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A Comparison of Statistical Disclosure Control Methods: Multiple Imputation Versus Noise Multiplication

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A Comparison of Statistical Disclosure Control Methods: Multiple Imputation Versus Noise Multiplication

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When statistical agencies release microdata to the public, a major concern is the control of disclosure risk, while ensuring utility in the released data. Often some statistical disclosure control methods such as data swapping, multiple imputation, top coding, and perturbation with random noise, are applied before releasing the data. This article provides a comprehensive comparison of two methods, namely, multiple imputation and noise multiplication, for drawing inference about some useful parameters under the exponential, normal and lognormal models. The comparison is provided under two scenarios: (1) the entire data set is replaced by multiply imputed or noise multiplied data, and (2) only the top part of the data is similarly replaced. The latter scenario arises, for example, when top coding is used for disclosure control, especially with income data. Methodology is developed for the analysis of noise multiplied data under both scenarios. Under the situation where only the large values in the dataset are noise multiplied, data analysis methods are developed and compared under two types of data releases: (i) each released value includes an indicator of whether or not it has been noise perturbed, and (ii) no such indicator is provided. The comparison study shows that data analyses under the multiple imputation and noise multiplication methods can provide similar results in terms of accuracy of statistical inferences; and that noise multi-

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plication can provide either more or less accuracy than multiple imputation by appropriately adjusting the variance of the noise generating distribution. Extensive simulation results provide guidance as to how the noise variance affects accuracy of inference in several parametric settings. A comparison using data from the 2000 U.S. Current Population Survey highlights the similarities of the methods. Detailed tables summarizing simulation results and some technical derivations are available online as supplementary material.

KEY WORDS: Customized noise distribution; EM algorithm; Partially synthetic data; Synthetic data; Tuning parameter.

1. INTRODUCTION

When survey organizations and statistical agencies such as the U.S. Census Bureau release microdata to the public, a major concern is the control of disclosure risk, while simultaneously ensuring quality and utility of the released data. Very often some popular statistical disclosure control methods such as data swapping, multiple imputation (MI), top coding/bottom coding (especially for income data), and multiplication with random noise, are applied before releasing the data. Multiple imputation has been in existence for some time as a viable methodology to handle missing data (see Rubin, 1987); following the initial proposal by Rubin (1993), in a series of papers (e.g., Drechsler and Reiter, 2010; Raghunathan, Reiter, and Rubin, 2003; Reiter, 2003, 2004, 2005a, 2005b) Reiter and his colleagues expanded its scope and provided a solid and rigorous foundation for its use so much so that statistical agencies can now employ this method for sensitive data protection while data users can carry out the required inference in a valid way. When MI is applied for statistical disclosure control, the multiply imputed data that are ultimately released are usually referred to as *synthetic data*. More recently, multiple imputation has been cleverly used by

An and Little (2007) as an alternative to top coding. Recall that top coding consists of censoring the top part of the data above a specified threshold, and is commonly used in the context of income data so that the identity of those in the top income bracket is protected. We refer to the recent monograph by Drechsler (2011) for a detailed discussion of multiple imputation as a tool for disclosure control. Noise perturbation by addition or multiplication has also been advocated by some statisticians as a possible data confidentiality protection mechanism (Hwang, 1986; Little, 1993; Kim and Winkler, 2003); recently there has a been a renewed interest on this topic (Nayak, Sinha and Zayatz, 2011; Sinha, Nayak and Zayatz, 2011).

