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Proof of Equivalence of Webster's Method and Willcox's Method of Major Fractions

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1 Introduction

This note is a simplification and expansion of the proof of the equivalence of Webster's method and Willcox's method of major fractions found in *Fair Representation: Meeting the Ideal of One Man, One Vote*, by Michel Balinski and H. Peyton Young, p. 103-104. The proof in *Fair Representation* demonstrates that Webster's method minimizes a function, which Willcox's method of major fractions also happens to minimize; it doesn't directly refer to Willcox's method. In this paper I have modified the proof to explicitly show that the two methods are equivalent and to include details, such as unstated lines of reasoning and algebraic steps, omitted from the *Fair Representation* proof.

1.1 Background

The method of apportioning seats in the US House of Representatives in line with the Constitution's requirements has been a continuous conflict. In the early 1900s, Walter Willcox concluded that Webster's method was the best method for apportioning seats in terms of not favoring either small or large states. However, Huntington's method, named for its most vocal proponent, Edward Huntington, was eventually adopted in 1941.

For the purposes of the proof, we will make assumptions about the apportionment methods that mirror the apportionment requirements for the US House of Representatives: 50 states, with each state being guaranteed at least one seat.

1.2 Terminology

Given that $\{1, ..., 50\}$ is a set of states, $\{P_1, ..., P_{50}\}$ is a set of populations where P_i is the population of state i, $\sum_{i=1}^{50} P_i = P$, and H is the number of representatives we have to apportion to all states, we assign values to $\{H_1, ..., H_{50}\}$ where H_i is the number of seats granted to state i and $\sum_{i=1}^{50} H_i = H$. We assume that for all i, $H_i \geq 1$, as is required by the US Constitution.

1.3 Webster's Method

To apportion seats via Webster's method, we find a divisor D_W such that

$$\sum_{i=1}^{50} [\frac{P_i}{D_W}] = H$$

where $[x]$ is the closest integer to x, and we assign $H_i = [\frac{P_i}{D_W}]$ seats to state i (Balinski and Young, 99).

1.4 An Important Inequality

From the definition of $[x]$, we can see that Webster's method is equivalent to choosing a divisor D_W such that for any state i,

$$H_i - \frac{1}{2} \leq \frac{P_i}{D_W} \leq H_i + \frac{1}{2} \ .$$

We then use algebraic manipulation to obtain, for each i,

$$\frac{1}{H_i - \frac{1}{2}} \geq \frac{D_W}{P_i} \geq \frac{1}{H_i + \frac{1}{2}}$$

$$\frac{P_i}{H_i - \frac{1}{2}} \geq D_W \geq \frac{P_i}{H_i + \frac{1}{2}} \ .$$

Hence, for $i \neq j$, $D_W \geq \frac{P_j}{H_j + \frac{1}{2}}$ and $D_W \leq \frac{P_i}{H_i - \frac{1}{2}}$. Equivalently, for all pairs of states i, j,

$$\frac{P_i}{H_i - \frac{1}{2}} \geq \frac{P_j}{H_j + \frac{1}{2}} \ . \qquad (1)$$

Thus, an apportionment using Webster's method is equivalent to an apportionment where (1) holds. This inequality will prove critical to the proof in both directions: showing that Willcox's method of major fractions leads to Webster's method, and vice versa.

2 Willcox's Method Implies Webster's Method

2.1 Willcox's Method

Willcox's method of major fractions apportions seats by picking the entire first column (to account for the restriction that every state has at least one seat) plus the $H - 50$ smallest other terms from the following array:

$$\begin{array}{cccc} \frac{0.5}{P_1} & \frac{1.5}{P_1} & \frac{2.5}{P_1} & \cdots \\ \frac{0.5}{P_2} & \frac{1.5}{P_2} & \frac{2.5}{P_2} & \cdots \\ \vdots & & & \\ \frac{0.5}{P_{50}} & \frac{1.5}{P_{50}} & \frac{2.5}{P_{50}} & \cdots \end{array}$$

State i gets an additional seat each time one of the values in row i is among the $H - 50$ smallest terms (Huntington (1928)). Note that Willcox's method of major fractions is guaranteed to produce at least one apportionment.

