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Unit Root Properties of Seasonal Adjustment and Related Filters: Special Cases

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1 Introduction

Linear filters used in seasonal adjustment contain various unit root factors. Seasonal unit root factors are those of the seasonal summation operator $U_s(B) = 1 + B + \cdots + B^{s-1}$ where B is the backshift operator ($By_t = y_{t-1}$ for any time series y_t) and s is the seasonal period. The presence of $U_s(B)$ in a filter means it will annihilate fixed seasonal effects, a desirable property for seasonal adjustment, trend, and irregular filters. The other unit root factors of interest are powers of the differencing operator $1 - B$. The presence of $(1 - B)^d$ for $d > 0$ in a filter means it will annihilate polynomials in t up to degree $d-1$. This is generally the case for seasonal and irregular filters, and it implies that the corresponding seasonal adjustment and trend filters will reproduce polynomials of degree $d-1$. This property has been of significant interest historically, as many empirical trend filters were explicitly designed to reproduce polynomials of a certain degree. For example, the symmetric Henderson trend filters will reproduce cubic polynomials (Kenny and Durbin 1982).

Bell (2012) gave general results on unit root factors contained in linear filters used in model-based and X-11 seasonal adjustment. It was noted there that special cases could arise for model-based adjustment where the filters contain more unit root factors than is obvious from the general results. The present article focuses on this point, examining some special cases for canonical ARIMA model-based adjustment (Hillmer and Tiao 1982, Burman 1980, Gomez and Maravall 1997) where the symmetric seasonal filters include two extra differencing operators, written as $(1 - B)(1 - F)$, where $F = B^{-1}$ is the forward shift operator ($Fy_t = y_{t+1}$). In these cases the symmetric seasonal adjustment filters will reproduce polynomials of two degrees higher than is indicated by the general results given in Bell (2012).

Section 2 defines notation and the framework used for linear model-based seasonal adjustment. Sections 3 and 4 provide results showing when the extra $(1 - B)(1 - F)$ factor occurs in two models considered explicitly by Hillmer and Tiao (1982), which we hereafter cite as HT: the ARIMA$(0, 0, 1)(0, 1, 1)_s$ model and the ARIMA$(0, 1, 1)(0, 1, 1)_s$ (airline) model. Values considered for the seasonal period s are 2 (biannual), 4 (quarterly), and 12 (monthly). Section 5 discusses some additional related results for canonical ARIMA model-based adjustment, while Section 6 briefly considers special cases for structural component models. Technical details of the derivations for Sections 3 and 4 are reserved to two Appendices.

2 Notation and framework for model-based seasonal adjustment

The additive decomposition used in seasonal adjustment is

$$y_t = S_t + T_t + I_t \tag{1}$$

where y_t is the observed time series (possibly after transformation, e.g., taking logarithms), and S_t, T_t, and I_t are the seasonal, trend, and irregular components. We also let $N_t = T_t + I_t = y_t - S_t$ denote the nonseasonal component, the estimate of which is known as the seasonally adjusted series. Many of the models proposed for model-based seasonal adjustment use component models that can be written in the following form:

$$\begin{aligned} U_s(B)S_t &= u_t \\ (1-B)^d T_t &= v_t \\ I_t &\sim i.i.d.\ N(0,\sigma_I^2) \end{aligned} \tag{2}$$

where u_t and v_t are stationary time series that are independent of each other and of I_t. Often u_t and v_t are assumed to follow stationary autoregressive-moving average models (Box and Jenkins 1970), in which case y_t follows an ARIMA (autoregressive-integrated-moving average) model that can be written

$$\phi(B)(1-B)^{d-1}(1-B^s)y_t = \theta(B)a_t \tag{3}$$

where $\phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p$ is the AR operator, $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$ is the MA operator, and a_t is white noise, $i.i.d.$ $N(0, \sigma_a^2)$. The operators $\phi(B)$ and $\theta(B)$, which may be products of nonseasonal and seasonal polynomials in B, are assumed to have all their zeros outside the unit circle. The expression of the model as in (3) requires $d \geq 1$, which is standard in seasonal adjustment practice. Note that $1 - B^s = (1-B)U_s(B)$ so $(1-B)^{d-1}(1-B^s) = (1-B)^d U_s(B)$.

This model framework covers the ARIMA model-based approach to seasonal adjustment as developed in HT and Burman (1980), and implemented in the TRAMO-SEATS software of Gomez and Maravall (1997) and in the X-13-ARIMA-SEATS program (Monsell 2007). It also covers the structural components models of Harvey (1989), Durbin and Koopman (2001), and Kitagawa and Gersch (1984). Though Harvey did not formulate all his component models in ARIMA form, they can generally be written this way – see Bell (2004).

Let $w_t = (1-B)^d U_s(B) y_t$ be the differenced observed series. From (1) and (2),

$$w_t = (1-B)^d u_t + U_s(B) v_t + (1-B)^d U_s(B) I_t. \tag{4}$$

Let $\gamma_w(k) = \mathrm{Cov}(w_t, w_{t+k})$ and let $\gamma_w(B)$ be the autocovariance generating function (ACGF) of w_t, defined as $\gamma_w(B) \equiv \sum_{k=-\infty}^{\infty} \gamma_w(k) B^k$, where we treat B for this purpose as a complex variable. Given the ARMA model $\phi(B) w_t = \theta(B) a_t$, and the orthogonality of the components in (4), it follows that (Box and Jenkins 1970, p. 49)

$$\begin{aligned} \gamma_w(B) &= \sigma_a^2 \theta(B)\theta(F)/\phi(B)\phi(F) & (5) \\ &= (1-B)^d(1-F)^d \gamma_u(B) + U_s(B)U_s(F)\gamma_v(B) + (1-B)^d(1-F)^d U_s(B) U_s(F) \sigma_I^2. & (6) \end{aligned}$$

Given ARMA models for u_t and v_t, analogous expressions to (5) can be given for their ACGFs, $\gamma_u(B)$ and $\gamma_v(B)$. From $w_t = (1-B)^d U_s(B) y_t$, the pseudo ACGF of y_t is defined as $\gamma_y(B) = \gamma_w(B)/(1-B)^d(1-F)^d U_s(B) U_s(F)$. We also define $z_t = (1-B)^d N_t = v_t + (1-B)^d I_t$ with ACGF $\gamma_z(B) = \gamma_v(B) + (1-B)^d(1-F)^d \sigma_I^2$.