In this paper we provide a comprehensive comparison of the above two methods, namely, multiple imputation (MI) and noise multiplication, for drawing inference about some useful parameters under three parametric models. Both of these methods have been applied to Census Bureau data products. For instance, multiply imputed synthetic data products have been developed and are available for the following: OnTheMap database of commuting patterns (Machanavajjhala et al., 2008), the Longitudinal Business Database (Kinney et al., 2011), the Survey of Income and Program Participation (Abowd et al., 2006), and American Community Survey group quarters data (Rodriguez, 2007; Hawala, 2008). Regarding application of noise multiplied data, the first public use microdata sample (PUMS) produced from the Survey of Business Owners (SBO) was released in August 2012 (`http://www.census.gov/econ/sbo/`), and noise multiplication was employed for confidentiality protection of some variables. Here each record corresponds to a business surveyed in the 2007 SBO, and a number of variables are provided relating to firm size, business characteristics, and business owner characteristics. In this data product, a number of steps are taken to protect confidentiality of businesses, and the variables relating to receipts, payroll, and employment are rounded and multiplied by random noise prior to release (2007 Survey of Business Owners (SBO) Public Use Microdata Sample (PUMS) Data Users Guide, 2012).

Given that both multiple imputation and noise multiplication are actually being employed for disclosure limitation, a comparison between these two is clearly of practical interest. It should however be noted that while methodology for data analysis is now well developed in the context of multiple imputation, such is not the case for noise multiplication. Thus a major goal of this work is the development of rigorous data analysis methodology for noise multiplied data. We have considered three popular univariate parametric models: exponential, normal, and lognormal, and have dealt with the problem of estimation of some standard parametric functions such as mean, variance and percentiles. We show that data analyses under the multiple imputation method as well as under the noise multiplication method can provide similar results in terms of accuracy of inferences, and under noise multiplication the inferences can be made either more or less accurate through adjustment of the variance of the noise generating distribution; in this sense, noise multiplication does offer some flexibility compared to multiple imputation.

When multiple imputation (MI) and noise multiplication (NM) are applied to the *entire* data at hand, we will refer to the scenario as *total MI* and *fully NM* case. Often times there are situations when a part of the data is sensitive and *must* not be released while the rest of the data can be used/released without any compromise. This is the set up of top coding mentioned above, where values above a certain threshold C are suppressed and only the number of values in the data set above C are reported along with the actual values below C. This is precisely the scenario considered by An and Little (2007), and they have developed data analysis methods based on multiple imputation of the data above C, in combination with the original values below C. We can also consider noise multiplication of the values above C, and release the data consisting of such noise multiplied values, along with the actual observations below C. Again, we will provide a comparison of An and Little's (2007) multiple imputation along with the noise multiplication idea just mentioned, for the three parametric models mentioned earlier. Under the scenario of noise multiplication of only values above the threshold C, we consider two types of data releases referred to as cases (i) and (ii). In case (i) data release, each released variable includes an indicator of whether or not it has been perturbed, while in case (ii), no such indicator is provided. Data analysis methods for both of these cases are developed in Section 3. Naturally, case (i) data appear to carry more information than case (ii) data, and so one would expect that case (i) would lead to more accurate inference than case (ii), perhaps at an increased disclosure risk. The case (i) data always yield more (or equally) accurate inferences than a top coded sample, whereas depending on the magnitude of the noise variance, case (ii) data can yield either more or less accurate inference than a top coded sample; we elaborate and justify these points in Section 6.

Here is the organization of the paper. In Section 2 we provide details of the statistical analysis for the case of *fully* noise perturbed data. Section 3 deals with developing statistical analysis based on noise perturbation of only top-coded data, keeping the rest of the data (below C) as it is. Appendix A provides proof of a technical result used in Section 3 and states some related results. Following the work of Reiter (2003) and An and Little (2007), we provide a brief review of the multiple imputation method for both the *total* MI and top-coded data situations in Section 4. Section 5 briefly mentions statistical data analysis under usual top coding so that we have censored data. Conclusions based on extensive numerical results are given in Section 6. In Section 7, an application using data from the 2000 U.S. Current Population Survey is given to illustrate the similarities of the methods. The data comprise of seventeen (17) variables measured on 51,016 heads of households (see Table 1 in Drechsler and Reiter, 2010, for a description of ten variables) including age, race, sex and marital status as key identifiers, and a mix of other categorical and numerical variables with non-Gaussian distributions, and with large percentages of values equal to zero. We have selected two such numerical variables with extreme sample sizes, namely, *total household income* with sample size = 50,661 and *household alimony payment* with sample size 206, and have carried out the analysis (after deleting the zero values). It turns out that a lognormal distribution fits both the datasets well, and the inferences drawn upon the original mean and variance and the log-scale mean and variance are comparable under the two methods: multiple imputation and noise multiplication with noise variance not exceeding 10%. In fact, the inferences closely reproduce those based on the original data. We conclude the paper with a summary and recommendations in Section 8.

