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Measurement Error in Small Area Estimation: Functional Versus Structural Versus Naïve Models

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ABSTRACT

Small area estimation using area-level models can sometimes benefit from covariates that are observed subject to random errors, such as covariates that are themselves estimates drawn from another survey. Given estimates of the variances of these measurement (sampling) errors for each small area, one can account for the uncertainty in such covariates using measurement error models (e.g., Ybarra and Lohr, 2008). Two types of area-level measurement error models have been examined in the small area estimation literature. The functional measurement error model assumes that the underlying true values of the covariates with measurement error are fixed but unknown quantities. The structural measurement error model assumes that these true values follow a model, leading to a multivariate model for the covariates observed with error and the original dependent variable. We compare and contrast these two models with the alternative of simply ignoring measurement error when it is present (naïve model), exploring the consequences for prediction mean squared errors of use of an incorrect model under different underlying assumptions about the true model. Comparisons done using analytic formulas for the mean squared errors assuming model parameters are known yield some surprising results. We also illustrate results with a model fitted to data from the U.S. Census Bureau's Small Area Income and Poverty Estimates (SAIPE) Program.

Key Words: sample survey, area level model, covariate, prediction

1 Introduction

Linear mixed models, particularly that of Fay and Herriot (1979), have gotten great attention in small area estimation. The Fay-Herriot (FH) model can be written

$$Y_i = \theta_i + e_i \qquad \theta_i = z_i'\delta + u_i \qquad i = 1, \ldots, m \tag{1}$$

where, for areas i indexed from 1 to m, the Y_i are direct survey estimates of population quantities θ_i, the sampling errors e_i in Y_i are assumed independent $N(0, D_i)$ with the D_i taken as known (they are actually estimated using survey micro-data), the z_i are $q \times 1$ vectors of regression covariates with corresponding coefficient vector δ, and the random effects u_i are distributed *i.i.d.* $N(0, \sigma_u^2)$ and independently of the e_i.

In some cases it may be desired to augment the model for θ_i with one or more covariates X_i that are themselves estimates taken from another survey that estimates characteristics believed to be related to θ_i. One approach is to simply ignore the sampling error in X_i, treating it like the covariates in z_i which we shall assume are not subject to sampling or other measurement errors. We shall call this the *naïve Fay-Herriot model*, which, taking for simplicity the case of a single such covariate X_i, we write as

$$Y_i = \theta_i + e_i \qquad \theta_i = \beta_N X_i + z_i'\delta_N + u_{i,N}. \tag{2}$$

We add the "N" subscripts to the regression coefficients and the random effects ($u_{i,N}$) to distinguish this model from the measurement error models to come. The model assumes that $u_{i,N} \sim$ *i.i.d.* $N(0, \sigma_{u,N}^2)$, although with heteroscedastic sampling error in X_i the assumption that $\text{var}(u_{i,N})$ is constant is incorrect, implying that the model (2) is misspecified. This point is discussed further below.

An alternative to the naïve FH model is to use a measurement error model to account for the sampling (measurement) error in X_i. Assume x_i denotes the population characteristic being estimated by X_i with sampling error η_i, where the η_i are assumed distributed independently $N(0, C_i)$ and with the C_i taken as known (actually estimated using survey micro-data). A generalization of the model (1) to include the covariate X_i while accounting for its sampling error is

$$Y_i = \theta_i + e_i \qquad \theta_i = \beta x_i + z_i'\delta + u_i \tag{3}$$

$$X_i = x_i + \eta_i. \tag{4}$$

If the x_i are assumed to be fixed unknown quantities, then the model defined by (3)–(4) is known as the *functional measurement error model* (FME model). This model is discussed by Fuller (1987) and has been studied for small area estimation by Ybarra and Lohr (2008), Arima, Datta, and Liseo (2015, 2016), and Arima, et al. (2017). Analogous unit level measurement error models for small area estimation have been studied by Ghosh and Sinha (2007), Datta, Rao, and Torabi (2010), and Arima, Datta, and Liseo (2012).

Another alternative to the naïve FH model is to specify a model for x_i in (4) which, with (3), implies bivariate models for $(\theta_i, x_i)'$ and $(Y_i, X_i)'$. This is known as a *structural measurement error model* (SME model). If x_i follows the regression model $x_i = z_{xi}'\delta_x + v_i$, with covariates z_{xi} and residuals $v_i \sim i.i.d.\ N(0, \sigma_v^2)$ independent of u_i, then the resulting model for $(Y_i, X_i)'$ can be written as

$$\begin{bmatrix} Y_i \\ X_i \end{bmatrix} = \begin{bmatrix} \theta_i \\ x_i \end{bmatrix} + \begin{bmatrix} e_i \\ \eta_i \end{bmatrix} \qquad \begin{bmatrix} e_i \\ \eta_i \end{bmatrix} \sim i.i.d.\ N(0, \Omega) \quad \Omega = \begin{bmatrix} D_i & 0 \\ 0 & C_i \end{bmatrix} \tag{5}$$

$$= \left(\begin{bmatrix} z_i' & \beta z_{xi}' \\ 0 & z_{xi}' \end{bmatrix} \begin{bmatrix} \delta \\ \delta_x \end{bmatrix} + \begin{bmatrix} u_i + \beta v_i \\ v_i \end{bmatrix} \right) + \begin{bmatrix} e_i \\ \eta_i \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} u_i + \beta v_i \\ v_i \end{bmatrix} \sim i.i.d.\ N(0, \Sigma) \quad \Sigma = \begin{bmatrix} \sigma_u^2 + \beta^2 \sigma_v^2 & \beta \sigma_v^2 \\ \beta \sigma_v^2 & \sigma_v^2 \end{bmatrix}. \tag{7}$$

This model differs from a standard bivariate FH model in that the parameter β affects both the regression mean function for Y_i and the random effect covariance matrix Σ. However, if the covariates z_{xi} are linear functions of the covariates z_i, then the fixed effects regression part of (6) can be reparameterized to unrestricted linear regression effects $[z_i'\delta_y \;\; z_{xi}'\delta_x]'$ with regression covariates z_i for the first equation and z_{xi} for the second. With this reparameterization β no longer affects the regression fixed effects, so the matrix Σ can then be reparameterized in the general form $\Sigma = [\sigma_{jk}]$, or by σ_{11}, σ_{22}, and $\rho = \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}} \in [-1, 1]$, as there is now a 1-1 correspondence between $(\sigma_u^2, \sigma_v^2, \beta)$ and $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ or $(\sigma_{11}, \sigma_{22}, \rho)$. Two instances where this condition on z_{xi} holds are (i) if the regression covariates are the same in both equations ($z_{xi} = z_i$), or (ii) if z_{xi} is just an intercept term ($z_{xi} = 1$) and z_i also includes an intercept.

Datta, et al. (2018) study the area level SME model, while Huang and Bell (2012) present a study examining use of general bivariate models for small area estimation. Analogous unit level models have been studied by Ghosh, Sinha, and Kim (2006) and Torabi, Datta, and Rao (2009). Fuller (1987) and Buonaccorsi (2010) discuss additional measurement error models including nonlinear models and the Berkson model.

Note that the FME and SME models model the relation between the true unobserved quantities θ_i and x_i, whereas the naïve FH model models the relation between θ_i and the observed X_i. The X_i contain noise in the form of generally heteroscedastic sampling error, and this heteroscedasticity produces the naïve model's misspecification noted earlier.

If Y_i and X_i are estimates from the same survey sample their sampling errors e_i and η_i are likely to be correlated. This can be accommodated by replacing the off-diagonal 0 of Ω in (5) by the appropriate $\text{cov}(e_i, \eta_i)$ (estimated using survey micro-data). While this works for the SME model, correlation between e_i and η_i implies that the regressor X_i and sampling error e_i are correlated, violating a fundamental assumption of the FH model and causing potentially severe problems for the naïve FH model. Hence, we do not consider that situation here.

In this paper we compare the three alternative models – naïve FH, functional measurement error, and structural measurement error – focusing on their predictive performance for small area estimation. One motivating case involves the use of the naïve FH model when measurement error (η_i) is present, comparing the naïve FH model's predictive accuracy with those of the other two models. We also compare the predictive performance of the functional versus structural measurement error models. We make these comparisons using analytic formulas for the mean squared errors (MSEs) for the case where model parameters are known (first order approximations). This provides good approximations for the case when the number of areas m is large. It is also relevant as the typically dominant term in the MSEs for smaller values of m. Since the naïve FH model is misspecified, we make precise in what sense its parameters are "known".

Section 2 summarizes some theoretical results for the three alternative models, first on convergence of parameter estimates and then on small area prediction, covering both the point predictors and their MSEs. We provide results for the three models first for the case where the FME model is true, and then for the case where the SME model is true. Derivations of these results are deferred to the Appendix. Section 3 compares, via contour plots, the theoretical MSEs of small area predictors for the three models across ranges of the parameters of a true SME model. Section 4 uses the theoretical MSE formulas to compare prediction MSEs from the three models when they are applied to an empirical example of modeling poverty rates of school-age children for U.S. counties. The example is taken from the U.S. Census Bureau's Small Area Income and Poverty Estimates (SAIPE) program. Section 5 then gives general conclusions.

2 Theoretical Results

To facilitate interpretation of the results, here we use the simplest possible versions of the models outlined in the Introduction, specifically, models where the vector of non-measurement error covariates reduces to just an intercept term, i.e., $z_i = 1$. To revert to fairly standard notation, we use α for the intercept coefficient instead of δ, so the simplified model for θ_i in the FME and SME models (from (3)) becomes

$$\theta_i = \alpha + \beta x_i + u_i. \tag{8}$$

For the SME model we assume that $x_i \sim i.i.d.\ N(\mu, \sigma_x^2)$ so there are no regression terms other than the mean μ in the model for x_i.

For the naïve FH model (2), the simplified model for θ_i becomes

$$\theta_i = \alpha_N + \beta_N X_i + u_{i,N} \tag{9}$$

where, as before, we use the "N" subscript to distinguish the coefficients and random effects in the naïve model (9) for θ_i, since this model differs from (8) by substituting X_i in place of x_i.

We now give some results on parameter estimation and small area prediction for these models. The Appendix provides derivations of these results.

2.1 Parameter estimators and their large sample limits

The Appendix details unbiased estimating equations for the parameters of the three models considered here. The resulting estimators of α, β, and σ_u^2 for the simplified versions of the FME and SME models are the same, and are given by

$$\begin{aligned} \hat{\beta} &= \frac{\frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X})(Y_i - \bar{Y})}{\frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X})^2 - \bar{C}} \\ \hat{\alpha} &= \bar{Y} - \hat{\beta}\bar{X} \\ \hat{\sigma}_u^2 &= \frac{1}{m}\sum_{i=1}^{m}(Y_i - \hat{\alpha} - \hat{\beta}X_i)^2 - \bar{D} - \hat{\beta}^2\bar{C} \end{aligned} \tag{10}$$

where $\bar{X} = m^{-1}\sum_{i=1}^{m} X_i$, with analogous definitions of $\bar{Y}$, $\bar{C}$, and $\bar{D}$. For fitting the SME model, we also have $\hat{\mu} = \bar{X}$ and $\hat{\sigma}_x^2 = \frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X})^2 - \bar{C}$. Result 1 gives the probability limits of all these parameter estimators.

Result 1: For the FME and SME models given by (3)–(4) and by (5)–(7), respectively, but with the simplified model for θ_i as in (8), the parameter estimators given in (10) are consistent for the true model parameters, that is,

$$\hat{\beta} \xrightarrow{P} \beta \qquad \hat{\alpha} \xrightarrow{P} \alpha \qquad \hat{\sigma}_u^2 \xrightarrow{P} \sigma_u^2 \tag{11}$$

where $\xrightarrow{P}$ denotes convergence in probability as $m \to \infty$ under the true model (whether FME or SME). For fitting the SME model when it is true, we also have $\hat{\mu} \xrightarrow{P} \mu$ and $\hat{\sigma}_x^2 \xrightarrow{P} \sigma_x^2$. For fitting the SME model when the FME model is true, we have $\hat{\mu} \xrightarrow{P} \bar{x} = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m x_i$ and $\hat{\sigma}_x^2 \xrightarrow{P} s_x^2 = \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^m (x_i - \bar{x})^2$, with both limits assumed to exist.