Bell (1984 and 2012, p. 445) notes that the minimum mean squared error (MMSE) linear signal extraction estimate of S_t given the full doubly infinite realization of the series $\{y_t\}$ is

$$\hat{S}_t = \omega_S(B) y_t \quad \text{where} \quad \omega_S(B) = \frac{\gamma_u(B)}{\gamma_w(B)} (1-B)^d (1-F)^d. \tag{7}$$

Analogous to (7), the linear filters for the MMSE estimates of N_t, T_t, and I_t are

$$\omega_N(B) = \frac{\gamma_z(B)}{\gamma_w(B)} U_s(B) U_s(F) \tag{8}$$

$$\omega_T(B) = \frac{\gamma_v(B)}{\gamma_w(B)} U_s(B) U_s(F) \tag{9}$$

$$\omega_I(B) = \frac{\sigma_I^2}{\gamma_w(B)} U_s(B) U_s(F)(1-B)^d(1-F)^d. \tag{10}$$

Note also that since $\hat{N}_t = y_t - \hat{S}_t$ and $\hat{T}_t = \hat{N}_t - \hat{I}_t$, it follows that $\omega_N(B) = 1 - \omega_S(B)$ and $\omega_T(B) = 1 - \omega_S(B) - \omega_I(B)$.

Simple inspection of (7)–(10) led to the results reported in Bell (2012) for unit root factors contained in these symmetric filters. The specific result of interest here is that $\omega_S(B)$ contains $(1-B)^d(1-F)^d$, implying that $\omega_S(B)$ annihilates, and $\omega_N(B)$ thus reproduces, polynomials up to degree $2d-1$. The models most commonly used in seasonal adjustment have $d = 2$, in which case the symmetric seasonal adjustment filter must reproduce cubic polynomials in t. Less commonly used models have $d = 1$, in which case the symmetric seasonal adjustment filter must reproduce linear polynomials in t. Values of d other than 1 or 2 are uncommon in practice.

Bell (2012, pp. 446–447) also noted that:

> Something not clear from [(7)–(10)] is whether these filters contain additional unit root factors beyond those obvious from inspection. Bell (2010) notes that $\omega_I(B)$ will not include additional unit root factors, while for $\omega_S(B)$, $\omega_N(B)$, and $\omega_T(B)$, additional unit root factors are possible if they appear in the MA polynomials of the ARIMA models for S_t, N_t, or T_t. For example, Hillmer and Tiao (1982, p. 67) examine a model for which the canonical trend component has a factor of $(1+B)$ in its MA polynomial. While potential additional unit root factors in the filters considered can obviously be examined for any particular model, general results are difficult to give.

The polynomial factors in the MA operator of any ARMA model, such as $\theta(B)$ in (3), correspond to double factors in the numerator of the autocovariance generating function – note $\theta(B)\theta(F)$ in equation (5). So $1 - B$ is a factor of the MA polynomial of the model for u_t if and only if the numerator of $\gamma_u(B)$ contains $(1-B)(1-F)$.

Sections 3 and 4 examine special cases that occur with canonical ARIMA model-based seasonal adjustment where, for two commonly used models, and depending on the seasonal period s and on the model parameter values, $\gamma_u(B)$ indeed contains a factor of $(1-B)(1-F)$. From (7) this implies that $\omega_S(B)$ contains an extra $(1-B)(1-F)$ so it will annihilate, and $\omega_N(B)$ will reproduce, polynomials in t up to degree $2d+1$, which is two degrees higher than would otherwise be the case. For the common cases of $d=1$ or 2, the extra $(1-B)(1-F)$ means that the seasonal adjustment filter will reproduce cubic and quintic polynomials, respectively, instead of just linear and cubic polynomials. This property will not be shared by the corresponding trend filter $\omega_T(B) = 1 - \omega_S(B) - \omega_I(B)$ because, as noted in the quotation above, the corresponding canonical irregular filter will not include the extra $(1-B)(1-F)$ factor.

3 Results for the ARIMA$(0,0,1)(0,1,1)_s$ model

The ARIMA$(0,0,1)(0,1,1)_s$ model is

$$(1-B^s)y_t = (1-\theta_1 B)(1-\theta_2 B^s)a_t. \tag{11}$$

The nonseasonal and seasonal MA parameters θ_1 and θ_2 are both restricted to lie in the interval $(-1,1)$, though for seasonal adjustment interest focuses on the case of $\theta_2 \geq 0$, for which the existence of the canonical decomposition is assured (HT, p. 68). Without loss of generality for the derivations and results presented here, we assume that $\mathrm{Var}(a_t)=1$.

HT's canonical decomposition starts with a partial fractions decomposition of the ACGF for y_t. For the model (11), HT (p. 68) observe that the seasonal part of this partial fractions decomposition can be expressed as $Q_s^*(B)/U_s(B)U_s(F)$, where

$$Q_s^*(B) = \frac{(1-\theta_2)^2(1-\theta_1 B)(1-\theta_1 F)}{(1-B)(1-F)}\left\{1-\frac{1}{s^2}U_s(B)U_s(F)\right\}. \tag{12}$$

Appendix A observes that $1-1/s^2 U_s(B)U_s(F)$ contains $(1-B)(1-F)$, and so can be expressed as