We end this section with two general observations. First, while standard and often *optimum* parametric inference can be drawn for the three chosen simple probability models, such an analysis is far from being close to optimum or even simple when noise multiplication (NM) is used, either with full data or with top-coded data. We have essentially relied on the asymptotic theory, providing enough computational details of the MLEs and observed Fisher information matrices in each case. Second, we should point out that our approach to modify the microdata to protect the confidentiality of all records and carry out the analysis based on noise-modified microdata data is different from modifying the microdata when the goal is to release tables (Evans, Slanta, and Zayatz, 1998). Moreover, the focus of this paper is on data analysis methods based on noise-modified microdata rather than on a study of the effectiveness of the procedures in protecting the data.

2. DATA ANALYSIS UNDER FULL NOISE MULTIPLICATION

The problem of statistical data analysis under full noise perturbation, although old (see Hwang, 1986; Little, 1993; Kim and Winkler, 2003), has been recently revisited, and some *new* and *interesting* results have emerged in a *nonparametric* setup for estimation of the moments and for inference about the quantiles of a variable Y based on noise *multiplied* data (Nayak, Sinha and Zayatz, 2011; Sinha, Nayak and Zayatz, 2011). Briefly, Nayak et al. (2011) discussed at length various issues related to the statistical properties of random noise perturbation methods for data masking. Under the noise multiplication scenario, issues such as confidentiality protection, moment estimation, properties of balanced noise distribution, and effects on data quality and privacy protection in the context of tabular data were addressed at length. In a subsequent paper, Sinha et al. (2011) proposed some inferential procedures for quantile estimation based on noise multiplied micro data. It turns out that this is indeed a difficult inferential problem, and an empirical Bayes solution based on a nonparametric model was developed by the authors.

As far as we know, *efficient* model-based parametric inference procedures based on noise perturbed data are still lacking, with the sole exception of the work by Little (1993) where inference under noise multiplication is briefly mentioned, with a remark that *...this approach does not yield valid inferences for parameters....* Denoting by $y_1, \ldots, y_n$ a random sample of size n from the distribution of $Y \sim f_\theta(y)$, with unknown parameter θ, and by $R \sim h(r)$ the underlying noise variable (with a completely known distribution $h(r)$), the noise perturbed data $\boldsymbol{z} = (z_1, \ldots, z_n)$ with $z_i = y_i \times r_i$ can be thought of as a random sample from $Z \sim g_\theta(z) = \int f_\theta(\frac{z}{r}) h(r) \frac{dr}{r}$. The noise variables $r_1, \ldots, r_n$ are independent and identically distributed (iid) as R, and it is assumed that R is a nonnegative random variable. We assume that both Y and R are continuous random variables. When the parameters θ have moment-type interpretations based on Y, they admit simple unbiased (not necessarily *optimum*) estimates based on $\boldsymbol{z}$ (see Hwang, 1986; Nayak et al., 2011); however, efficient inference for θ based on $\boldsymbol{z}$ is far from being simple due mainly to the possible complexity of $g_\theta(z)$. In the context of masking by noise multiplication, unlike our setup where R has a specified noise distribution, independent of the data Y, Kim (1986), Sullivan and Fuller (1989, 1990) and Little (1993) dealt with the case when R is made data-dependent! This procedure, while it keeps intact certain basic moments of the original data, obviously renders considerable difficulty in the inference process. In this context, it is rather interesting to quote Little (1993), which indeed provides a compelling motivation for our research: *Although a full likelihood-based* *analysis may not be feasible in many settings, I think the modeling perspective provides a useful basis for assessing simpler approximate methods. Future work might provide more detailed applications of the modeling approach to specific masking procedures.* Our goal here is to provide exact and approximate efficient inference procedures when noise multiplication is used as a data masking mechanism.