Remark 1: The estimators in (10) are the same for the two models despite being obtained from different estimating equations – see equations (19) and (20) versus equations (36)–(38) in the Appendix. The consistency results in (11) thus hold whether the true model is the FME or the SME. These consistency results also hold for the more general versions of these models considered in the Appendix.

The parameter estimators for fitting the naïve FH model with the simplified model for θ_i as in (9) are given by

$$\begin{aligned} \hat{\beta}_N &= \frac{\frac{1}{m}\sum_{i=1}^m (X_i - \bar{X})(Y_i - \bar{Y})}{\frac{1}{m}\sum_{i=1}^m (X_i - \bar{X})^2} = \frac{\frac{1}{m}\sum_{i=1}^m (X_i - \bar{X})^2 - \bar{C}}{\frac{1}{m}\sum_{i=1}^m (X_i - \bar{X})^2} \times \hat{\beta} \\ \hat{\alpha}_N &= \bar{Y} - \hat{\beta}_N \bar{X} \\ \hat{\sigma}_{u,N}^2 &= \frac{1}{m}\sum_{i=1}^m (Y_i - \hat{\alpha}_N - \hat{\beta}_N X_i)^2 - \bar{D} \end{aligned} \tag{12}$$

Result 2: When the FME model (or SME model) is true, the parameter estimators in (12) have the following probability limits:

$$\hat{\beta}_N \xrightarrow{P} a\beta \qquad \hat{\alpha}_N \xrightarrow{P} \alpha + (1-a)\beta\bar{x} \qquad \hat{\sigma}^2_{u,N} \xrightarrow{P} \sigma^2_u + a\beta^2\bar{C} \qquad \text{(FME model true)}$$

$$\hat{\beta}_N \xrightarrow{P} a_*\beta \qquad \hat{\alpha}_N \xrightarrow{P} \alpha + (1-a_*)\beta\mu \qquad \hat{\sigma}^2_{u,N} \xrightarrow{P} \sigma^2_u + a_*\beta^2\bar{C} \qquad \text{(SME model true)}$$

where the "attenuation factors" are given by

$$a = \frac{s_x^2}{s_x^2 + \bar{C}} \quad \text{(FME model true)} \qquad\qquad a_* = \frac{\sigma_x^2}{\sigma_x^2 + \bar{C}} \quad \text{(SME model true)}. \qquad (13)$$

Remark 2: The results for convergence of $\hat{\beta}_N$ in Result 2 are versions of the well-known attenuation of the estimate of the regression parameter when measurement error is ignored – see Theil (1971, p. 608) for the FME case and Fuller (1987, p. 3) for the SME case. The limit for $\hat{\sigma}^2_{u,N}$ shows that the naïve FH model inflates the estimate of the model error variance σ^2_u for the true model (whether FME or SME) by the amount $a\beta^2\bar{C}$ (or $a_*\beta^2\bar{C}$) due to the failure of the naïve model to account for the measurement error.

2.2 Small area predictors and their MSEs

Result 3 lists the formulas for the predictors of θ_i for the three models. Note that any of these formulas will apply whenever the corresponding model is assumed and used for prediction, regardless of whether the true model is the FME, the SME, or some other model. The FME predictor $\hat{\theta}_{i,F}$ is given by Theorem 1 of Ybarra and Lohr (2008), while the naïve FH predictor $\hat{\theta}_{i,N}$ is simply the predictor of Fay and Herriot (1979) for the case of our simplified naïve FH model. Derivations for the more general models are given in the Appendix.

Result 3: The predictors of θ_i from the simple versions of the FME, SME, and naïve FH models considered here are as follows:

$$\begin{aligned}
\text{FME predictor} &: \quad \hat{\theta}_{i,F} = Y_i - \frac{D_i\left\{Y_i - \hat{\alpha} - \hat{\beta}X_i\right\}}{D_i + \hat{\sigma}_u^2 + \hat{\beta}^2 C_i} \\
\text{SME predictor} &: \quad \hat{\theta}_{i,S} = Y_i - \frac{D_i\left\{Y_i - \hat{\alpha} - \hat{\beta}X_i + \hat{\beta}(C_i/(\hat{\sigma}_x^2 + C_i))(X_i - \bar{X})\right\}}{D_i + \hat{\sigma}_u^2 + \hat{\beta}^2 C_i \hat{\sigma}_x^2/(\hat{\sigma}_x^2 + C_i)} \\
\text{naïve FH predictor} &: \quad \hat{\theta}_{i,N} = Y_i - \frac{D_i\left\{Y_i - \hat{\alpha}_N - \hat{\beta}_N X_i\right\}}{D_i + \hat{\sigma}_{u,N}^2}.
\end{aligned}$$

These are the empirical versions of the optimal predictors (best linear unbiased predictors) under their respective assumed models.

Remark 3: Several special cases are worth noting from these results. First, as $D_i \to 0$ all the predictors converge to the direct survey estimate Y_i, and since its sampling variance is then going to 0, all the predictors achieve design consistency assuming that Y_i is itself design consistent. Second, if $C_i = \bar{C}$ it can be shown that the SME and naïve FH predictors agree while the FME predictor generally remains different. (The Appendix shows that, for the more general model considered there, the SME and naïve FH predictors agree asymptotically when $C_i = \bar{C}$.) Third, it can be seen that as $\hat{\sigma}_x^2 \to \infty$ the SME predictor converges to the FME predictor. The same holds as $C_i \to 0$, which implies in the limit that x_i is known. We can put these together and say that the SME and FME predictors behave similarly when C_i/σ_x^2 is small.

Remark 4: It can be shown that the formula for $\hat{\theta}_{i,S}$ can be obtained by taking the formula for $\hat{\theta}_{i,F}$ and replacing X_i in the numerator of the fraction by $E(x_i|X_i) = X_i - E(\eta_i|X_i) = X_i - [C_i/(\hat{\sigma}_x^2 + C_i)](X_i - \bar{X})$, and $C_i = \text{var}(X_i - x_i)$ in the denominator of the fraction by $\text{var}(x_i|X_i) = C_i\hat{\sigma}_x^2/(\hat{\sigma}_x^2 + C_i)$, these being the conditional mean and variance of x_i given X_i under the estimated model.

Table 1 gives, for the case when the FME model is true, the first order biases and prediction error variances of the three predictors. The MSEs are then the squared biases plus the variances. (The prediction error variance, and thus the MSE, for the FME model is given by Theorem 1 of Ybarra and Lohr (2008).) The results assume the true FME model parameters are known, but for the naïve FH model they account for the fact that the estimates of the parameters are biased as shown in Result 2. This gives a realistic approximation for the case when m, the number of areas, is large. The Table 1 entries for the SME model use the quantity

$$F_i = (\sigma_u^2 + D_i)(s_x^2 + C_i) + \beta^2 s_x^2 C_i. \tag{14}$$

Table 1. Biases and prediction error variances when the FME model is true

Prediction model	**bias**	**prediction error variance**
FME	0	$\frac{(\sigma_u^2+\beta^2 C_i)D_i}{\sigma_u^2+\beta^2 C_i+D_i}$
SME	$\frac{-\beta D_i C_i}{F_i}(x_i - \bar{x})$	$D_i - \frac{D_i^2[(s_x^2+C_i)+\beta^2 s_x^2 C_i^2/F_i]}{F_i}$
naïve FH	$\frac{-\beta D_i(1-a)}{\sigma_u^2+a\beta^2\bar{C}+D_i}(x_i - \bar{x})$	$\frac{(\sigma_u^2+a\beta^2\bar{C})D_i}{\sigma_u^2+a\beta^2\bar{C}+D_i} + \left(\frac{\beta D_i}{\sigma_u^2+a\beta^2\bar{C}+D_i}\right)^2 a(aC_i - \bar{C})$

Table 2 gives the results for the case when the SME model is true. In this case all the predictors are unbiased in the sense that $E(\hat{\theta}_i - \theta_i) = 0$. Hence, the table just gives the prediction error variances, which are also the MSEs. For F_i^* in Table 2, we substitute σ_x^2 for s_x^2 in the expression (14), analogous to the definition of a_* in (13).

Table 2. Prediction error variances when the SME model is true

Prediction model	**prediction error variance = MSE**
FME	$\frac{(\sigma_u^2+\beta^2 C_i)D_i}{\sigma_u^2+\beta^2 C_i+D_i}$
SME	$D_i - \frac{D_i^2(\sigma_x^2+C_i)}{F_i^*}$
naïve FH	$\frac{(\sigma_u^2+a_*\beta^2\bar{C})D_i}{\sigma_u^2+a_*\beta^2\bar{C}+D_i} + \left(\frac{\beta D_i}{\sigma_u^2+a_*\beta^2\bar{C}+D_i}\right)^2 a_*^2(C_i-\bar{C})$

Several points are worth noting about the results of Tables 1 and 2.

1. The results for the FME predictor are the same in both cases, i.e., whether the FME or SME model is true. To achieve unbiasedness under the assumption that the x_i are fixed, unknown quantities, the FME predictor eliminates them from the prediction error. Hence, its prediction error results are not affected by whether the x_i actually are fixed and unknown or are random variables following some distribution, as the SME model assumes.
2. When the FME model is true, the biases of the SME and naïve FH predictors are proportional to $(x_i - \bar{x})$, which is unconstrained, and so can be arbitrarily large in magnitude. Hence, for areas where $|x_i - \bar{x}|$ is large the squared bias can dominate the prediction MSE. Since the x_i are unobserved it will typically be difficult to estimate the squared bias (unless the C_i are small so the X_i are very good estimators of the x_i, in which case the motivation to use a measurement error model diminishes).
3. The MSEs in Table 2 for the SME and naïve models can be obtained by taking the expressions for squared bias plus prediction error variance from Table 1, substituting σ_x^2 for s_x^2 and also for $(x_i - \bar{x})^2$, and simplifying. This is the difference between assuming the x_i fixed and unknown versus assuming $x_i \sim i.i.d.\ N(\mu, \sigma_x^2)$.
4. As noted earlier, if an area has $C_i = \bar{C}$ then the SME and naïve FH predictors agree. Hence, when $C_i = \bar{C}$ the biases, error variances, and MSEs of the SME and

naïve FH predictors are the same. If the SME model is true then the SME predictor is optimal, and thus so is the naïve FH predictor for areas with $C_i = \bar{C}$, in which case both are superior to the FME predictor. In fact, comparing the MSEs of the naïve and FME predictors from Table 2, and given that $0 \leq a_* < 1$, one can show directly that the FME predictor's MSE is larger when $C_i = \bar{C}$.

5. When the SME model is true, for areas with $C_i = \bar{C}$ the "reported MSE" for the naïve FH model will agree with the true MSE. The reported MSE is the MSE one would compute assuming the naïve model to be true, and is given by the first term in the naïve FH MSE expression in Table 2. The second term is obviously zero when $C_i = \bar{C}$, and is positive when $C_i > \bar{C}$. We thus see that when $C_i > \bar{C}$ the reported MSE understates the true MSE, while when $C_i < \bar{C}$ the second term in the MSE is negative so the reported MSE overstates the true MSE. The misspecification of the naïve FH model when the SME model is true can thus lead to substantial misstatement of the MSEs except for areas for which C_i is close to $\bar{C}$.

An implication of points 4 and 5, and the analogous result stated earlier for the point predictors, is that if $C_i = \bar{C}$ for all $i = 1, \ldots, m$, then the prediction results for the SME and naïve FH models are the same. This provides some basis for the statement sometimes made that measurement error in covariates doesn't affect model prediction. Put another way, this statement is true only if the C_i are constant for all areas, and only when comparing prediction results for the naïve FH model to those for the SME model. Prediction results for the FME model will be different.

3 Comparing MSEs for the Alternative Predictors

We now compare the performance of the three alternative model predictors when applied to data from a true SME model, making such comparisons across a range of values for the model parameters and the D_i and C_i values. The comparisons use the MSE results of Table 2, examining percentage differences in the MSEs calculated, for example, as

$$100\left(\frac{MSE_F}{MSE_S}-1\right) \tag{15}$$

for comparing MSEs of the FME and SME predictors. We similarly define the analogs to (15) for comparing MSEs for the naïve FH and FME predictors, and of the naïve FH and SME predictors, as well as for comparing the reported and actual MSEs of the naïve FH predictor. Assuming that the SME model is true facilitates the comparisons. Assuming that the FME model is true leads to the complication that the MSEs for the structural and naïve models depend on x_i (Table 1), which has unrestricted variation over areas. Section 4 nonetheless makes some MSE comparisons under a true FME model.