$(1-B)(1-F)\alpha_s(B)$, where $\alpha_s(B)$ is a symmetric polynomial in B and F. Appendix A also gives $\alpha_s(B)$ for the cases of $s = 2$, 4, and 12. Cancelling the $(1-B)(1-F)$ factors in the numerator and denominator, $Q_s^*(B)$ simplifies to $(1-\theta_2)^2(1-\theta_1 B)(1-\theta_1 F)\alpha_s(B)$. The spectrum of the canonical seasonal is then $(2\pi)^{-1}$ times $f_s(\lambda) = Q_s^*(e^{i\lambda})/|U_s(e^{i\lambda})|^2 - \epsilon_s$, where

$$\epsilon_s \equiv \min_{\lambda \in [0,\pi]} \frac{Q_s^*(e^{i\lambda})}{|U_s(e^{i\lambda})|^2} = \min_{\lambda \in [0,\pi]} \frac{(1-\theta_2)^2[(1+\theta_1^2) - 2\theta_1 \cos(\lambda)]\alpha_s(e^{i\lambda})}{|U_s(e^{i\lambda})|^2}. \tag{13}$$

The value ϵ_s becomes part of the canonical irregular variance. If the minimum value ϵ_s occurs at $\lambda = 0$, then the resulting canonical seasonal spectrum $(2\pi)^{-1} f_s(\lambda)$ will be zero at $\lambda = 0$, and the pseudo-ACGF of S_t, which is $\gamma_u(B)/U_s(B)U_s(F)$, must include a $1 - B$ factor in $\gamma_u(B)$ (so that $\gamma_u(e^{i0}) = \gamma_u(1) = 0$). By symmetry of $\gamma_u(B)$, it must then also include a $1 - F$ factor, and so in such cases the canonical seasonal filter $\omega_S(B)$ given by (7) will include an extra $(1-B)(1-F)$ in its numerator. In these cases the canonical $\omega_S(B)$ for the $(0,0,1)(0,1,1)_s$ model includes in total $(1-B)^2(1-F)^2$. Then $\omega_S(B)$ will annihilate, and $\omega_N(B)$ will reproduce, cubic polynomials in t, not just linear polynomials (the standard result for this model which has $d = 1$).

For given values of the nonseasonal MA parameter θ_1, the value of λ that minimizes $f_s(\lambda)$ was determined through inspection by computing $f_s(\lambda)$ over a detailed grid of λ values (from 0 to π in increments of .01) and picking off the minimizing value of λ. Examining the results for a detailed set of θ_1 values revealed those values of θ_1 for which the minimum of $f_s(\lambda)$ occurs at $\lambda = 0$, so that $\omega_S(B)$ from the $(0,0,1)(0,1,1)_s$ model contains $(1-B)^2(1-F)^2$ and not just $(1-B)(1-F)$. Table 1 gives the results. Note that, for $s = 2$, $\omega_S(B)$ contains $(1-B)^2(1-F)^2$ for any value of θ_1, while for $s = 4$ and $s = 12$, $\omega_S(B)$ contains $(1-B)^2(1-F)^2$ only for limited intervals of θ_1. In fact, the result for $s = 2$ can be established analytically since it is easy to show that $f_2(\lambda)$ is increasing in λ over $[0,\pi]$ for any value of θ_1. Another point worth noting is that, for $\theta_1 > 0$, the $(1+\theta_1^2) - 2\theta_1\cos(\lambda)$ factor in (13), which does not depend on s, is an increasing function of λ on $[0,\pi]$, while $\alpha_s(e^{i\lambda})/|U_s(e^{i\lambda})|^2$, which does not depend on θ_1, has a global minimum at $\lambda = 0$. Hence, for each s and for all $\theta_1 > 0$, the minimum of $f_s(\lambda)$ occurs at $\lambda = 0$. Finally, note that the results of Table 1 are not affected by the value of θ_2.

Table 1. Range of values of θ_1 for which the canonical seasonal filter $\omega_S(B)$ from (7) for the ARIMA$(0,0,1)(0,1,1)_s$ model (11) includes $(1-B)^2(1-F)^2$, not just $(1-B)(1-F)$.

seasonal period s	2	4	12
range of values of θ_1	all $\theta_1 \in (-1,1)$	$-.35 < \theta_1 < 1$	$-.28 < \theta_1 < 1$

To provide further insight into the results of Table 1, Figure 1 shows plots of $f_s(\lambda)$ (but omitting the $(1-\theta_2)^2$ factor since it does not depend on λ) for both the quarterly and monthly cases, for three values of θ_1: $-.2$, $-.3$, and $-.4$. Features common to these plots, and to plots of $f_s(\lambda)$ for other values of θ_1, include: a local minimum at $\lambda = 0$; infinite peaks at the seasonal frequencies; and, necessarily, dips between the seasonal frequencies. The plots also show, consistent with Table 1, that *(i)* for $\theta_1 = -.2$, $f_s(\lambda)$ is minimized at $\lambda = 0$ for both the quarterly and monthly cases, *(ii)* for $\theta_1 = -.3$, this occurs for the quarterly but not the monthly case, and *(iii)* for $\theta_1 = -.4$, this occurs for neither the quarterly nor the monthly case. In fact, as θ_1 decreases from 1 towards -1, the dips in $f_s(\lambda)$ between the seasonal frequencies decrease relative to the local minimum at $\lambda = 0$. Eventually a θ_1 value is reached beyond which the global minimum of $f_s(\lambda)$ occurs at the dip between the last two seasonal frequencies, rather than at $\lambda = 0$. These θ_1 values define the lower limits of the ranges given by Table 1.

4 Results for the ARIMA$(0,1,1)(0,1,1)_s$ (airline) model

The ARIMA$(0,1,1)(0,1,1)_s$ (airline) model is (Box and Jenkins 1970, sec. 9.2)

$$(1-B)(1-B^s)y_t = (1-\theta_1 B)(1-\theta_2 B^s)a_t. \tag{14}$$

As with the $(0,0,1)(0,1,1)_s$ model, the nonseasonal and seasonal MA parameters θ_1 and θ_2 are restricted to lie in the interval $(-1,1)$, though again interest focuses on the case of $\theta_2 \geq 0$, for which existence of the canonical decomposition is assured. We again assume without loss of generality that $\mathrm{Var}(a_t) = 1$.