Here is a brief outline of our approach. In view of the anticipated complexity of the *marginal* likelihood based on $\boldsymbol{z}$, we can apply the familiar EM algorithm to compute the MLE of θ (Little and Rubin, 2002). It is also possible to use a parametric bootstrap procedure to study the frequentist properties of the MLE (Efron and Tibshirani, 1994). However we instead derive the observed Fisher information matrices and use them to estimate standard deviation in our extensive simulations. We construct 95% confidence intervals as (MLE) $\pm$ (1.96 $\times$ estimated standard deviation). Under the EM framework, we can write $\boldsymbol{u}_{\text{obs}} = (z_1, ..., z_n)$, $\boldsymbol{u}_{\text{mis}} = (r_1, ..., r_n)$, $\boldsymbol{u}_{\text{c}} = (\boldsymbol{u}_{\text{obs}}, \boldsymbol{u}_{\text{mis}})$, to denote the observed data, missing data and complete data, respectively. Using the notations in the previous paragraph, the complete data likelihood can obviously be expressed as $L(\theta|\boldsymbol{u}_{\text{c}}) = \prod_{i=1}^{n}[f_\theta(\frac{z_i}{r_i})\frac{h(r_i)}{r_i}]$ and the observed data likelihood as $L(\theta|\boldsymbol{u}_{\text{obs}}) = \prod_{i=1}^{n}[\int f_\theta(\frac{z_i}{r_i})\frac{h(r_i)}{r_i}dr_i]$.

Taking logarithm, if $\ell(\theta|\boldsymbol{u}_{\text{c}}) = \ln L(\theta|\boldsymbol{u}_{\text{c}})$ (ignoring constants), the E-step is then carried out by starting with the estimate $\theta^{(t)}$ (at the tth step) and computing

$$Q(\theta|\theta^{(t)}) = E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{u}_{\text{c}})|\boldsymbol{u}_{\text{obs}}] = E_{\theta^{(t)}}\left[\sum_{i=1}^{n} \ln f_\theta(\frac{z_i}{r_i})|z_1, ..., z_n\right] = \sum_{i=1}^{n} E_{\theta^{(t)}}\left[\ln f_\theta(\frac{z_i}{r_i})|z_i\right] \quad (1)$$

and the M-step is carried out by maximizing $Q(\theta|\theta^{(t)})$ with respect to θ, resulting in $\theta^{(t+1)}$. It would be rather easy to evaluate the one dimensional integral (with respect to r) in $Q(\theta|\theta^{(t)})$ above, either explicitly or numerically. The iteration defined through the E and M-steps can then be run until a stopping criterion is met. For many choices of $f_\theta(y)$, the M-step will have a closed form expression. The E-step is also quite feasible since the one-dimensional integrals appearing in (1) are straightforward to evaluate using numerical or Monte Carlo methods.

As for the noise distribution, we have taken a popular choice $R \sim h(r) \sim$ Uniform$(1-\epsilon, 1+\epsilon)$ for some $0 < \epsilon < 1$ with $E(R) = 1$ and $\text{var}(R) = \sigma_r^2 = \epsilon^2/3$. Computation of the MLEs, and the derivation of the observed Fisher information are both described in Appendix B of the supplementary material for the exponential, normal and lognormal models. For exponential and lognormal distributions, we have also used what we call *customized* noise distributions that can provide simplified inference because they permit closed form evaluation of $g_\theta(z)$. These are given by

$$\text{Under the exponential distribution :} \quad R \sim h_\delta(r) = \frac{\delta^{\delta+1}}{\Gamma(\delta+1)} r^{-\delta-2} e^{-\frac{\delta}{r}}, \delta > 1,$$

$$\text{Under the lognormal distribution :} \quad R \sim \text{lognormal with } \ln R \sim N(-\frac{\psi^2}{2}, \psi^2).$$