For making relative comparisons as in (15) the scale of the data doesn't matter, so rescaling to Y_i/σ_u will not affect these comparisons. This is also true for rescaling X_i to X_i/σ_x. These rescalings reduce the number of varying parameters we need to consider by two, which lets us express $1/\sigma_u^2$ times MSE_F, MSE_S, MSE_N, and $\widehat{MSE}_N$ (the reported MSE for the naïve FH predictor), all computed assuming the SME model is true, in terms of the following four scale independent quantities:

$$r_D=\frac{D_i}{\sigma_u^2}, \qquad r_C=\frac{C_i}{\sigma_x^2}, \qquad \rho=\operatorname{corr}(\theta_i,x_i), \qquad \bar{r}_C=\frac{\bar{C}}{\sigma_x^2}. \tag{16}$$

The Appendix illustrates such re-expression for the calculation of MSE_S/σ_u^2. To simplify the notation, we omit the i subscript from r_D and r_C, though except in the unusual situation where the D_i and C_i are actually constant over areas, one needs to compute the MSE expressions separately for each area i. To compare MSEs we examine contour plots over (r_D, r_C) for each of the MSE percentage differences defined as in (15), viewing the MSE percentage difference as a function of (r_D, r_C). We examine such plots for fixed values of ρ and $\bar{r}_C$ (which do not vary over i).

Figure 1 gives contour plots of (15) for $\rho = 0.3$ and $\rho = 0.7$. The x- and y-axes of the plots, representing the values of r_D and r_C, range from 0.1 to 10, and are shown with log scaling. We need not set $\bar{r}_C$ for these comparisons because MSE_F and MSE_S do not depend on $\bar{C}$. The results in the plots are easy to summarize: the percentage differences are all positive, favoring the SME model which here is assumed to be true, and the differences increase with both r_D and r_C, so that the more sampling or measurement error is present, the larger is the advantage to use of the SME predictor. When either r_D or r_C is small, say generally below 1, the MSE percentage differences are small, which is why no contours show up plotted in this area, and choice of model has little effect on prediction accuracy. In fact, when both r_D and r_C are sufficiently small the FME and SME predictors are both close to the direct estimator Y_i, leading to small MSE differences, a pattern repeated in subsequent graphs. Towards the upper right corner the MSE percentage differences become substantial in both graphs, larger for $\rho = 0.7$. Analysis of the formula for MSE_F/MSE_S reveals that, for given values of r_C and r_D, the MSE percent differences increase with $\rho > 0$ to the point where $\rho = [1 + r_C/\sqrt{(1 + r_C)(1 + r_D)}]^{-.5}$, and then they decline to 0 as ρ increases to 1. Over the range of values $[1, 10]$ for r_C and r_D, this maximum point varies from about $\rho = 0.57$ to $\rho = 0.91$. Note that the results for ρ and

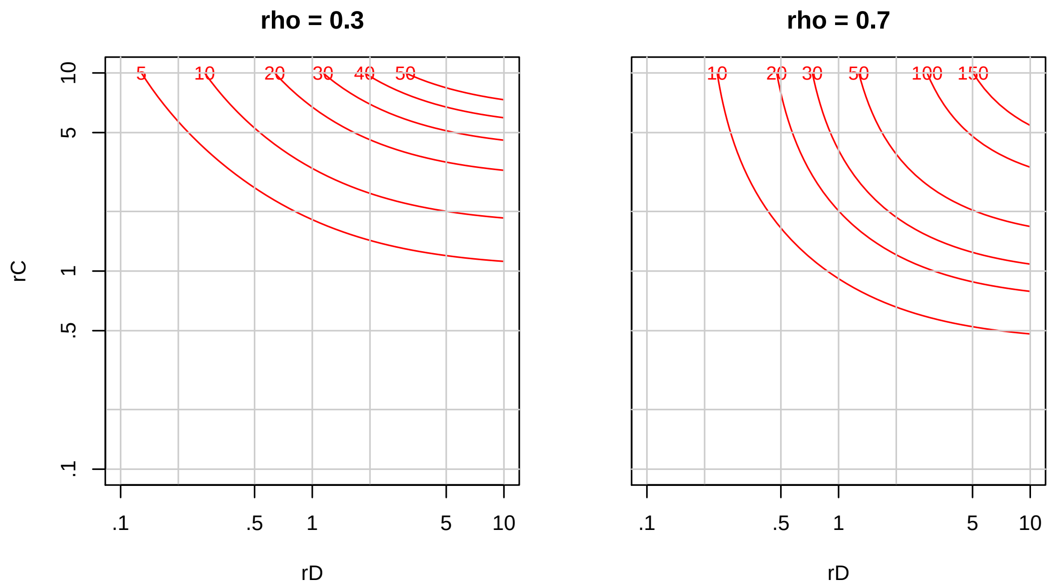

Figure 1: Contours of $100\,(MSE_F/MSE_S - 1)$ for two values of ρ when the SME model is true.

for $-\rho$ would be the same since the MSEs actually depend on ρ^2.

For the other MSE comparisons, which are shown in Figures 2-4, the percentage differences depend on all four quantities in (16). To get a general idea of how the comparisons vary, we take $\rho = 0.7$ as a representative value, and examine contour plots for $\bar{r}_C = 0.1$, 1, and 10. It is worth noting that the analogous plots done for $\rho = 0.3$, 0.5, and 0.9, not shown here, present similar patterns, though with the patterns generally shifted somewhat in location on the plots, and typically with contours representing lower or higher percentage differences.

Figure 2 shows contour plots of $100\,(MSE_N/MSE_F - 1)$, comparing MSEs for the naïve and FME predictors. In these plots we see both positive and negative contours, indicating regions where the FME predictor does better, and other parts where the naïve FH predictor does better. The patterns in these plots can be understood by keeping in

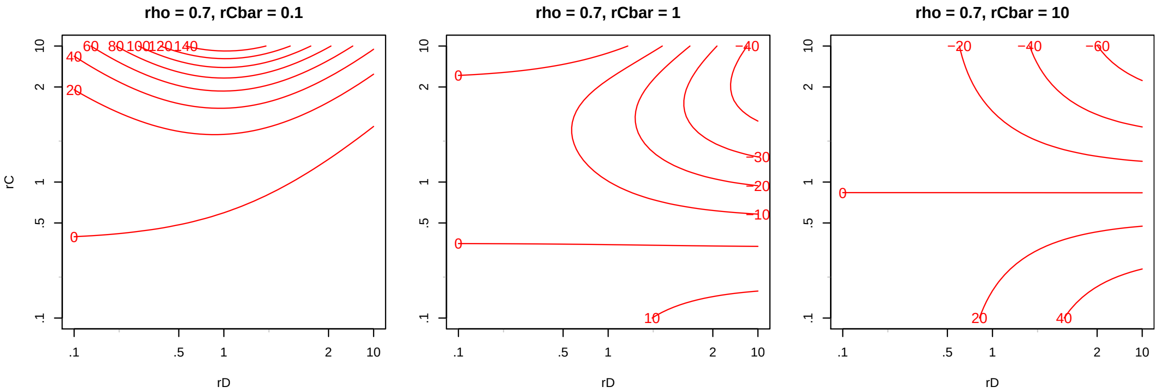

Figure 2: Contours of $100\,(MSE_N/MSE_F - 1)$ for $\rho = 0.7$ and $\bar{r}_C = 0.1, 1,$ and 10 when the SME model is true.

mind that (i) for small values of r_C the FME predictor acts like the SME predictor, which here is optimal, so the FME predictor performs well, and (ii) for r_C close to $\bar{r}_C$ the naïve FH predictor acts like the SME predictor and so performs well. Thus, in the plot for $\bar{r}_C = 0.1$, both the FME and naïve FH predictors perform similarly to the optimal SME predictor for small values of r_C, so there is little difference in their MSEs. Apart from this case where both perform well, the naïve FH predictor performs better when r_C is sufficiently close to $\bar{r}_C$, where the meaning of "sufficiently close" depends on the values of $\bar{r}_C$ and r_D.

The results in Figure 2 showing that for certain regions the naïve FH predictor has lower prediction MSE than the FME predictor may at first seem surprising given that, when the SME model is true, the naïve model is misspecified since it ignores the measurement error in X_i. In contrast, the FME model accounts for the measurement error in X_i and, since it makes no assumptions about the x_i, it is not inconsistent with the true SME model. In fact, as we move towards larger amounts of measurement error overall (larger values of $\bar{r}_C$), the MSE advantages of the naïve FH predictor become more substantial

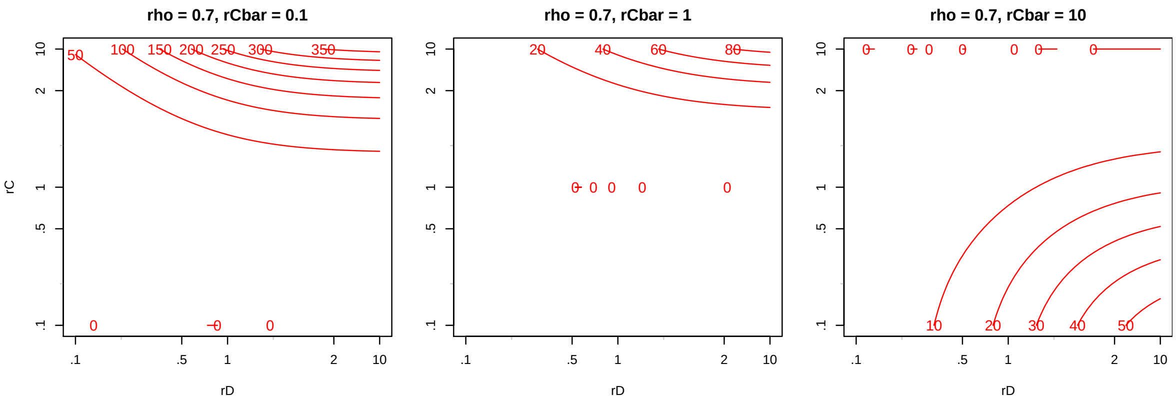

Figure 3: Contours of $100\,(MSE_N/MSE_S - 1)$ for $\rho = 0.7$ and $\bar{r}_C = 0.1, 1,$ and 10 when the SME model is true.

and cover larger ranges of the r_C and r_D values. The general explanation for this is that, when measurement error is substantial, the FME model's avoidance of any modeling assumptions about the x_i can lead to rather inefficient use of the data X_i, while the naïve FH predictor makes suboptimal but better use of the X_i unless r_C is very different from $\bar{r}_C$ (equivalently, C_i is very different from $\bar{C}$.)

Figure 3 gives contour plots of $100\,(MSE_N/MSE_S - 1)$, comparing MSEs for the naïve and SME predictors. Since the SME model is assumed true for the purposes of these computations, all the contours shown are positive, with the exception of a zero line in each plot (represented here by the contour plotting function of R (R core team 2016) as a set of "0" labels not joined by a line). These zero contours occur as horizontal lines for $\bar{r}_C = 0.1$, 1, and 10 on the three plots, these being where $r_C = \bar{r}_C$, implying $C_i = \bar{C}$, which is when the naïve FH and SME predictors agree. Apart from this, the plots for $\bar{r}_C = 0.1$ and 1 show substantial positive contours for large values of r_C that also increase with r_D, while for $\bar{r}_C = 10$ the substantial positive contours occur for small r_C as r_D grows large.