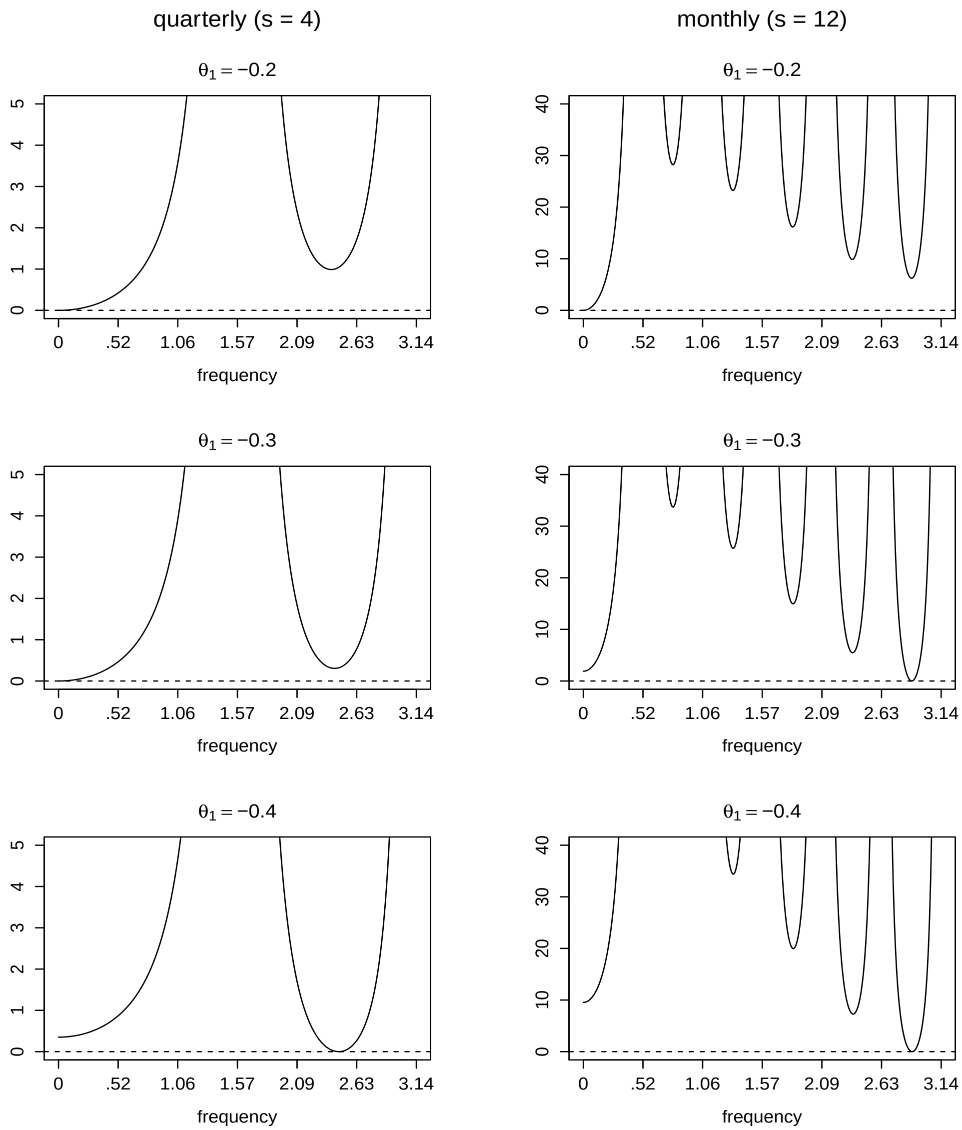

Figure 1: Plots of the (rescaled) canonical seasonal component spectrum, $f_s(\lambda)/(1-\theta_2)^2$, for the ARIMA$(0,0,1)(0,1,1)_s$ model. Plots are given for both the quarterly (left) and monthly (right) cases, for three values of θ_1: $-.2$, $-.3$, and $-.4$. When the minimum of $f_s(\lambda)$ occurs at frequency zero, the canonical symmetric seasonal filter includes $(1-B)^2(1-F)^2$. When the minimum occurs at a nonzero frequency, the canonical symmetric seasonal filter includes only $(1-B)(1-F)$.

HT (p. 67) observe that, for y_t following the model (14) with $\theta_2 \geq 0$, the seasonal part of the partial fractions decomposition of $\gamma_y(B)$ can be expressed as $Q_s^*(B)/U_s(B)U_s(F)$, where now

$$Q_s^*(B) = \frac{(1-\theta_2)^2}{(1-B)^2(1-F)^2} \times$$

$$\left\{ \frac{(1-\theta_1)^2}{4}(1+B)(1+F)\left[1 - \frac{1}{s^2}U_s(B)U_s(F) - \frac{s^2-1}{12s^2}(1-B^s)(1-F^s)\right] \right. \tag{15}$$

$$\left. + \frac{(1+\theta_1)^2}{4}(1-B)(1-F)\left[1 - \frac{1}{4s^2}U_s(B)U_s(F)(1+B)(1+F)\right]\right\}.$$

Appendix B simplifies the expression in braces in (15), showing that both of its terms contain $(1-B)^2(1-F)^2$, so that after cancellation with the $(1-B)^2(1-F)^2$ of the denominator, $Q_s^*(B)$ simplifies to

$$Q_s^*(B) = (1-\theta_2)^2 \left\{ \frac{(1-\theta_1)^2}{4}(1+B)(1+F)m_{s1}(B) + \frac{(1+\theta_1)^2}{4}m_{s2}(B) \right\}$$

where $m_{s1}(B)$ and $m_{s2}(B)$ are symmetric polynomials given in Appendix B. The spectrum of the canonical seasonal is then $(2\pi)^{-1}$ times $f_s(\lambda) = Q_s^*(e^{i\lambda})/|U_s(e^{i\lambda})|^2 - \epsilon_s$, where now

$$\epsilon_s = \min_{\lambda \in [0,\pi]} \frac{(1-\theta_2)^2}{|U_s(e^{i\lambda})|^2} \left\{ \frac{(1-\theta_1)^2}{4} 2[1+\cos(\lambda)]m_{s1}(e^{i\lambda}) + \frac{(1+\theta_1)^2}{4}m_{s2}(e^{i\lambda}) \right\}.$$