3. DATA ANALYSIS UNDER NOISE MULTIPLICATION OF EXTREME VALUES

We now consider the set up where top coding is used for disclosure control. The random variables y_i, r_i, and z_i, $i = 1, ..., n$, are as defined in Section 2. Now, any observation y_i which exceeds a specified threshold $C > 0$ is considered sensitive and, under top coding such values are simply not reported. Another option is to report z_i, the noise perturbed version of y_i, along with an indicator that the true value has been perturbed. More precisely, for $i = 1, ..., n$, let us define

$$\Delta_i = I(y_i \le C), \quad \text{and} \quad x_i = \begin{cases} y_i, & \text{if } y_i \le C, \\ z_i, & \text{if } y_i > C. \end{cases} \tag{2}$$

Inference for θ will be based on $\{(x_1, \Delta_1), ..., (x_n, \Delta_n)\}$, or based on $\{x_1, ..., x_n\}$. We note that the latter data do not directly identify which observations have been perturbed, and hence provide more disclosure control than the former. The expression for the joint distribution of (x_i, r_i, Δ_i), which is crucial for our purpose, is given by the following proposition, whose proof appears in Appendix A.

Proposition 1. The joint *pdf* of (x_i, r_i, Δ_i) is given by

$$k_\theta(x_i, r_i, \delta_i) = \begin{cases} f_\theta(x_i)h(r_i), & \text{if } x_i < C,\ 0 < r_i < \infty,\ \delta_i = 1, \\ \\ f_\theta(x_i/r_i)h(r_i)(1/r_i), & \text{if } 0 < r_i < x_i/C,\ \delta_i = 0, \\ \\ 0, & \text{if } \delta_i = 0,\ r_i > x_i/C, \\ & \text{or if } \delta_i = 1, x_i > C. \end{cases} \tag{3}$$

Proposition 1 can be used to derive the likelihood function of θ based on data of the two types: $\{(x_1, \Delta_1), ..., (x_n, \Delta_n)\}$ and $(x_1, ..., x_n)$; details appear in Appendix A. We refer to the former data type as case (i), and the latter as case (ii). Letting $k_\theta(x, \delta)$ and $k_\theta(x)$ be the densities defined in Appendix A, we now describe details of the computation of the maximum likelihood estimate of θ based on the EM algorithm. We shall separately consider cases (i) and (ii).

Case (i): Observed data consist of $\{(x_1, \Delta_1), ..., (x_n, \Delta_n)\}$.

In order to describe the EM algorithm under this setting, we define the observed, missing, and complete data, respectively, as follows: $\boldsymbol{v}_{\text{i,obs}} = (x_1, ..., x_n, \Delta_1, ..., \Delta_n)$, $\boldsymbol{v}_{\text{i,mis}} = (r_1, ..., r_n)$, $\boldsymbol{v}_{\text{c}} = (\boldsymbol{v}_{\text{i,obs}}, \boldsymbol{v}_{\text{i,mis}})$. By Proposition 1, the joint density function of $\boldsymbol{v}_{\text{c}}$ is $\prod_{i=1}^n [f_\theta(x_i)h(r_i)]^{\Delta_i}[f_\theta(x_i/r_i)h(r_i)(1/r_i)]^{1-\Delta_i}$, and thus we define the complete data likelihood function as $L(\theta|\boldsymbol{v}_{\text{c}}) = \prod_{i=1}^n f_\theta(x_i)^{\Delta_i} f_\theta(x_i/r_i)^{1-\Delta_i}$, and the complete data log-likelihood function as

$$\ell(\theta|\boldsymbol{v}_{\text{c}}) = \sum_{i=1}^n \{\Delta_i \ln f(x_i;\theta) + (1-\Delta_i) \ln f_\theta(x_i/r_i)\}\,. \tag{4}$$

The E and M-steps are then computed as follows.