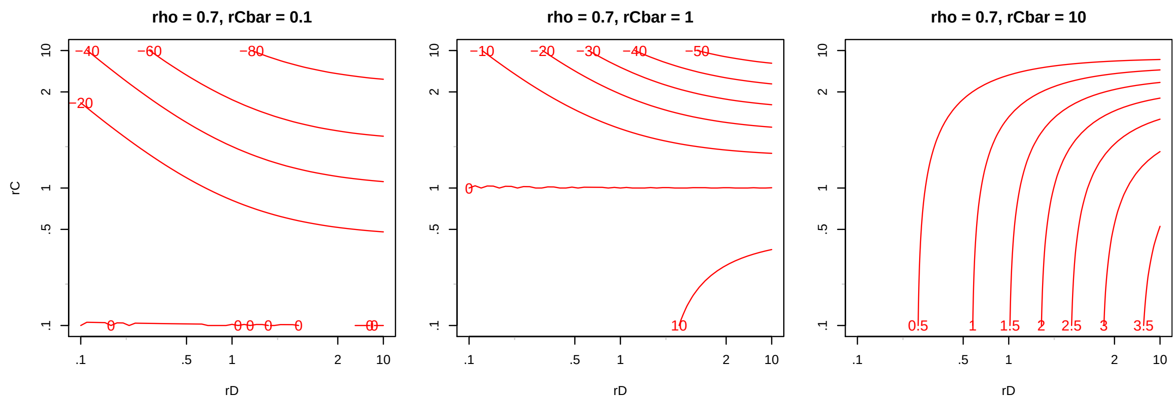

Figure 4: Contours of $100\left(\widehat{MSE}_N/MSE_N - 1\right)$ for $\rho = 0.7$ and $\bar{r}_C = 0.1, 1,$ and 10 when the SME model is true.

Figure 4 gives contour plots of $100\left(\widehat{MSE}_N/MSE_N - 1\right)$, comparing the reported and actual MSEs for the naïve FH predictor. As with Figure 3, the three plots should show zero contours at the values $\bar{r}_C = 0.1$, 1, and 10, respectively (which are poorly represented in the first two plots, and absent from the third). In these plots the regions above the zero contours have negative values that reflect understatement of the true MSE by the reported MSE, while the regions below the zero contours have positive values that reflect overstatement of the true MSE. The first two plots show regions for $r_C > \bar{r}_C$ with significant understatement of the true MSE, while the second two reflect at most very minor overstatement of the true MSE when $r_C < \bar{r}_C$. This pattern remains when the axis ranges are expanded to include larger values of r_D and r_C. While further extrapolation of these results to more general cases than those considered here is questionable, they nonetheless suggest that understatement of MSE by the naïve FH model may be a potentially more serious problem than overstatement.

4 SAIPE Illustration

The previous section compared the performance of the three alternative model predictors across a range of values for the model parameters and the D_i and C_i values for a true SME model. Here we take a model developed for an important small area application – modeling county poverty rates of school-age children for U.S. counties – to determine realistic values of the model parameters and the D_i and C_i values. We take the fitted model as a true model, and then use the theoretical formulas from Section 2 to compare small area prediction MSEs for the three alternative model predictors. We do this using the fitted SME model as truth, and then repeat the exercise using the corresponding FME model as truth. For the latter we simulate the true covariate values x_i from the fitted SME model, since the prediction biases and MSEs depend on these values which are not observed. We emphasize that our objective here is not in producing county poverty estimates; we use the poverty rate data merely to get a realistic model for illustrating the results from Section 2.

We fit the SME model to estimates of poverty rates of school-age children for U.S. counties from the American Community Survey, or ACS (U.S. Census Bureau 2014), the largest U.S. household survey. ACS produces annual estimates based on one year or five years of data collection. Here, we use 2010 ACS one-year estimates of county poverty rates of school-age children as the primary response variable Y_i. We center the analogous 2005-2009 ACS five-year estimates, and treat them as a covariate X_i which is subject to measurement error. We also use covariates from administrative records as the covariates z_i not subject to measurement error. These are drawn from two sources – tabulations of income tax records obtained under an agreement with the U.S. Internal

Revenue Service, as well as recipient counts from the Supplemental Nutrition Assistance Program, a program that provides food subsidies to low income households. The specific covariates used are the same as those of Arima et al. (2017), though that paper jointly modeled two years of poverty rates using a multivariate FME model. All covariates used here are centered about their means. The model we use here is similar to models applied to such data by Bell et al. (2007), and is related to the county production model used by the SAIPE Program. SAIPE produces poverty estimates at the state, county, and school district level for different age groups, including the school-age group 5-17. For more information about SAIPE, see Bell et al. (2016) or the SAIPE web page at https://www.census.gov/programs-surveys/saipe.html/.

We fitted the SME model to the poverty rate data via maximum likelihood using R (R core team 2016) to obtain values of the parameters defining our "true model." This yielded $\hat{\sigma}_u^2 = 0.0012$, $\hat{\sigma}_x^2 = 0.0064$, and $\hat{\beta} = 0.407$. These parameter values imply that $\hat{\rho} = \hat{\beta}[\hat{\sigma}_x^2/(\hat{\sigma}_u^2 + \hat{\beta}^2\hat{\sigma}_x^2)]^{.5}$ (see Appendix) is about 0.7. We omit the estimates of the other model parameters since they do not affect the first order MSE calculations done here.

For the D_i and C_i values we used estimates from a Generalized Variance Function (GVF, see Wolter, 1985) developed for the sampling variances of the 2010 one-year and 2005-2009 five-year ACS county school-age poverty rate estimates, respectively. The specifics of the GVF are described in Franco and Bell (2013). After the SME model fitting, but for use when computing the MSEs, the D_i and C_i values were altered to protect against their disclosure by adding zero mean bivariate normal noise to the $\log(D_i)$ and $\log(C_i)$ values, and exponentiating the results. The noise terms added to the D_i and C_i had a correlation of 0.5 and variances of $2/n_i$ and $2/5n_i$, respectively, where the n_i are the 2010 one-year ACS county sample sizes. Thus, more noise was added to the

$\log(D_i)$ than to the $\log(C_i)$, and more noise was added for counties with smaller sample sizes. The resulting D_i values range from about 0.00005 to 0.12 with a median of 0.0046, while C_i ranges from 7×10^{-6} to 0.013 with a median of 0.0009. Resulting values of the ratios $r_D = D_i/\hat{\sigma}_u^2$ range from about 0.04 to 100 with a median of about 4, and values of $r_C = C_i/\hat{\sigma}_x^2$ range from about 0.001 to 2 with a median of about 0.14. The noise altered values still provide a practically plausible range of values for the D_i and C_i, and the general appearance of the plots that follow was not materially changed by the noise infusion.

Figure 5 panels (a)–(c) display ratios comparing first order approximations of the three model predictors' MSEs plotted against C_i on the log scale, with a vertical line at $\bar{C} = 0.0014$ shown for reference. Panel (a) shows the ratios of MSEs for the SME and naïve predictors. We note that, due to their optimality under the assumed SME model, the SME model predictors always have lower prediction MSEs than the naïve model predictors. Because the C_i's are strongly related to the D_i's (with a correlation of about 0.9), for small C_i's the D_is are also likely to be small, and all three model predictors are then approximately equal to the direct estimators, so that the MSEs of the naïve and SME predictors are similar. We will see this trend in all four panels of Figure 5. In panel (a), the ratio reaches its maximum of approximately one when $C_i \approx \bar{C}$. This agrees with a result given in Remark 3 of Section 2.2, where we noted that the two predictors are equivalent at this point. For $C_i > \bar{C}$ the ratios decline rapidly to values approaching 30% larger MSEs for the naïve predictors.

Panel (b) shows the ratios of the SME and FME predictors' MSEs. Again, the SME predictor performs best since we are assuming the SME model is true. For $C_i < \bar{C}$ the MSE differences are small, but the differences become pronounced for high values of C_i,

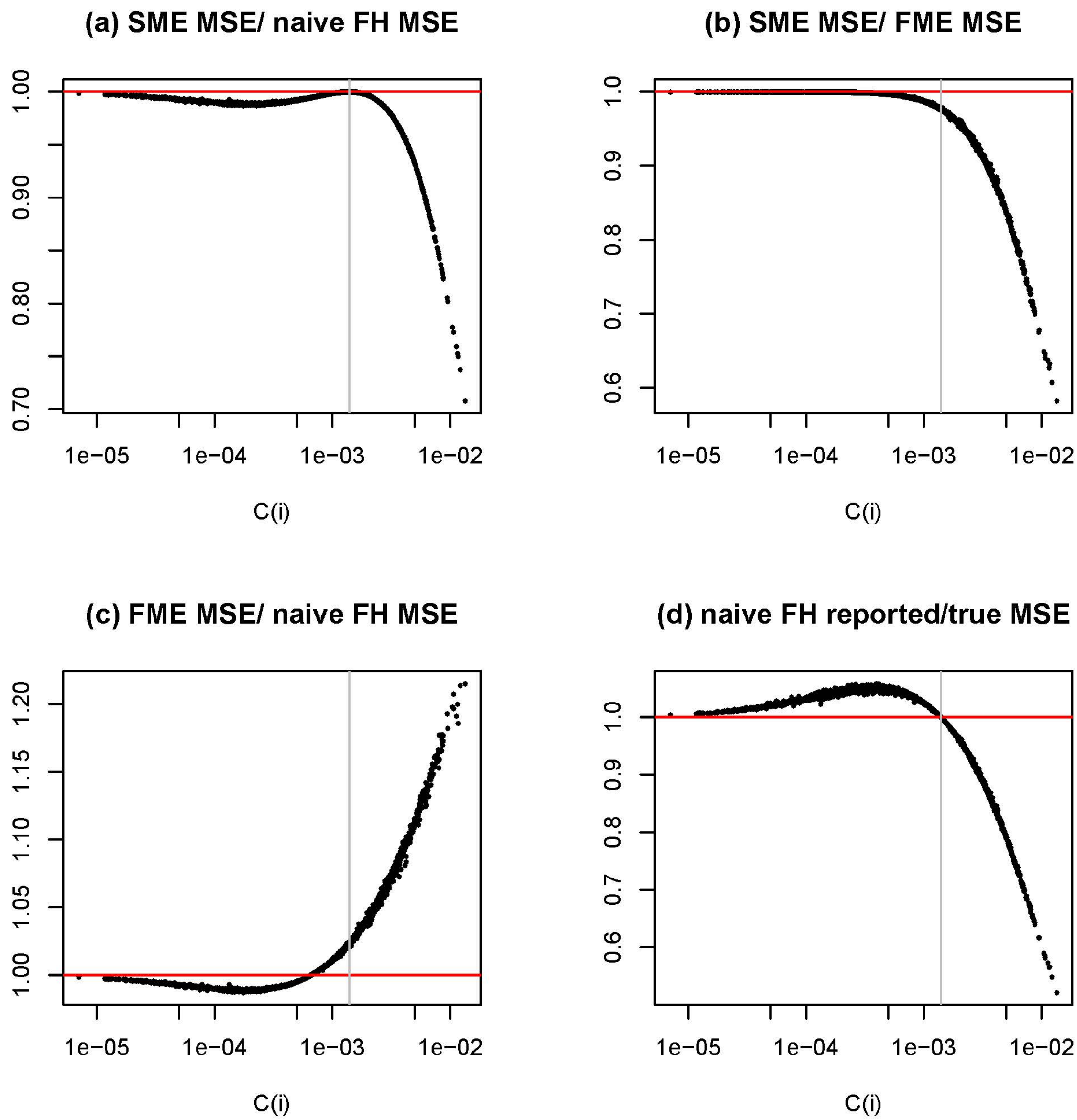

Figure 5: First order approximations of MSE ratios plotted against C_i for the U.S. county school-age children in poverty model when the SME model is true. For panels (a)-(c), the ratios are of the true MSEs of the SME and naive, SME and FME, and FME and naive models, respectively. Panel (d) shows the ratios of the reported MSEs and the true MSEs of the naive model. The vertical lines mark $\bar{C}$.

with the FME predictor MSEs up to 40% or more higher than those for the SME predictor.

Panel (c) shows the corresponding MSE ratios for the FME and naïve FH predictors. The naïve predictor has slightly higher MSEs than the FME predictor for small C_i but lower MSEs for large C_i's, a pattern expected from the results in panels (a) and (b). The two predictors' MSEs are approximately equal at some value of $C_i < \bar{C}$. The FME predictor's MSEs are larger by about 20% or more than the naïve predictor's MSEs for the largest C_i values.

Note that the MSE that is obtained for the FME predictor when the SME model is actually true is still correct to the first order, though the FME predictor is not optimal. However, the MSE obtained assuming the naïve model is true, what we call the "reported" MSE, differs from the naïve model predictor's true MSE. Panel (d) plots the ratios of the first order approximations of the reported and true MSEs of the naïve model predictor when the SME model is true. As noted in Section 2.2, the naïve model overstates the MSEs for small C_i's and understates them for large C_i's, while correctly estimating the MSE at $C_i = \bar{C}$. The overstatement for $C_i < \bar{C}$ is relatively small, less than 10%, while the understatement for $C_i > \bar{C}$ becomes large, increasing with increasing C_i to more than 40%.