For $s = 2$, 4, and 12, and for a detailed set of values of θ_1, the minima ϵ_s were again determined by inspection, noting cases when the minimum occurs at $\lambda = 0$, so $\gamma_u(B)$ contains $(1-B)(1-F)$, implying that $\omega_S(B)$ contains $(1-B)^3(1-F)^3$ and not just $(1-B)^2(1-F)^2$. Table 2 gives the results which, as for Table 1, are unaffected by the value of θ_2. Analogously to Table 1, we see that, for $s = 2$, $\omega_S(B)$ contains $(1-B)^3(1-F)^3$ for any value of θ_1, while for $s = 4$ and $s = 12$, this occurs only for limited intervals of θ_1. This is unsurprising since plots of $f_s(\lambda)$ (not shown) reveal broadly similar patterns to the plots of Figure 1. However, the limited intervals for $s = 4$ and $s = 12$ given in Table 2 are much smaller than the corresponding intervals given in Table 1, and they exclude some positive values of θ_1.

Table 2. Range of values of θ_1 for which the canonical seasonal filter $\omega_S(B)$ from (7) for the ARIMA$(0,1,1)(0,1,1)_s$ (airline) model (14) includes $(1-B)^3(1-F)^3$, not just $(1-B)^2(1-F)^2$.

seasonal period s	2	4	12
range of values of θ_1	all $\theta_1 \in (-1,1)$	$.11 < \theta_1 < 1$	$.58 < \theta_1 < 1$

Figure 2 illustrates the result of Table 2 for the quarterly ($s = 4$) case. It shows plots of what results when the symmetric seasonal filter from the canonical decomposition of various airline models is applied to polynomials of degrees four and five, with the results plotted against the value of the airline model parameter θ_1, for values of θ_1 covering the interval $-.5 \le \theta_1 \le .5$. The two polynomials are of the form $100 \times (t-1)^k/30^k$ for $k = 4$ or $k = 5$. They both take the values 0 at $t = 1$ and 100 at $t = 31$, while at $t = 61$, the last time point used, they take the values $1,600$ (for $k = 4$) and $3,200$ (for $k = 5$). The parameter θ_2 was set to zero to minimize the effective length of the filter $\omega_S(B)$, so that its application at the mid-point of the series ($t = 31$) would be negligibly affected by the absence of data prior to $t = 1$ and after $t = 61$. Computations were done with the X-13-ARIMA-SEATS program.

Table 2 says that the values shown in Figure 2 should be zero for $\theta_1 > .11$, which is indeed the case. For $\theta_1 \le .11$ the values are positive, and they increase as θ_1 decreases further and further below .11. However, considering that the series value is 100 at the midpoint, and increases as t increases past 31, the seasonally filtered values seem quite small, and are smaller for the fourth degree polynomial than for the fifth. Thus, even for $\theta_1 \le .11$, the symmetric canonical seasonal filter comes close to reproducing the fourth and fifth degree polynomials.

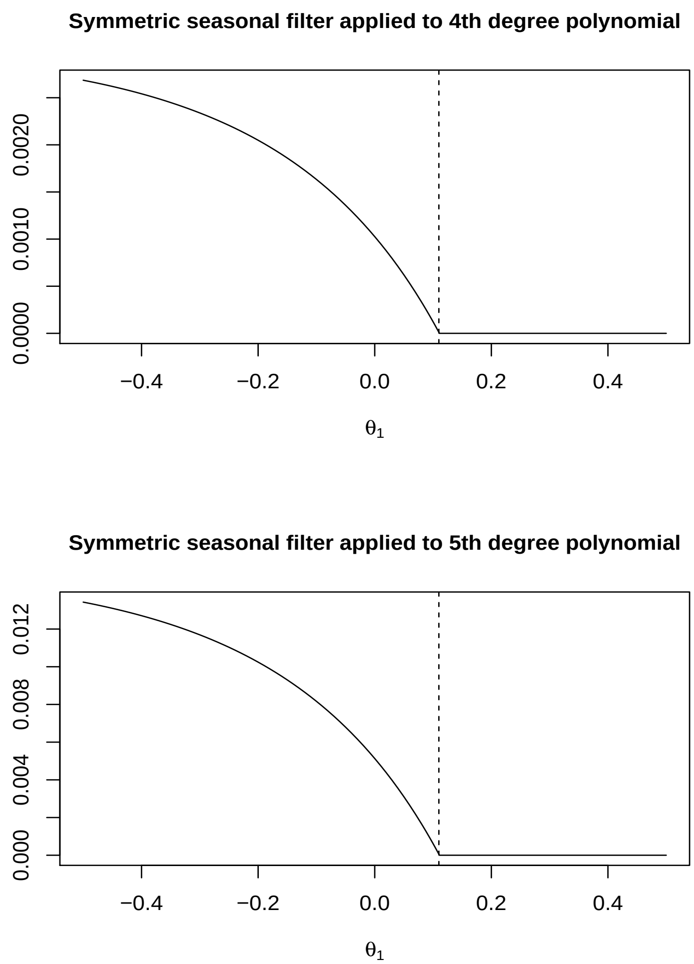

Figure 2: Canonical decomposition of quarterly airline model for various values of θ_1: Results from applying the symmetric seasonal filter to fourth (top) and fifth (bottom) degree polynomials in t. The dotted vertical lines are at $\theta_1 = .11$. See text for further details.