2.2 Proof

Choosing terms in the previous array via Willcox's method is equivalent to minimizing

$$\sum_{i=1}^{50} \sum_{j=1}^{H_i} \frac{j - 0.5}{P_i}$$

subject to the constraints $\sum_{i=1}^{50} H_i = H$ and for all i, $H_i \geq 1$. Multiplying the top by 2, we see that this is equivalent to minimizing

$$\sum_{i=1}^{50} \sum_{j=1}^{H_i} \frac{2j - 1}{P_i} \ .$$

We can easily show through mathematical induction that $H_i^2 = \sum_{j=1}^{H_i} (2j - 1)$, so the above double sum is equivalent to minimizing

$$\sum_{i=1}^{50} \frac{H_i^2}{P_i} \ . \tag{2}$$

We know that a minimizing solution $\{H_1, ..., H_{50}\}$ for (2) exists, because Willcox's method is guaranteed to produce an apportionment. This means if we start from the apportionment $\{H_1, ..., H_{50}\}$, transferring one representative from any one state to any other state cannot reduce the value of (2). Equivalently, for all pairs of states i, j with $i \neq j$,

$$\frac{(H_i - 1)^2}{P_i} + \frac{(H_j + 1)^2}{P_j} \geq \frac{H_i^2}{P_i} + \frac{H_j^2}{P_j} \ . \tag{3}$$

We can take the following algebra steps to prove that (3) is equivalent to (1).

$$\frac{H_i^2 - 2H_i + 1}{P_i} + \frac{H_j^2 + 2H_j + 1}{P_j} \geq \frac{H_i^2}{P_i} + \frac{H_j^2}{P_j}$$

$$\frac{-2H_i + 1}{P_i} + \frac{2H_j + 1}{P_j} \geq 0$$

$$\frac{2H_j + 1}{P_j} \geq \frac{2H_i - 1}{P_i}$$

$$\frac{P_j}{2H_j + 1} \leq \frac{P_i}{2H_i - 1}$$

$$\frac{P_j}{H_j + \frac{1}{2}} \leq \frac{P_i}{H_i - \frac{1}{2}}$$

So (3) is equivalent to (1), and (1) is equivalent to a Webster's method apportionment. Therefore, Willcox's method of major fractions implies Webster's method.

3 Webster's Method Implies Willcox's Method

The *Fair Representation* proof does not simply reverse the steps of the proof in Section 2. If the authors saw a problem with reversing the proof, they didn't include it in the text. No such problem is apparent, but for the sake of being conservative, this section will use the proof method found in the text.

3.1 What We Will Prove

Let $\{H_1, ..., H_{50}\}$ be a Webster's method apportionment. We will show that $\{H_1, ..., H_{50}\}$ minimizes (2), which, as we have demonstrated, is equivalent to showing that $\{H_1, ..., H_{50}\}$ is an apportionment via Willcox's method of major fractions.

3.2 New Notation

Let $\{x_1, ..., x_{50}\}$ be any apportionment such that $\sum_{i=1}^{50} x_i = H$ and for all i, $x_i \geq 1$. We can partition the fifty states into three sets by comparing the number of seats each state is apportioned in $\{x_1, ..., x_{50}\}$ versus $\{H_1, ..., H_{50}\}$:

$$S^+ = \{i | x_i > H_i\}$$

$$S^- = \{i | x_i < H_i\}$$

$$S^0 = \{i | x_i = H_i\} \ .$$

So a state is assigned to one of these sets depending on whether $\{x_1, ..., x_{50}\}$ apportions the state more, fewer, or the same number of seats as compared to Webster's method. If state i is in S^+, then there exists an integer $\delta_i \geq 1$ such that

$$x_i = H_i + \delta_i \ .$$

Similarly, if state j is in S^-, then there exists an integer $\mu_j \geq 1$ such that

$$x_j = H_j - \mu_j \ .$$

Both $\{x_1, ..., x_{50}\}$ and $\{H_1, ..., H_{50}\}$ must apportion H representatives. Therefore, if we wanted to change our apportionment from $\{x_1, ..., x_{50}\}$ to $\{H_1, ..., H_{50}\}$, the number of seats that states in S^+ gain must equal the number of seats that states in S^- lose. This means that there exists an integer $\alpha \geq 0$ such that

$$\alpha = \sum_{S^+} \delta_i = \sum_{S^-} \mu_j \ .$$

This new notation will prove critical to the proof.