One might argue that the SME model is more reasonable than the FME model for this application, because if one is willing to assume a model for the true poverty rates as measured by the ACS 2010 estimates, why not assume a model for the true five-year average poverty rates as essentially measured by the ACS 2005–2009 estimates? Still, it is of interest to investigate the performance of each of the predictors when the FME model holds. This presents a further challenge because the true x_i's are not known. For this illustration, we generate them as $x_i \stackrel{i.i.d.}{\sim} N(0, \hat{\sigma}_x^2)$, and then treat these as the true values. (Recall that we centered the X_i values so that $E(x_i) \equiv \hat{\mu} = \bar{X} = 0$.) For the FME model parameters we used the estimates obtained from fitting the SME model since the parameter estimators we developed in (10) agree for the FME and SME models. While we have no explicit proof, we expect the ML parameter estimators used in this illustration would converge for $m \to \infty$ to the same quantities for both the FME and SME models.

Figure 6 panels (a)–(d) are analogous to Figure 5, but assume for the first order approximations that the FME model is true. Panel (a) plots the ratios of the SME and FME predictor MSEs. Although our assumption that the FME model is true makes the FME predictor "optimal," for many counties it actually performs worse than the SME predictor with respect to the MSE. This is because the FME predictor's optimality is in the class of unbiased predictors, and both the SME and the naïve predictors are biased, so there is no mathematical contradiction. The difference in MSEs can be up to about 50% in either direction. However, there are relatively few points for which the SME MSE is more than 20% higher than the FME MSE, while there are a substantial number where the SME MSE is more than 20% lower than the FME MSE. Computing the bias and variance terms of the MSE separately reveals that for this application when the SME predictor performs worse than the FME predictor in panel (a) it is due to the bias of the former.

Panel (b) of Figure 6 plots ratios of the MSEs of the SME and naïve predictors against C_i. It shows that when the FME model is true, the SME predictor sometimes performs better and sometimes performs worse than the naïve predictor in terms of MSE. The same statement can be made about the functional and naïve predictors based on panel (c), which shows the ratios of the FME and naïve MSEs plotted against C_i. However, as C_i increases beyond $\bar{C}$ the FME MSE is almost always higher than the naïve predictor's MSE.

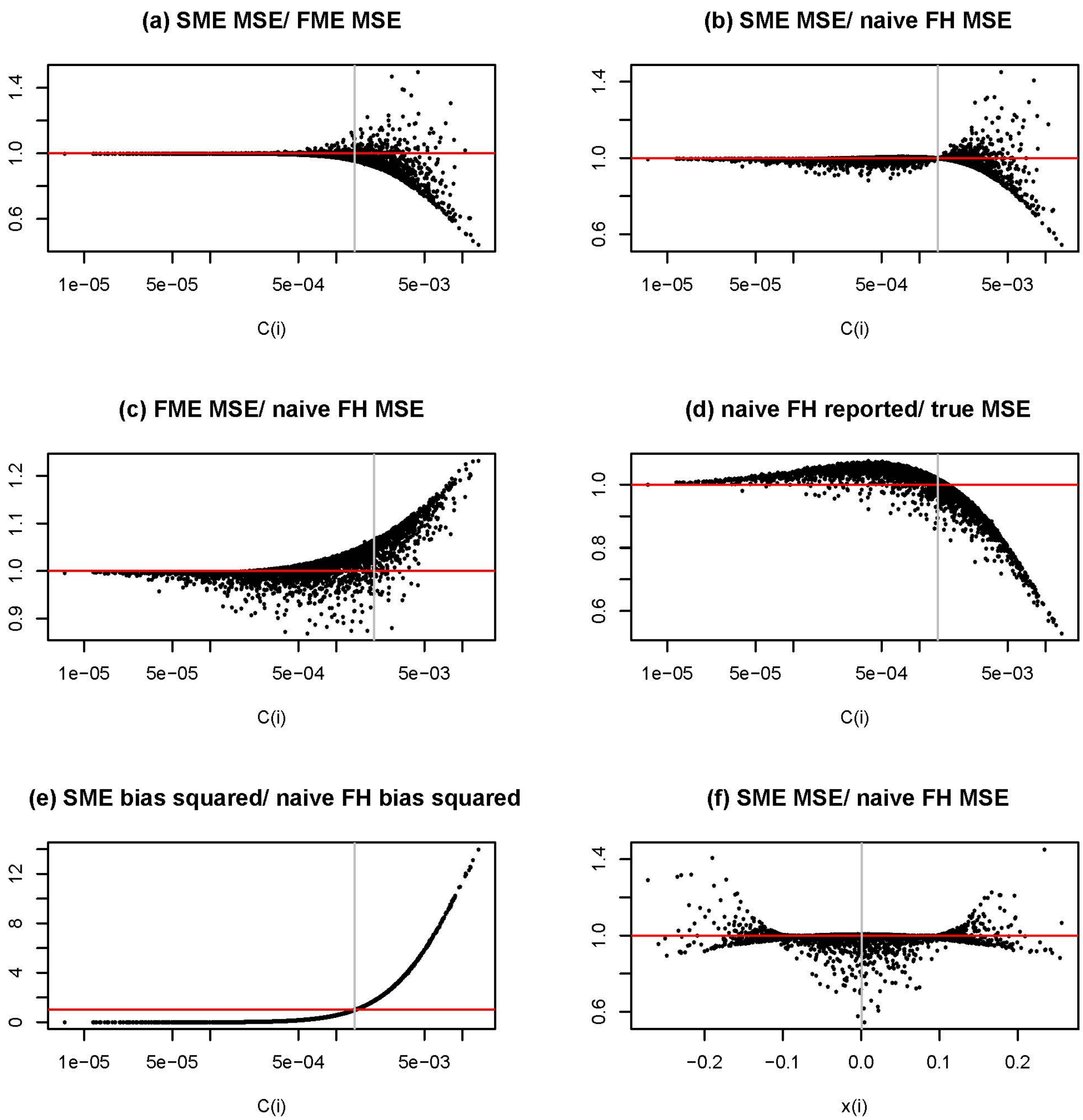

Figure 6: First order approximations of MSE and bias squared ratios for the U.S. county school-age children in poverty model when the FME model is true, plotted against C_i or x_i. Panels (a)-(c) show the ratios of the true MSEs of the SME and FME, SME and naive, and FME and naive models, respectively. Panel (d) shows the ratios of the reported MSEs and the true MSEs of the naive model. Panel (e) shows the ratios of the biases squared of the SME and naive models. All panels plot the ratios against C_i except panel (f), which plots the ratio of the true MSEs of the SME and naive models against x_i. The vertical lines mark $\bar{C}$ or $\bar{x}$, as appropriate.

Panel (d), which plots the ratio of reported to true MSE of the naïve predictor, reminds us that the naïve model will misstate the mean squared error, sometimes overstating it and sometimes understating it. The overstatement is small, up to about 10%, but the understatement is more considerable, up to and beyond 40%. Overstatement is most likely for $C_i < \bar{C}$ and understatement for $C_i > \bar{C}$, though these tendencies do not hold for every county (as they do for the SME model) due to the variations in the squared bias terms under the FME model.

Since both the SME and naïve predictors are biased under the FME model, we can analyze the relationship between their respective biases. Panel (e) of Figure 6 shows the ratio of the bias squared of the SME predictors and the naïve predictors. It shows that the SME predictor has lower squared bias for $C_i < \bar{C}$, and higher squared bias for $C_i > \bar{C}$, with equality when $C_i = \bar{C}$, where the two predictors are equal. This suggests that the extreme points in the top right quadrant in panel (b) are due to the bias. The specific realization of x_i in our generation of the data will also influence these extreme points. Panel (f) plots the ratio of the SME and naïve true MSEs plotted against x_i. The vertical line represents the mean of x_i, which is approximately 0 due to how the x_i's were generated. Note that the extreme points in the top quadrants have high deviations of x_i from its mean. On the other hand, the most extreme points in the bottom quadrants correspond to values where x_i is close to $\bar{x}$. This suggests large deviations of x_i's from $\bar{x}$ will have more impact on the true MSEs of the SME predictors than on those of the naïve predictors. For the majority of points, however, the SME model's MSEs are lower than those of the naïve model based on our first order approximations.

5 Conclusions

This paper considered three models proposed for small area estimation when one or more regression covariates are measured with error: the functional and structural measurement error models (FME and SME), and the naïve Fay-Herriot model. Section 2 established certain theoretical results for these models about parameter estimation, their small area predictions, and their corresponding prediction biases, error variances, and MSEs. This led to several observations relating the models including (i) the naïve and SME model predictions and MSEs agree, at least asymptotically, for areas with $C_i = \bar{C}$, (ii) SME prediction results converge to FME prediction results as $\sigma_x^2 \to \infty$, and (iii) in the presence of measurement error the naïve model is misspecified, so it will misstate the prediction MSE except for areas with $C_i = \bar{C}$.

Section 3 made prediction MSE comparisons between the three models over ranges of the true model's parameter values for the case when the true model was the SME. Section 4 made such comparisons taking as truth a particular SME model obtained by fitting it to data on poverty rates of school-age children for U.S. counties. This model is very similar to models used by the Census Bureau's SAIPE program, so its use provided results for a realistic case of a true SME model. MSE comparisons were also obtained for an analogous FME model by simulating values of the unobserved true covariate values x_i.

The MSE comparisons of Sections 3 and 4 tended to favor the SME model overall. Comparisons to the naïve model showed that the naïve predictor can fare poorly for C_i not near $\bar{C}$, with substantial MSE increases compared to the SME model predictor. Regarding the naïve model's additional problem of misstatement of MSE for C_i not near $\bar{C}$, understatement of MSE when $C_i > \bar{C}$ appeared more serious than overstatement of

MSE for $C_i < \bar{C}$.

From the comparisons of the SME and FME models, it was noted that when the SME model is true the FME predictor can have substantially higher prediction MSE when sampling and measurement error are large (D_i and C_i are large). While the FME predictor can be best when the FME model is true, it was also not unusual in this case for the SME and naïve FH predictors to actually have lower MSEs than the "optimal" FME predictor. This is because the optimality of the FME predictor for the FME model is among the class of unbiased predictors given fixed x_i, while the SME and naïve FH predictors, being biased, fall outside this class and so can and sometimes do have lower MSE. Though more research is needed on this point, it appears that while the avoidance of modeling assumptions for the x_i gives the FME model some potential for robustness, this can come at a significant cost in terms of higher prediction error variances for some areas.

A practical consideration related to this last point is that, in small area estimation, the most likely candidates for useful covariates with quantified measurement error (the C_i being known, or actually estimated) are other survey estimates X_i of population quantities x_i thought to be related to the population quantities θ_i whose direct estimates Y_i we seek to improve with our model. This leads to the question of why, if we believe we can adequately model θ_i through Y_i, we would choose the FME model over the SME model to avoid modeling x_i through X_i?