5 Additional results for canonical ARIMA model-based seasonal adjustment

For any particular seasonal ARIMA model for which the canonical decomposition exists one can obviously check for the presence of additional unit root factors in the various filters by examining the component models from the canonical decomposition. The computations can be done with the original SEATS program (Gomez and Maravall 1997) or the X-13-ARIMA-SEATS program (Monsell 2007), either of which will provide output tables giving the roots of the AR and MA polynomials of the component models. This approach was used to check some results of the previous two sections.

This approach was also applied to the $(1,1,0)(0,1,1)_{12}$ model $(1-\phi B)(1-B)(1-B^{12})y_t = (1-\theta B^{12})a_t$, to check for the presence of an extra $(1-B)(1-F)$ factor in the symmetric seasonal filter. For this model the results turn out to depend on the value of the seasonal moving average parameter θ, as well as on the value of ϕ. For $\theta = .7$, the extra $(1-B)(1-F)$ factor was found to be present for $\phi < -.6$, while for $\theta = .8$, it was found for $\phi \leq -.5$. This serves to illustrate that the extra $(1-B)(1-F)$ factor in the seasonal filter can indeed occur for models other than the two considered explicitly in Sections 3 and 4.

As noted earlier, for models of the form of (2) with $\sigma_I^2 > 0$, extra unit root factors are not present in the symmetric canonical irregular filter, and so the symmetric canonical trend filter will reproduce only polynomials up to degree $2d-1$, not degree $2d+1$. For models with $d=2$ and when $\omega_S(B)$ does contain the extra $(1-B)(1-F)$, $\omega_S(B)$ then contains $(1-B)^3(1-F)^3$ while $\omega_I(B)$ contains only $(1-B)^2(1-F)^2$, so $\omega_N(B)$ reproduces quintic polynomials in t while $\omega_T(B)$ reproduces only cubic polynomials. This matches analogous results for X-11 symmetric filters reported in Bell (2012, p. 449).

The quotation in Section 2 noted that HT considered a model for which the canonical trend model had a $1+B$ factor in its MA polynomial. This implies that $\gamma_v(B)$ contains $(1+B)(1+F)$, so that $\omega_T(B)$ given by (9) has this extra $(1+B)(1+F)$. The quotation refers to HT's treatment of the $(0,0,0)(0,1,1)_s$ model, which is the $(0,0,1)(0,1,1)_s$ model with $\theta_1 = 0$. In fact, HT's derivations for the $(0,0,1)(0,1,1)_s$ and the $(0,1,1)(0,1,1)_s$ models (the latter with $\theta_2 \geq 0$) show that the canonical trend spectrum is minimized at $\lambda = \pi$. Thus, for both these models $\gamma_v(B)$ contains $(1+B)(1+F)$, so that $\omega_T(B)$, which always contains $U_s(B)U_s(F)$, has this extra $(1+B)(1+F)$, and so includes $(1+B)^2(1+F)^2$.

Extra $1-B$ factors will not be present in asymmetric seasonal filters because application of such filters is equivalent to application of the corresponding symmetric seasonal filter $\omega_S(B)$ after forecast and backcast extension of the time series. Since the forecast and backcast extension will reproduce polynomials only up to degree $d-1$, this becomes the limiting factor in the degree of polynomials reproduced by the asymmetric seasonal adjustment and trend filters (Bell 2012, p. 447). The same argument applies to seasonal unit root factors contained in the asymmetric seasonal adjustment, trend, and irregular filters. For example, though we just noted that $\omega_T(B)$ from the models examined by HT will include $(1+B)^2(1+F)^2$ instead of just the expected $(1+B)(1+F)$, the asymmetric trend filters will include just the single $1+B$ factor.

The symmetric finite filters (the filters applied at $t = m+1$ for a time series of length $2m+1$) provide some further exceptions to the results for model-based adjustment from both canonical ARIMA and from structural component models. For the case of $d = 1$, all the finite seasonal and irregular filters will include $1-B$, so all will annihilate constants, which are then reproduced by the corresponding finite seasonal adjustment and trend filters (Bell 2012, Table 1). However, the finite symmetric seasonal and irregular filters must, by symmetry, then include $(1-B)(1-F)$, so they will annihilate linear polynomials in t, which are then what is reproduced by the symmetric finite seasonal adjustment and trend filters. The symmetry argument extends to odd values of $d > 1$. Thus, for $d = 3$, the symmetric finite seasonal and irregular filters cannot include just $(1-B)^3$, so they must include $(1-B)^2(1-F)^2$. Hence, they will annihilate cubics, not just quadratics. Values of $d \geq 3$ are seldom used in practice, however. Finally, since all the finite trend filters include $U_s(B)$ which includes the factor $1+B$, the symmetric finite trend filters must include $(1+B)(1+F)$ (Findley and Martin 2006, p. 29).

6 Special cases for structural component models

Special case results for the structural models proposed by the references cited in Section 2 differ from the special case results presented for canonical ARIMA seasonal adjustment. For the structural models a zero in the spectrum of a component will, in most cases, arise only if model fitting estimates zero for the variance of the component's stationary part – u_t, v_t, or I_t in (2). If that happens, the component becomes deterministic, not stochastic. If $\hat{\sigma}_I^2 = 0$, then $I_t = 0$, so it can be dropped from the model, and $N_t = T_t$. Assuming no other components have zero variances, the formulas (7)–(9) still apply (although (8) and (9) are now the same), and the results of Bell (2012) still apply to signal extraction estimation of S_t and $N_t = T_t$.