3.3 An Important Inequality

Because $\{H_1, ..., H_{50}\}$ is a Webster's method apportionment, it fulfills (1). That is, for all pairs of states i and j,

$$\frac{P_i}{H_i - \frac{1}{2}} \geq \frac{P_j}{H_j + \frac{1}{2}} \ .$$

We can take the following algebra steps:

$$\frac{P_i}{2H_i - 1} \geq \frac{P_j}{2H_j + 1}$$

$$\frac{2H_j + 1}{P_j} \geq \frac{2H_i - 1}{P_i} \ . \tag{4}$$

Assume that state j is in S^+, and assume that state i is in S^-. That means that there exists a $\delta_j \geq 1$ associated with state j and a $\mu_i \geq 1$ associated with state i. Because $\delta_i \geq 1$ and $\mu_j \geq 1$, we can substitute these terms into (4) to get the following important inequality:

$$\frac{2H_j + \delta_j}{P_j} \geq \frac{2H_i - \mu_i}{P_i} \ . \tag{5}$$

This inequality will prove critical to the proof.

3.4 Proof

As noted earlier, we want to show that a Webster's method apportionment $\{H_1, ..., H_{50}\}$ will minimize (2). This is equivalent to showing that for any apportionment $\{x_1, ..., x_{50}\}$ satisfying $\sum_{i=1}^{50} x_i = H$ and for all i, $x_i \geq 1$,

$$\sum_{i=1}^{50} \frac{x_i^2}{P_i} \geq \sum_{i=1}^{50} \frac{H_i^2}{P_i} \ .$$

This is equivalent to showing that

$$\sum_{i=1}^{50} \frac{x_i^2 - H_i^2}{P_i} \geq 0 \ . \tag{6}$$

We can rewrite this sum to exclude x_i terms by taking advantage of our partition of states into S^+, S^- and S^0. If state i is in S^+, then $x_i = H_i + \delta_i$ and

$$x_i^2 - H_i^2 = (H_i + \delta_i)^2 - H_i^2 = H_i^2 + 2\delta_i H_i + \delta_i^2 - H_i^2 = (2H_i + \delta_i)\delta_i \ .$$

If state j is in S^-, then $x_j = H_j - \mu_j$ and

$$x_j^2 - H_j^2 = (H_j - \mu_j)^2 - H_j^2 = H_j^2 - 2\mu_j H_j + \mu_j^2 - H_j^2 = -(2H_j - \mu_j)\mu_j \ .$$

If state k is in S^0, then $x_k = H_k$ and

$$x_k^2 - H_k^2 = H_k^2 - H_k^2 = 0 \ .$$

From this, we can see that showing that (6) holds is equivalent to showing the following:

$$\sum_{i \in S^+} \frac{(2H_i + \delta_i)\delta_i}{P_i} - \sum_{j \in S^-} \frac{(2H_j - \mu_j)\mu_j}{P_j} + \sum_{k \in S^0} 0 \geq 0 \ .$$

This is equivalent to

$$\sum_{i \in S^+} \frac{(2H_i + \delta_i)\delta_i}{P_i} \geq \sum_{j \in S^-} \frac{(2H_j - \mu_j)\mu_j}{P_j}$$

which is equivalent to

$$\sum_{i \in S^+} \sum_{x=1}^{\delta_i} \frac{2H_i + \delta_i}{P_i} \geq \sum_{j \in S^-} \sum_{y=1}^{\mu_j} \frac{2H_j - \mu_j}{P_j} . \tag{7}$$

Thus, if (7) holds, then a Webster's method apportionment leads to an apportionment via Willcox's method of major fractions. We can combine two facts to prove that (7) must hold:

1. The number of terms on each side of (7) are equal. There are $\sum_{S^+} \delta_i$ terms on the left side and $\sum_{S^-} \mu_j$ terms on the right hand side. We have already shown that $\sum_{S^+} \delta_i = \sum_{S^-} \mu_j = \alpha$. Thus, there are α terms on each side of (7).

2. Each term on the left hand side of (7) is greater than or equal to each term on the right hand side of (7). This is true because the terms on the left and right of (7) are identical to the terms in (5), which we have already proved must hold for a Webster's method apportionment.

From these two facts, we conclude that (7) holds. This means that Webster's method solves the minimization problem equivalent to Willcox's method of major fractions. Therefore, Webster's method implies Willcox's method of major fractions.

4 Conclusion

We demonstrated in Section 2 that Willcox's method of major fractions implies Webster's method, and we demonstrated in Section 3 that Webster's method implies Willcox's method of major fractions. Thus, we have demonstrated the equivalence of the two methods.

References

Balinski, M. L. and Young, H. P. (2001). *Fair Representation: Meeting the Ideal of One Man, One Vote (Second Edition)*, Washington, D.C.: Brookings Institution Press.

Huntington, E. V. (1928). "The Apportionment of Representatives in Congress", *Transactions of the American Mathematical Society, Vol. 30*, 85-110.