Appendix

Asymptotic evaluations of estimators and predictors under the FME model

We consider the following functional measurement error (FME) model:

$$Y_i = \theta_i + e_i, \quad \theta_i = x_i'\beta + z_i'\delta + u_i, \quad X_i = x_i + \eta_i, \quad i = 1, \dots, m, \tag{17}$$

where X_i and z_i are p and q component vectors, respectively, and the e_i's, u_i's and η_i's are independently distributed, each with zero mean, and with respective variances given by D_i, σ_u^2 and C_i. We define $Y = (Y_1, \dots, Y_m)'$, $C_i^* = \mathrm{Diag}(C_i, \mathbf{0}_{q,q})$, and

$$A = \begin{bmatrix} X_1' & z_1' \\ \vdots & \vdots \\ X_m' & z_m' \end{bmatrix}, \tag{18}$$

which is the observed design matrix of covariates. We use $E_F(\cdot)$ and $V_F(\cdot)$ to denote the expectation and variance operators under the FME model. Let $(\beta', \delta') = \gamma'$. Under the model in (17)

$$E_F\left[(A'A - \sum_{i=1}^{m} C_i^*)\gamma\right] = E_F(A'Y),$$

which leads to an unbiased estimating equation,

$$(A'A - \sum_{i=1}^{m} C_i^*)\gamma = A'Y, \tag{19}$$

for γ. Also, for $i = 1, \ldots, m$, $E_F\left[(Y_i - X_i'\beta - z_i'\delta)^2\right] = D_i + \sigma_u^2 + \beta' C_i \beta$ leads to another unbiased estimating equation,

$$\frac{1}{m}\sum_{i=1}^{m}(Y_i - X_i'\beta - z_i'\delta)^2 - \bar{D} - \beta'\bar{C}\beta - \sigma_u^2 = 0. \tag{20}$$

Assuming non-singularity of the matrix $A'A - \sum_{i=1}^{m} C_i^*$, an estimator of γ follows from (19). We denote this estimator by $\hat{\gamma}_F$ (the subscript F is used to denote the FME model fitting), which is

$$\hat{\gamma}_F = (A'A - \sum_{i=1}^{m} C_i^*)^{-1} A'Y. \tag{21}$$

Using this in (20), an estimator of σ_u^2 follows, given by

$$\hat{\sigma}_{u,F}^2 = \frac{1}{m}\sum_{i=1}^{m}(Y_i - X_i'\hat{\beta}_F - z_i'\hat{\delta}_F)^2 - \bar{D} - \hat{\beta}_F'\bar{C}\hat{\beta}_F. \tag{22}$$

We assume that the following limits exist, where

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m}\begin{pmatrix} x_i \\ z_i \end{pmatrix}\begin{pmatrix} x_i \\ z_i \end{pmatrix}' = K, \tag{23}$$

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=1}^{m} C_i^* = \mathrm{Diag}(\bar{C}, \mathbf{0}_{q,q}), \tag{24}$$

and K and $\bar{C}$ are assumed positive definite matrices. Let $\overset{P_F}{\to}$ denote weak convergence under the FME model. Then under the model in (17), with the assumptions in (23) and (24), and by weak laws of large numbers, we get

$$\frac{1}{m}(A'A - \sum_{i=1}^{m} C_i^*) \overset{P_F}{\to} K, \quad \frac{1}{m}A'Y \overset{P_F}{\to} K\gamma, \quad \hat{\gamma}_F \overset{P_F}{\to} K^{-1}K\gamma = \gamma.$$

Using the consistency of $\hat{\gamma}_F$ and standard arguments, it follows that

$$\hat{\sigma}^2_{u,F} - \left[\frac{1}{m}\sum_{i=1}^{m}(Y_i - X_i'\beta - z_i'\delta)^2 - \bar{D} - \beta'\bar{C}\beta\right] \overset{P_F}{\to} 0.$$

Finally, using this and weak laws of large numbers we get

$$\frac{1}{m}\sum_{i=1}^{m}(Y_i - X_i'\beta - z_i'\delta)^2 - \bar{D} - \beta'\bar{C}\beta \overset{P_F}{\to} \sigma_u^2 \qquad \Rightarrow \qquad \hat{\sigma}^2_{u,F} \overset{P_F}{\to} \sigma^2_u.$$

This establishes consistency of the estimators $\hat{\gamma}_F$ and $\hat{\sigma}^2_{u,F}$ derived from the unbiased estimating equation approach.

We know from Ybarra and Lohr (2008) that, under the FME model with known parameters γ and σ^2_u, the best linear unbiased predictor of θ_i is

$$\tilde{\theta}_{i,F} = Y_i - \frac{D_i}{D_i + \sigma_u^2 + \beta' C_i \beta}(Y_i - X_i'\beta - z_i'\delta).$$

Replacing the unknown model parameters in $\tilde{\theta}_{i,F}$ with the parameter estimators derived above, the EBLUP of θ_i is obtained as

$$\hat{\theta}^{EB}_{i,F} = Y_i - \frac{D_i}{D_i + \hat{\sigma}^2_{u,F} + \hat{\beta}_F' C_i \hat{\beta}_F}(Y_i - X_i'\hat{\beta}_F - z_i'\hat{\delta}_F). \tag{25}$$

Simple calculations lead to $E_F(\hat{\theta}_{i,F}^{EB} - \theta_i) \to 0$, implying asymptotic unbiasedness of the EBLUP predictor under the FME model. Again by simple calculations we obtain

$$MSE_F(\hat{\theta}_{i,F}^{EB}) = E_F(\hat{\theta}_{i,F}^{EB} - \theta_i)^2 \to \frac{D_i(\sigma_u^2 + \beta' C_i \beta)}{D_i + \sigma_u^2 + \beta' C_i \beta}. \tag{26}$$

If we ignore the measurement error in X_i, then we fit the following naïve FH model

$$Y_i = \theta_i + e_i, \quad \theta_i = X_i'\beta_N + z_i'\delta_N + u_{i,N}, \quad i = 1, \ldots, m, \tag{27}$$

where, as usual, we assume that e_i's and $u_{i,N}$'s are independently distributed with means zero and variances D_i and $\sigma_{u,N}^2$, respectively. To avoid confusion, we use the subscript N for the parameters in the naïve model. Using $(\beta_N', \delta_N') = \gamma_N'$, we estimate γ_N and $\sigma_{u,N}^2$ by

$$\hat{\gamma}_N = (A'A)^{-1}A'Y, \tag{28}$$

$$\hat{\sigma}_{u,N}^2 = \frac{1}{m}\sum_{i=1}^{m}(Y_i - X_i'\hat{\beta}_N - z_i'\hat{\delta}_N)^2 - \bar{D}. \tag{29}$$

Using these estimators, the naïve EBLUP of θ_i would be obtained as

$$\hat{\theta}_{i,N}^{EB} = Y_i - \frac{D_i}{D_i + \hat{\sigma}_{u,N}^2}(Y_i - X_i'\hat{\beta}_N - z_i'\hat{\delta}_N). \tag{30}$$

With respect to the true model in (17), the naïve FH model given by (27) is misspecified. We study the limiting behavior of the naïve model parameter estimators $\hat{\gamma}_N$ and $\hat{\sigma}_{u,N}^2$, and of the naïve predictor $\hat{\theta}_{i,N}^{EB}$, under the true model in (17). It easily follows that

$$\frac{1}{m}A'A \overset{P_F}{\to} K + \mathrm{Diag}(\bar{C}, \mathbf{0}_{q,q}) := K_*, \quad \frac{1}{m}A'Y \overset{P_F}{\to} K\gamma.$$

We partition the matrices K and K_* as

$$K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \; K_* = \begin{bmatrix} K_{*11} & K_{*12} \\ K_{*21} & K_{*22} \end{bmatrix}.$$

Note that $K_{ij} = K_{*ij}$, for $(i,j) \neq (1,1)$ and $K_{*11} = K_{11} + \bar{C}$. Using the inversion formula for partitioned matrices it can be shown, using $K_{11.2}$ to denote $K_{11} - K_{12}K_{22}^{-1}K_{21}$, that

$$K_*^{-1} = \begin{bmatrix} (K_{11.2}+\bar{C})^{-1} & -(K_{11.2}+\bar{C})^{-1}K_{12}K_{22}^{-1} \\ -K_{22}^{-1}K_{21}(K_{11.2}+\bar{C})^{-1} & K_{22}^{-1} + K_{22}^{-1}K_{21}(K_{11.2}+\bar{C})^{-1}K_{12}K_{22}^{-1} \end{bmatrix},$$

and, after writing $(K_{11.2}+\bar{C})^{-1}K_{11.2} = \tilde{A}$, we find that

$$K_*^{-1}K = \begin{bmatrix} \tilde{A} & \mathbf{0}_{p,q} \\ K_{22}^{-1}K_{21}(I_p - \tilde{A}) & I_q \end{bmatrix}.$$

Putting all these together, it follows that

$$\hat{\gamma}_N \overset{P_F}{\to} K_*^{-1}K\gamma = \begin{bmatrix} \tilde{A}\beta \\ \delta + K_{22}^{-1}K_{21}(I_p - \tilde{A})\beta \end{bmatrix} := \bar{\gamma}. \tag{31}$$

Thus, $\hat{\beta}_N$ estimates a function of β which is shrinking β to $\mathbf{0}_p$. After substantial simplifications it follows that

$$\hat{\sigma}_{u,N}^2 \overset{P_F}{\to} \sigma_u^2 + \beta'\bar{C}\tilde{A}\beta := \sigma_u^2 + \tau. \tag{32}$$

Note that $\tau = \beta'(\bar{C}^{-1} + K_{22.1}^{-1})^{-1}\beta > 0$ unless $\beta = 0$ (since $\bar{C}$ is assumed to be positive definite). So $\hat{\sigma}^2_{u,N}$ overestimates σ^2_u with probability converging to 1.

Using (31) and (32) in (30) we get, after simplifications, that

$$E_F(\hat{\theta}^{EB}_{i,N} - \theta_i) \overset{P_F}{\to} -\frac{D_i(x_i - K_{12}K_{22}^{-1}z_i)'(I - \tilde{A})\beta}{D_i + \sigma^2_u + \tau}, \qquad (33)$$

an asymptotic expression for the bias of the naïve EBLUP $\hat{\theta}^{EB}_{i,N}$. After more simplification, we also get that

$$V_F(\hat{\theta}^{EB}_{i,N} - \theta_i) \overset{P_F}{\to} \frac{D_i(\sigma^2_u + \tau)}{D_i + \sigma^2_u + \tau} + \frac{D_i^2\{\beta'(\tilde{A}'C_i - \bar{C})\tilde{A}\beta\}}{(D_i + \sigma^2_u + \tau)^2}. \qquad (34)$$

Combining the square of (33) and (34) gives the limiting expression for $MSE_F(\hat{\theta}^{EB}_{i,N}) = E_F(\hat{\theta}^{EB}_{i,N} - \theta_i)^2$.

We now investigate the impact of model misspecification when, instead of the true FME model in (17), we fit the following structural measurement error (SME) model:

$$\begin{aligned} Y_i &= \theta_i + e_i, \qquad \theta_i = x_i'\beta + z_i'\delta + u_i, \\ X_i &= x_i + \eta_i, \qquad x_i = \mu + v_i, \quad i = 1, \ldots, m, \end{aligned} \qquad (35)$$

where the e_i's, u_i's, η_i's and v_i's are independently distributed with zero means, and with variances D_i, σ^2_u, C_i and Σ_x, respectively. We assume that Σ_x is a positive definite matrix. In our SME formulation in (35) we assume that $z_{i1} = 1$ for an intercept, and for simplicity of calculations that $z_{i2}, \ldots, z_{iq}$ are centered about their respective means.

Under the model in (35), we can create the following unbiased estimating equations:

$$\sum_{i=1}^{m}(Y_i - X_i'\beta - z_i'\delta)z_i = \mathbf{0}_q, \tag{36}$$

$$\sum_{i=1}^{m}(X_i - \mu) = \mathbf{0}_p, \tag{37}$$

$$\sum_{i=1}^{m}\left\{\begin{bmatrix} Y_i - z_i'\delta \\ X_i - \mu \end{bmatrix}\begin{bmatrix} Y_i - z_i'\delta \\ X_i - \mu \end{bmatrix}' - \begin{bmatrix} D_i + \sigma_u^2 + \beta'(\mu\mu' + \Sigma_x)\beta & \beta'\Sigma_x \\ \Sigma_x\beta & \Sigma_x + C_i \end{bmatrix}\right\} = \mathbf{0}_{p+1,p+1}. \tag{38}$$

Let $\hat{\mu}_S$, $\hat{\delta}_S$, $\hat{\beta}_S$, $\hat{\Sigma}_{x,S}$ and $\hat{\sigma}^2_{u,S}$ denote a solution to (36)–(38). The subscript S is used to indicate the SME model is being fitted. Note that $\hat{\mu}_S = \bar{X}$, and

$$\hat{\Sigma}_{x,S} = \frac{1}{m}\sum_{i=1}^{m}(X_i - \bar{X})(X_i - \bar{X})' - \bar{C}, \tag{39}$$

$$\hat{\Sigma}_{x,S}\hat{\beta}_S = \frac{1}{m}\sum_{i=1}^{m}(Y_i - z_i'\hat{\delta}_S)(X_i - \bar{X}), \tag{40}$$

$$\hat{\sigma}^2_{u,S} = \frac{1}{m}\sum_{i=1}^{m}(Y_i - z_i'\hat{\delta}_S)^2 - \bar{D} - \hat{\beta}_S'(\bar{X}\bar{X}' + \hat{\Sigma}_{x,S})\hat{\beta}_S, \tag{41}$$

$$\sum_{i=1}^{m}(Y_i - X_i'\hat{\beta}_S - z_i'\hat{\delta}_S)z_i = \mathbf{0}_q. \tag{42}$$