If $\text{var}(v_t)$ is estimated to be zero, the fitted model then has $(1-B)^d T_t = 0$, implying that T_t is a polynomial in t of degree $d-1$. We cannot leave the component model as $(1-B)^d T_t = v_t$ with $\text{var}(v_t) = 0$ and apply (9) since, from (6), setting $\gamma_v(B) = 0$ will produce a factor of $(1-B)^d(1-F)^d$ in $\gamma_w(B)$, violating an assumption that underlies the symmetric signal extraction formulas (7)–(10), as well as the corresponding asymmetric infinite filter formulas. Instead we replace the stochastic component T_t in the model by a polynomial regression function $\beta_0 + \beta_1 t + \cdots + \beta_{d-1} t^{d-1}$. The fitted value of this function provides $\hat{T}_t$, and the signal extraction estimate of N_t is then $\hat{T}_t + \omega_I(B)[y_t - \hat{T}_t]$ (assuming $\hat{\sigma}_I^2 > 0$). If this form of signal extraction estimation (including regression estimation of the $\beta_j s$) is applied to a time series y_t that is exactly a polynomial in t of degree $d-1$ or less, the polynomial will be reproduced in $\hat{T}_t$, and thus also in $\hat{N}_t$. This contrasts with the symmetric infinite filter estimates for seasonal adjustment and trend estimation that apply with $\text{var}(v_t) > 0$, which reproduce polynomials of degree $2d-1$. For related discussion on treatment of trend constants, see Bell (2010, pp. 5-6), including the proof given of Theorem 2.

Having $\text{var}(v_t) = 0$ is acceptable for finite sample signal extraction, but will produce the same results as modeling T_t as a $d-1$ degree polynomial regression function. Analogous results to those just described hold if u_t is estimated to have zero variance so S_t becomes fixed seasonal effects. See Harvey (1981) and Bell (1987) for discussion related to these two points.

Special case results are more involved for the local linear trend model of Harvey (1989, p. 37), which is

$$(1-B)T_t = \beta_t + \varepsilon_{1t} \qquad \text{where} \qquad (1-B)\beta_t = \varepsilon_{2t}$$

with ε_{1t} and ε_{2t} independent white noise series with variances $\sigma^2_{\varepsilon_1}$ and $\sigma^2_{\varepsilon_2}$. This model can be rewritten as ARIMA(0,2,1): $(1-B)^2 T_t = (1-\eta B)c_t$, with $\eta \in [0,1]$. If $\sigma^2_{\varepsilon_2} > 0$ then $\eta < 1$ and the usual results of Bell (2012) apply: the symmetric infinite filters $\omega_S(B)$ and $\omega_I(B)$ contain $(1-B)^2(1-F)^2$, and $\omega_N(B)$ and $\omega_T(B)$ reproduce cubics. If both $\sigma^2_{\varepsilon_1}$ and $\sigma^2_{\varepsilon_2}$ equal 0, then $(1-B)^2 T_t = 0$ and, from the previous discussion, $T_t = \beta_0 + \beta_1 t$ and estimation of N_t and T_t reproduces only linear functions of t. If $\sigma^2_{\varepsilon 1} > 0$ but $\sigma^2_{\varepsilon 2} = 0$, then β_t becomes a fixed trend constant β, and the model becomes a random walk with a trend constant, for which estimation of N_t and T_t again reproduces just linear polynomials. To summarize, if $\sigma^2_{\varepsilon_2} > 0$, then $\omega_N(B)$ and $\omega_T(B)$ in (8) and (9) reproduce cubics, while if $\sigma^2_{\varepsilon_2} = 0$, then asymmetric and finite sample signal extraction estimation of N_t and T_t reproduce only linear functions of t. Note that estimating $\sigma^2_{\varepsilon_2} = 0$ but $\sigma^2_{\varepsilon 1} > 0$, which is equivalent to estimating $\eta = 1$ in the ARIMA(0,2,1) formulation, occurs frequently in practice (Bell and Pugh 1990, Shephard 1993).

Appendix A: Derivation details for the ARIMA $(0,0,1)(0,1,1)_s$ model

We consider (12):

$$Q_s^*(B) = \frac{(1-\theta_2)^2(1-\theta_1 B)(1-\theta_1 F)}{(1-B)(1-F)}\left\{1 - \frac{1}{s^2}U_s(B)U_s(F)\right\}.$$

Applying $U_s(B)$ or $U_s(F)$ to the constant 1 yields s, so that applying $1 - 1/s^2 U_s(B)U_s(F)$ to 1 yields 0. This shows that $1 - 1/s^2 U_s(B)U_s(F)$ contains a factor $(1-B)$. Since

$$\begin{aligned} 1 - \frac{1}{s^2}U_s(B)U_s(F) \quad &= \quad \frac{1}{s^2}\left[s(s-1) - (s-1)(B+F) - (s-2)(B^2+F^2) - \right. \\ &\qquad \left. \cdots - 2(B^{s-2}+F^{s-2}) - (B^{s-1}+F^{s-1})\right] \end{aligned}$$

has symmetric coefficients, it must also contain $(1-F)$, and so can be expressed as $(1-B)(1-F)\alpha_s(B)$, where the polynomial $\alpha_s(B)$, which is of degree $s-2$ in B and F, also has symmetric coefficients. Cancelling the $(1-B)(1-F)$ factors in the numerator and denominator of $Q_s^*(B)$ then simplifies it to $(1-\theta_2)^2(1-\theta_1 B)(1-\theta_1 F)\alpha_s(B)$.

The coefficients of $\alpha_s(B)$ can be obtained using the following Lemma on division of polynomials in B by $1-B$ and $1-F$.

Lemma: Let $a(B) = a_0 + a_1 B + \cdots + a_k B^k$ be a polynomial in B of degree $k > 0$. Then

(i) $\frac{a(B)}{1-B} = a_0 + (a_0+a_1)B + \cdots + (a_0+\cdots+a_{k-1})B^{k-1} + \frac{(a_0+\cdots+a_k)B^k}{1-B}$, and

(ii) $\frac{a(B)}{1-F} = a_k B^k + (a_k + a_{k-1})B^{k-1} + \cdots + (a_k+\cdots+a_1)B + \frac{(a_k+\cdots+a_0)}{1-F}$.

If $a_0 + \cdots + a_k = 0$, then $a(B)$ contains $1-B$ (equivalently, contains $1-F$) as a factor.

Proof: Results (i) and (ii) are easily verified by writing their right-hand side expressions as single fractions and simplifying. The last statement follows since the remainder terms in (i) and (ii) are zero if $a_0 + \cdots + a_k = 0$.

