not amenable to high volume production.

22. Model-Based SA versus X-11 Filter SA

Maravall and Peréz (2012) illustrates the application by experts of AMBSA to an important economic indicator at a time of economic instability. It was stimulated by preceding results obtained at a different central bank with different software using X-11 filter estimates. Its focus is not the comparison of estimation methods but rather the versatility of the tools available in the AMBSA software and the greater versatility of the model-based approach. It cannot be assumed *a priori* that model-based filters will provide a better seasonal adjustment than X-11 filters (for X-11 details see Ladiray and Quenneville (2001)). Although AMBSA has a wider range of filters, the results are often very similar, as the filters can also be, see Bell, Chu and Tiao (2012). What is clearly advantageous about the AMBSA approach with an admissible decomposition is that it specifies ARIMA models and other properties of the canonical seasonal and nonseasonal components and their estimates. This provides a context for the seasonally adjusted series that is rich in auxiliary information of interest, for example statistical precision, not only for seasonal adjustments (for X-11 adjustments, see Bell and Kramer (1999) and Scott, Pfeffermann and Sverchkov (2012)) but also for derived quantities, such as seasonally adjusted growth rates, and covariances of component estimates, from which quality diagnostics can be derived, see Maravall (1987) and Findley, McElroy and Wills (2005). The X-11 approach has no such rich and coherent context. In particular, its traditional adjustment quality diagnostics are *ad hoc* and difficult to validate.

Advantageous features of the X-11 filter method include the directness of its time-tested multiplicative decomposition procedure, which avoids the level bias of log-additive adjustment, and the conceptual simplicity of its filters and iterative procedure (if its complicated extreme value procedure is not considered, for which the outlier identification and adjustment procedure of AMBSA software is a possible substitute). This simplicity makes it easier to explain SA to non-experts and reduces the time series background and amount of training required for new users, compared to AMBSA.

Modern software makes it easy to obtain and compare AMBSA and X-11 method adjustments. When they are close, confidence in each adjustment is increased.

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