We assume $\hat{\Sigma}_{x,S}$ is positive definite. We show below that the estimators $\hat{\delta}_S$, $\hat{\beta}_S$ and $\hat{\sigma}^2_{u,S}$ obtained by solving (39)–(42) also solve the estimating equations (19) and (20), which correspond to the FME model fitting. Since $z_{i1} = 1$, from (36) it follows that $\bar{Y} - \bar{X}'\hat{\beta}_S - \bar{z}'\hat{\delta}_S = 0$. Using this, it follows from (39) and (40), after simplification, that

$$\sum_{i=1}^{m}(X_iX_i' - C_i)\,\hat{\beta}_S + \left(\sum_{i=1}^{m} X_i z_i'\right)\hat{\delta}_S = \sum_{i=1}^{m} Y_i X_i. \tag{43}$$

From (42), we get

$$\left(\sum_{i=1}^{m} z_i X_i'\right)\hat{\beta}_S + \left(\sum_{i=1}^{m} z_i z_i'\right)\hat{\delta}_S = \sum_{i=1}^{m} Y_i z_i. \tag{44}$$

Equations (43) and (44) can also be written as

$$\left(A'A - \sum_{i=1}^{m} C_i^*\right)\begin{bmatrix}\hat{\beta}_S \\ \hat{\delta}_S\end{bmatrix} = A'Y.$$

From this equation, and the assumption that $A'A - \sum_{i=1}^{m} C_i^*$ is non-singular, it follows that $(\hat{\beta}_S', \hat{\delta}_S')'$ also satisfies (19), so $\hat{\gamma}_S = \hat{\gamma}_F$. Using $\hat{\gamma}_S = \hat{\gamma}_F$ in (41) and equation (39), it can be shown after a lot of simplification that

$$\begin{aligned}
\hat{\sigma}_{u,S}^2 &= \frac{1}{m}\sum_{i=1}^{m}\left(Y_i - z_i'\hat{\delta}_F\right)^2 - \bar{D} - \hat{\beta}_F'\left\{\frac{1}{m}\sum_{i=1}^{m}(X_i X_i' - C_i)\right\}\hat{\beta}_F \\
&= \frac{1}{m}\sum_{i=1}^{m}\left(Y_i - X_i'\hat{\beta}_F - z_i'\hat{\delta}_F\right)^2 + \frac{1}{m}\hat{\beta}_F'\left(\sum_{i=1}^{m} X_i X_i'\right)\hat{\beta}_F \\
&\quad + \frac{2}{m}\hat{\beta}_F'\left\{\sum_{i=1}^{m} X_i\left(Y_i - X_i'\hat{\beta}_F - z_i'\hat{\delta}_F\right)\right\} - \bar{D} - \hat{\beta}_F'\left(\frac{1}{m}\sum_{i=1}^{m} X_i X_i' - \bar{C}\right)\hat{\beta}_F.
\end{aligned} \tag{45}$$

Using $\sum_{i=1}^{m} X_i(Y_i - X_i'\hat{\beta}_S - z_i'\hat{\delta}_S) = -\sum_{i=1}^{m} C_i\hat{\beta}_S$ (from (43)), we get from (22) and (45) that

$$\hat{\sigma}_{u,S}^2 = \frac{1}{m}\sum_{i=1}^{m}(Y_i - X_i'\hat{\beta}_F - z_i'\hat{\delta}_F)^2 - \bar{D} - \hat{\beta}_F'\bar{C}\hat{\beta}_F = \hat{\sigma}_{u,F}^2.$$

These calculations show that fitting the SME model using the estimating equations given by (36)–(38) will lead to estimators of δ, β and σ_u^2 that are identical to the estimators of these parameters given in (21) and (22) that were obtained by fitting the FME model.

From our derivation that $\hat{\gamma}_S = \hat{\gamma}_F$ and $\hat{\sigma}^2_{u,S} = \hat{\sigma}^2_{u,F}$, and from the convergence under the FME model of the estimators $\hat{\gamma}_F$ and $\hat{\sigma}^2_{u,F}$ to γ and σ^2_u, we immediately conclude that the parameters estimated by fitting the SME model converge to their true values under the FME model. If we write $(Y_i, X_i') = Y_i^{*'}$, $(\theta_i, x_i') = \theta_i^{*'}$, $D_i^* = \text{Diag}(D_i, C_i)$ and

$$\Sigma = \begin{bmatrix} \sigma_u^2 + \beta'\Sigma_x\beta & \beta'\Sigma_x \\ \Sigma_x\beta & \Sigma_x \end{bmatrix},$$

from the multivariate Fay-Herriot model, the EBLUP of θ_i^* is given by

$$\hat{\theta}^{*EB}_{i,MFH} = Y_i^* - D_i^*\left(D_i^* + \hat{\Sigma}\right)^{-1} \begin{bmatrix} Y_i - \hat{\mu}'\hat{\beta}_S - z_i'\hat{\delta}_S \\ X_i - \hat{\mu} \end{bmatrix}. \tag{46}$$

If $\lambda = (1, \mathbf{0}_p')'$ denotes a unit vector with the first component 1 and all others zero, then the EBLUP of θ_i, denoted by $\hat{\theta}^{EB}_{i,S}$, is the first component of $\hat{\theta}^{*EB}_{i,MFH}$, which is, $\lambda'\hat{\theta}^{*EB}_{i,MFH}$. Using the block diagonal structure of D_i^* and using the inverse formula for partitioned matrices, it follows after substantial simplifications that

$$\lambda' D_i^*\left(D_i^* + \hat{\Sigma}\right)^{-1} \begin{bmatrix} Y_i - \bar{X}'\hat{\beta}_S - z_i'\hat{\delta}_S \\ X_i - \bar{X} \end{bmatrix} = \frac{D_i\left\{Y_i - X_i'\hat{\beta}_S - z_i'\hat{\delta}_S + \hat{\beta}_S' C_i(\hat{\Sigma}_{x,S} + C_i)^{-1}(X_i - \bar{X})\right\}}{D_i + \hat{\sigma}^2_{u,S} + \hat{\beta}_S' C_i(\hat{\Sigma}_{x,S} + C_i)^{-1}\hat{\Sigma}_{x,S}\hat{\beta}_S},$$

and finally,

$$\hat{\theta}^{EB}_{i,S} = Y_i - \frac{D_i\left\{Y_i - X_i'\hat{\beta}_S - z_i'\hat{\delta}_S + \hat{\beta}_S' C_i(\hat{\Sigma}_{x,S} + C_i)^{-1}(X_i - \bar{X})\right\}}{D_i + \hat{\sigma}^2_{u,S} + \hat{\beta}_S' C_i(\hat{\Sigma}_{x,S} + C_i)^{-1}\hat{\Sigma}_{x,S}\hat{\beta}_S}. \tag{47}$$

By similar simplification that was used to obtain equation (47), we can derive that

$$V_S(\hat{\theta}_{i,S}^{EB} - \theta_i) = \frac{D_i\{\sigma_u^2 + \beta' C_i(\Sigma_x + C_i)^{-1}\Sigma_x\beta\}}{D_i + \sigma_u^2 + \beta' C_i(\Sigma_x + C_i)^{-1}\Sigma_x\beta} + O(m^{-1}). \tag{48}$$

We note that

$$\hat{\Sigma}_{x,S} \overset{P_F}{\to} \lim_{m\to\infty} \frac{1}{m}\sum_{i=1}^{m}(x_i - \bar{x})(x_i - \bar{x})' := S_x. \tag{49}$$

By the convergence properties of $\hat{\gamma}_F$ and $\hat{\sigma}_{u,F}^2$, and the result in (49), it follows from (47) that the asymptotic bias of the predictor $\hat{\theta}_{i,S}^{EB}$ under the FME model is given by

$$E_F(\hat{\theta}_{i,S}^{EB} - \theta_i) = -\frac{D_i\beta' C_i(S_x + C_i)^{-1}(x_i - \bar{x})}{D_i + \sigma_u^2 + \beta' C_i(S_x + C_i)^{-1}S_x\beta} + O(m^{-1}). \tag{50}$$

After considerable simplifications, it can be shown that

$$\begin{aligned} V_F(\hat{\theta}_{i,S}^{EB} - \theta_i) &= \frac{D_i\{\sigma_u^2 + \beta'(C_i^{-1} + S_x^{-1})^{-1}\beta\}}{D_i + \sigma_u^2 + \beta'(C_i^{-1} + S_x^{-1})^{-1}\beta} \\ &- \left(\frac{D_i}{D_i + \sigma_u^2 + \beta'(C_i^{-1} + S_x^{-1})^{-1}\beta}\right)^2 \beta' \left\{(C_i^{-1} + S_x^{-1})S_x(S_x^{-1} + C_i^{-1})\right\}^{-1}\beta + O(m^{-1}). \end{aligned} \tag{51}$$

Combining the square of (50) and (51), we obtain an asymptotic expression for $MSE_F(\hat{\theta}_{i,S}^{EB})$.

Asymptotic evaluations of estimators and predictors under the SME model

We first evaluate the estimators and predictors derived under the FME model. Let $\stackrel{P_S}{\to}$ denote convergence in probability under the SME model, and let $E_S(\cdot)$ denote the expectation. Simple calculations and weak laws of large numbers imply that

$$\hat{\gamma}_F = \hat{\gamma}_S \stackrel{P_S}{\to} \gamma, \quad \hat{\sigma}^2_{u,F} = \hat{\sigma}^2_{u,S} \stackrel{P_S}{\to} \sigma^2_u.$$

Thus, the parameter estimators from fitting the SME and FME models both estimate the true model parameters consistently. With these results it follows from (25) that

$$E_S(\hat{\theta}^{EB}_{i,F} - \theta_i) = 0 + O(m^{-1}). \tag{52}$$

Moreover,

$$V_S(\hat{\theta}^{EB}_{i,F} - \theta_i) = \frac{D_i(\sigma^2_u + \beta' C_i \beta)}{D_i + \sigma^2_u + \beta' C_i \beta} + O(m^{-1}). \tag{53}$$

In the same fashion, for the SME predictor it follows from (47) that

$$E_S(\hat{\theta}^{EB}_{i,S} - \theta_i) = 0 + O(m^{-1})$$

and, after a little simplification of (48),

$$V_S(\hat{\theta}^{EB}_{i,S} - \theta_i) = D_i - \frac{D_i^2}{\sigma^2_u + D_i + \beta' C_i (\Sigma_x + C_i)^{-1} \Sigma_x \beta} + O(m^{-1}). \tag{54}$$

For the simpler case considered in the main text (scalar x_i), the result (54) can be seen to reduce to the SME model result given in Table 2 by noting that $F_i/(\sigma_x^2 + C_i) = \sigma_u^2 + D_i + \beta^2\sigma_x^2 C_i/(\sigma_x^2 + C_i)$.