Note from the Lemma that the coefficients of the $k-1$ degree polynomial that results from dividing $a(B)$ by $1-B$ can be obtained by cumulatively summing the coefficients of $a(B)$ or, for division by $1-F$, by cumulatively summing the coefficients of $a(B)$ in reverse order. Also, note that the Lemma can be applied to a polynomial in B and F. Thus, if $a(B) = a_{-j}F^j + \cdots + a_{-1}F + a_0 + a_1B + \cdots + a_kB^k$, we pre-multiply $a(B)$ by B^j, where j is the highest power of F in $a(B)$, then apply the Lemma, and then multiply the result from the division by $1-B$ or $1-F$ by F^j.

Using the Lemma, we obtained the coefficients of $\alpha_s(B)$ by cumulatively summing the coefficients of $1 - 1/s^2U_s(B)U_s(F)$, and then cumulatively summing the resulting coefficients in reverse order. The results of this are as follows for the three values of s that we consider:

$$
\begin{aligned}
s=2:&\qquad \alpha_2(B) = \frac{1}{4}\\
s=4:&\qquad \alpha_4(B) = \frac{1}{16}\left[10 + 4(B+F) + (B^2+F^2)\right]\\
s=12:&\qquad \alpha_{12}(B) = \frac{1}{144}[286 + 220(B+F) + 165(B^2+F^2) + 120(B^3+F^3)\\
&\qquad\qquad +84(B^4+F^4) + 56(B^5+F^5) + 35(B^6+F^6) + 20(B^7+F^7)\\
&\qquad\qquad +10(B^8+F^8) + 4(B^9+F^9) + (B^{10}+F^{10})].
\end{aligned}
$$

Appendix B: Derivation details for the ARIMA $(0,1,1)(0,1,1)_s$ (airline) model

For the airline model, we consider (15):

$$
\begin{aligned}
Q_s^*(B) = {} & \frac{(1-\theta_2)^2}{(1-B)^2(1-F)^2}\times\\
& \left\{\frac{(1-\theta_1)^2}{4}(1+B)(1+F)\left[1 - \frac{1}{s^2}U_s(B)U_s(F) - \frac{s^2-1}{12s^2}(1-B^s)(1-F^s)\right]\right.\\
& \left. + \frac{(1+\theta_1)^2}{4}(1-B)(1-F)\left[1 - \frac{1}{4s^2}U_s(B)U_s(F)(1+B)(1+F)\right]\right\}.
\end{aligned}
$$

We know from Appendix A that $1 - 1/s^2U_s(B)U_s(F) = (1-B)(1-F)\alpha_s(B)$. Also, $(1-B^s)(1-F^s) = (1-B)(1-F)U_s(B)U_s(F)$. The first term in brackets on the right-hand side above is thus $(1-B)(1-F)$ times $\alpha_s(B) - \frac{s^2-1}{12s^2}U_s(B)U_s(F)$. If, for each of the cases $s=2$, 4, and 12, we sum the coefficients of $\alpha_s(B) - \frac{s^2-1}{12s^2}U_s(B)U_s(F)$, and then reverse sum the resulting sequence, we find that the first and last values in this twice summed sequence are both zero. Thus, from the Lemma, $\alpha_s(B) - \frac{s^2-1}{12s^2}U_s(B)U_s(F) = (1-B)(1-F)m_{s1}(B)$, where $m_{s1}(B)$ is the symmetric polynomial whose coefficients are the nonzero terms of the sequence produced by this summing and reverse summing. For the second term in brackets on the right-hand side above, if we sum the coefficients of $1-(1/4s^2)U_s(B)U_s(F)(1+B)(1+F)$, and reverse sum the result, we get zero for the first and last coefficients, so that $1-(1/4s^2)U_s(B)U_s(F)(1+B)(1+F) = (1-B)(1-F)m_{s2}(B)$ for the symmetric polynomial $m_{s2}(B)$ whose coefficients we just produced. The terms in the second and third lines of the expression (15) for $Q_s^*(B)$ thus both contain $(1-B)^2(1-F)^2$, and cancelling this with the $(1-B)^2(1-F)^2$ in the denominator shows that

$$Q_s^*(B) = (1-\theta_2)^2 \left\{ \frac{(1-\theta_1)^2}{4}(1+B)(1+F)m_{s1}(B) + \frac{(1+\theta_1)^2}{4}m_{s2}(B) \right\}.$$

The polynomials $m_{s1}(B)$ and $m_{s2}(B)$ for the cases of $s = 2$, 4, and 12 are given below.

$$s=2: \qquad m_{2,1}(B) = \frac{1}{4} \qquad \text{and} \qquad m_{2,2}(B) = \frac{1}{16}(6+B+F)$$

$$\begin{aligned} s=4: \qquad m_{4,1}(B) &= \frac{3}{16}\left[26 + 16(B+F) + 5(B^2+F^2)\right] \\ m_{4,2}(B) &= \frac{1}{64}\left[44 + 19(B+F) + 6(B^2+F^2) + (B^3+F^3)\right] \end{aligned}$$

$$\begin{aligned} s=12: \qquad m_{12,1}(B) &= \frac{1}{1,728}[16,874 + 16,016(B+F) + 14,091(B^2+F^2) \\ &\quad +11,616(B^3+F^3) + 8,988(B^4+F^4) + 6,496(B^5+F^5) \\ &\quad +4,333(B^6+F^6) + 2,608(B^7+F^7) + 1,358(B^8+F^8) \\ m_{12,2}(B) &= \frac{1}{576}[1,156 + 891(B+F) + 670(B^2+F^2) + 489(B^3+F^3) \\ &\quad +344(B^4+F^4) + 231(B^5+F^5) + 146(B^6+F^6) \\ &\quad +85(B^7+F^7) + 44(B^8+F^8) + 19(B^9+F^9) \\ &\quad +6(B^{10}+F^{10}) + (B^{11}+F^{11})]. \end{aligned}$$

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