We now turn to estimators and predictors derived under the naïve FH model. Let $\delta = (\delta_1, \ldots, \delta_q)'$. We note that

$$E_S(\frac{1}{m}A'Y) = \begin{bmatrix} \mu\mu' + \Sigma_x & \mu\bar{z}' \\ \bar{z}\mu' & \frac{1}{m}\sum_{i=1}^m z_i z_i' \end{bmatrix} \begin{bmatrix} \beta \\ \delta \end{bmatrix} \tag{55}$$

and

$$E_S(\frac{1}{m}A'A) = \begin{bmatrix} \mu\mu' + \Sigma_x + \bar{C} & \mu\bar{z}' \\ \bar{z}\mu' & \frac{1}{m}\sum_{i=1}^m z_i z_i' \end{bmatrix}. \tag{56}$$

Note that $\bar{z} = (1, 0, \ldots, 0)'$ (a $q \times 1$ vector) and $m^{-1}\sum_{i=1}^m z_i z_i'$ is a block diagonal matrix, which we denote by $\mathrm{Diag}(1, G_{22})$. We assume that this matrix converges to a positive definite matrix $\mathrm{Diag}(1, \Gamma_{22})$. Let R denote the matrix on the right hand side of (56). Partition R^{-1} in conformity with R. Adopting the usual notation, $R^{-1} = ((R^{jk}))_{j,k=1,2}$, it can be checked that $(R^{11})^{-1} = (\mu\mu' + \Sigma_x + \bar{C}) - \mu\bar{z}'(\sum_{i=1}^m z_i z_i'/m)^{-1}\bar{z}\mu'$. Since, $\bar{z}'(\sum_{i=1}^m z_i z_i'/m)^{-1}\bar{z} = 1$, we get

$$R^{11} = (\Sigma_x + \bar{C})^{-1}, \tag{57}$$

and

$$R^{21} = -R_{22}^{-1}R_{21}R^{11} = -\mathrm{Diag}(1, G_{22}^{-1})(1, 0, \ldots, 0)'\mu' R^{11} = -(1, 0, \ldots, 0)'\mu'(\Sigma_x + \bar{C})^{-1}.$$

If we denote the square matrix on the right hand side of (55) by S, we note that $S = R - \text{Diag}(\bar{C}, \mathbf{0}_{q,q})$. Consequently,

$$R^{-1}S = \begin{bmatrix} I_p - R^{11}\bar{C} & \mathbf{0}_{p,q} \\ -R^{21}\bar{C} & I_q \end{bmatrix} \quad \text{and} \quad R^{-1}S\gamma = \begin{bmatrix} (I_p - R^{11}\bar{C})\beta \\ \delta - R^{21}\bar{C}\beta \end{bmatrix}.$$

Denote $(\Sigma_x + \bar{C})^{-1}\Sigma_x$ by $\breve{A}$. After simplification, we get

$$R^{-1}S\gamma = \begin{bmatrix} \breve{A}\beta \\ \delta_1 + \mu'(I_p - \breve{A})\beta \\ \delta_2 \\ \vdots \\ \delta_q \end{bmatrix} := \bar{\gamma}. \tag{58}$$

By (58), it follows that $\hat{\gamma}_N \overset{P_S}{\to} \bar{\gamma}$. This shows that under the naïve model while the parameters $\delta_2, \ldots, \delta_q$ are consistently estimated, estimates of the parameter vector β are shrunk to the null vector. We now find the probability limit of $\hat{\sigma}^2_{u,N}$. Recall that $\hat{\sigma}^2_{u,N} = m^{-1}\sum_{i=1}^m (Y_i - X_i'\hat{\beta}_N - z_i'\hat{\delta}_N)^2 - \bar{D}$. Since, letting $\bar{\beta} = \breve{A}\beta$ and $\bar{\gamma} = [\bar{\beta}', \bar{\delta}']'$,

$$\begin{aligned} & E_S(Y_i - X_i'\hat{\beta}_N - z_i'\hat{\delta}_N)^2 = E_S(Y_i - X_i'\bar{\beta} - z_i'\bar{\delta})^2 + O(m^{-1}) \\ & = D_i + \sigma_u^2 + (\beta - \bar{\beta})'\Sigma_x(\beta - \bar{\beta}) + \bar{\beta}'C_i\bar{\beta} + (\gamma - \bar{\gamma})' \begin{pmatrix} \mu \\ z_i \end{pmatrix} \begin{pmatrix} \mu \\ z_i \end{pmatrix}' (\gamma - \bar{\gamma}) + O(m^{-1}), \end{aligned}$$

after substantial simplification,

$$E_S(\hat{\sigma}^2_{u,N}) \to \sigma^2_u + \beta'\bar{C}(\Sigma_x + \bar{C})^{-1}\Sigma_x\beta = \sigma^2_u + \beta'\bar{C}\breve{A}\beta. \tag{59}$$

We know that the naïve EBLUP of θ_i is

$$\hat{\theta}^{EB}_{i,N} = Y_i - \frac{D_i}{D_i + \hat{\sigma}^2_{u,N}}(Y_i - X_i'\hat{\beta}_N - z_i'\hat{\delta}_N).$$

It can be checked that

$$E_S(\hat{\theta}^{EB}_{i,N} - \theta_i) \to 0 \quad \text{as} \quad m \to \infty.$$

Also, after substantial simplification, we obtain

$$V_S(\hat{\theta}^{EB}_{i,N} - \theta_i) = \frac{D_i(\sigma^2_u + \beta'\bar{C}\breve{A}\beta)}{D_i + \sigma^2_u + \beta'\bar{C}\breve{A}\beta} + \left(\frac{D_i}{D_i + \sigma^2_u + \beta'\bar{C}\breve{A}\beta}\right)^2 \bar{\beta}'(C_i - \bar{C})\bar{\beta} + O(m^{-1}).$$

Equality of the SME and naïve FH predictors when $C_i = \bar{C}$

Remark 3 noted that, for the simple form of the model considered there, the SME model predictor of θ_i for a small area for which $C_i = \bar{C}$ agrees with the corresponding predictor obtained from the naïve FH model. Here we establish this result asymptotically when one or both covariates X_i and z_i are vector valued. We assume that the first element of z_i is 1 for an intercept.

We noticed earlier that the proposed estimators of the model parameters of the SME model depend on $\bar{C}$ only and not on the individual C_i's. Since the EB predictor of θ_i in the SME model also depends on C_i, and we are interested in small areas with $C_i = \bar{C}$, we consider a special case of the SME model where all C_i's are equal to C. From this special case we explain a relationship among the parameters between the balanced SME model and the naïve FH model. Writing $\delta = (\delta_1, \delta_2')'$, the balanced SME model has $Y_i = \theta_i + e_i$ along with

$$\theta_i = x_i'\beta + \delta_1 + z_{2,i}'\delta_2 + u_i \quad \text{and} \quad X_i = x_i + \eta_i,$$

where u_i, x_i and η_i are independently and normally distributed with zero means except for x_i which has mean μ, and with respective variances σ_u^2, Σ_x and C. In the naïve FH model θ_i is expressed as

$$\theta_i = X_i'\beta_N + \delta_{1,N} + z_{2,i}'\delta_{2,N} + u_{i,N},$$

where the X_i's are treated as fixed and $u_{i,N} \sim N(0, \sigma_{u,N}^2)$. If we decompose x_i in the SME model as $x_i = E(x_i|X_i) + [x_i - E(x_i|X_i)]$, then, noting that $E(x_i|X_i) = \mu + \Sigma_x(\Sigma_x + C)^{-1}(X_i - \mu) = \mu + \breve{A}(X_i - \mu)$ and $V(x_i|X_i) = (\Sigma_x^{-1} + C^{-1})^{-1}$, we can write

$$\begin{aligned} \theta_i &= x_i'\beta + \delta_1 + z_{2,i}'\delta_2 + u_i \\ &= X_i'(\breve{A}'\beta) + \delta_1 + \mu'(\Sigma_x + C)^{-1}C\beta + z_{2,i}'\delta_2 + u_i + \beta'[x_i - E(x_i|X_i)]. \end{aligned}$$

Identifying $u_i + \beta'[x_i - E(x_i|X_i)]$ as $u_{i,N}$, $\breve{A}'\beta$ as β_N, $\delta_1 + \mu'(\Sigma_x + C)^{-1}C\beta$ as $\delta_{1,N}$, and δ_2 as $\delta_{2,N}$, we obtain that $\sigma_{u,N}^2 = \sigma_u^2 + \beta'(\Sigma_x^{-1} + C^{-1})^{-1}\beta$. We estimated β_N, $\delta_{1,N}$, $\delta_{2,N}$, and $\sigma_{u,N}^2$ by method of moments (in this case, ordinary least squares conditional on the X_i's). We know that these estimators are consistent for β_N, $\delta_{1,N}$, $\delta_{2,N}$, and $\sigma_{u,N}^2$.

We also know that our proposed estimators $\hat{\beta}_S$, $\hat{\delta}_{1,S}$, $\hat{\delta}_{2,S}$, $\hat{\sigma}_{u,S}^2$, $\hat{\mu}_S$ and $\hat{\Sigma}_{x,S}$ are all consistent. We estimate $\check{A}$ by $\hat{A} = \hat{\Sigma}_{x,S}(\hat{\Sigma}_{x,S} + C)^{-1}$. From consistency of the two sets of estimators under the two models (and the naïve FH model as induced by the balanced SME model), we have the following results:

(i) $\hat{\beta}_N - \hat{A}'\hat{\beta}_S \overset{P_S}{\to} 0$,

(ii) $\hat{\sigma}^2_{u,N} - \hat{\sigma}^2_{u,S} - \hat{\beta}_S(\hat{\Sigma}^{-1}_{x,S} + C^{-1})^{-1}\hat{\beta}_S \overset{P_S}{\to} 0$

(iii) $\hat{\delta}_{2,N} - \hat{\delta}_{2,S} \overset{P_S}{\to} 0$

(iv) $\hat{\delta}_{1,N} - \hat{\delta}_{1,S} - \hat{\mu}'_S(\hat{\Sigma}_{x,S} + C)^{-1}C\hat{\beta}_S \overset{P_S}{\to} 0$.

Using (i)-(iv), we obtain after some simplification that

$$
\begin{aligned}
X'_i\hat{\beta}_N + z'_i\hat{\delta}_N - X'_i\hat{\beta}_S - z'_i\hat{\delta}_S + (X_i - \hat{\mu}_S)'(\hat{\Sigma}_{x,S} + C)^{-1}C\hat{\beta}_S \quad &\overset{P_S}{\to} \quad 0, \\
(D_i + \hat{\sigma}^2_{u,N}) - [D_i + \hat{\sigma}^2_{u,S} + \hat{\beta}'_S(\hat{\Sigma}^{-1}_{x,S} + C^{-1})^{-1}\hat{\beta}_S] \quad &\overset{P_S}{\to} \quad 0.
\end{aligned}
$$

Using these two expressions in $\hat{\theta}^{EB}_{i,S}$ (given in equation (47)) and $\hat{\theta}^{EB}_{i,N}$ (given in equation (25)), we immediately obtain that

$$
\hat{\theta}^{EB}_{i,S} - \hat{\theta}^{EB}_{i,N} \overset{P_S}{\to} 0.
$$

To conclude this remark, we note that although $\delta_{2,N} = \delta_2$ and $\hat{\delta}_{2,N} - \hat{\delta}_{2,S} \overset{P_S}{\to} 0$, there is no guarantee that, in general, $\hat{\delta}_{2,N} = \hat{\delta}_{2,S}$. Equality may or may not hold, depending on the estimation method. For our proposed estimators of $\hat{\delta}_{2,S}$ and $\hat{\delta}_{2,N}$, they are not equal.

Re-expression of MSE formulas for doing contour plots

We illustrate by showing how we re-express MSE_S/σ^2_u in terms of $r_D = D_i/\sigma^2_u$, $r_C = C_i/\sigma^2_x$, and ρ. Given the result from Table 2 that $MSE_S = D_i - D^2_i(\sigma^2_x + C_i)/F^*_i$, we start by re-expressing F_i^*:

$$\begin{aligned} F_i^* &= (\sigma_u^2 + D_i)(\sigma_x^2 + C_i) + \beta^2\sigma_x^2 C_i \\ &= \sigma_u^2\sigma_x^2 \left\{ (1 + r_D)(1 + r_C) + \frac{\beta^2\sigma_x^2}{\sigma_u^2} r_C \right\}. \end{aligned}$$

From (7), noting that for our simplified model $\sigma_v^2 = \sigma_x^2$, we have

$$\begin{aligned} \rho^2 &= \operatorname{corr}(\theta_i, x_i)^2 = \frac{\beta^2\sigma_x^4}{(\sigma_u^2 + \beta^2\sigma_x^2)\sigma_x^2} = \frac{\beta^2\sigma_x^2}{\sigma_u^2 + \beta^2\sigma_x^2} \\ \Rightarrow & \ r_\rho := \frac{\rho^2}{1 - \rho^2} = \frac{\beta^2\sigma_x^2}{\sigma_u^2} \end{aligned}$$

which implies that $F_i^* = \sigma_u^2\sigma_x^2[(1 + r_D)(1 + r_C) + r_\rho r_C]$. Then

$$\begin{aligned} MSE_S &= \sigma_u^2 \left\{ \frac{D_i}{\sigma_u^2} - \frac{(1/\sigma_u^2)D_i^2(\sigma_x^2 + C_i)}{\sigma_u^2\sigma_x^2[(1 + r_D)(1 + r_C) + r_\rho r_C]} \right\} \\ \Rightarrow & \ \frac{MSE_S}{\sigma_u^2} = r_D - \frac{r_D^2(1 + r_C)}{(1 + r_D)(1 + r_C) + r_\rho r_C}. \end{aligned}$$

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