E-step. To carry out the E-step, we compute:

$$\begin{aligned} Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{v}_{\mathrm{c}})|\boldsymbol{v}_{\mathrm{i,obs}}] \\ &= \sum_{i=1}^{n}\{\Delta_i \ln f_\theta(x_i) + E_{\theta^{(t)}}[(1-\Delta_i)\ln f_\theta(x_i/r_i)|x_i,\Delta_i]\} \\ &= \sum_{i=1}^{n}\{\Delta_i \ln f_\theta(x_i) + (1-\Delta_i)E_{\theta^{(t)}}[\ln f_\theta(x_i/r_i)|x_i,\Delta_i=0]\} \\ &= \sum_{i=1}^{n}\left\{\Delta_i \ln f_\theta(x_i) + (1-\Delta_i)\psi(x_i,\theta^{(t)},\theta)\right\}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \psi(x_i,\theta^{(t)},\theta) &\equiv E_{\theta^{(t)}}[\ln f_\theta(x_i/r_i)|x_i,\Delta_i=0] \\ &= \int_0^\infty [\ln f_\theta(x_i/r_i)]\frac{k_{\theta^{(t)}}(x_i,r_i,\Delta_i=0)}{k_{\theta^{(t)}}(x_i,\Delta_i=0)}dr_i \\ &= \frac{\int_0^{x_i/C}[\ln f_\theta(x_i/r_i)]f_{\theta^{(t)}}(x_i/r_i)h(r_i)r_i^{-1}dr_i}{\int_0^{x_i/C} f_{\theta^{(t)}}(x_i/\omega)h(\omega)\omega^{-1}d\omega}. \end{aligned} \tag{6}$$

M-step. If $\theta^{(t)}$ is the current estimate of θ, then the M step updates the current estimate to $\theta^{(t+1)} = \arg\max_\theta Q(\theta|\theta^{(t)})$.

Case (ii): Observed data consist of only $(x_1, ..., x_n)$.

To describe the EM algorithm in this setting, we re-define the observed and missing data, respectively, as $\boldsymbol{v}_{\mathrm{ii,obs}} = (x_1, ..., x_n)$, $\boldsymbol{v}_{\mathrm{ii,mis}} = (r_1, ..., r_n, \Delta_1, ..., \Delta_n)$. The complete data $\boldsymbol{v}_{\mathrm{c}} = (\boldsymbol{v}_{\mathrm{ii,obs}}, \boldsymbol{v}_{\mathrm{ii,mis}})$ remains the same as in case (i), and hence the complete data log-likelihood function is given by (4). The E and M-steps are then modified as follows.

E-step. To carry out the E-step, we now compute:

$$
\begin{aligned}
Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{v}_{\mathrm{c}})|\boldsymbol{v}_{\mathrm{ii,obs}}] \\
&= \sum_{i=1}^{n} \{E_{\theta^{(t)}}[\Delta_i|x_i] \ln f_\theta(x_i) + E_{\theta^{(t)}}[(1-\Delta_i)\ln f_\theta(x_i/r_i)|x_i]\} \\
&= \sum_{i=1}^{n} \left\{\psi_1(x_i,\theta^{(t)}) \ln f_\theta(x_i) + \psi_2(x_i,\theta^{(t)},\theta)\right\}, \qquad (7)
\end{aligned}
$$

where

$$
\begin{aligned}
\psi_1(x_i,\theta^{(t)}) &= E_{\theta^{(t)}}[\Delta_i|x_i] = \mathrm{Pr}_{\theta^{(t)}}[\Delta_i = 1|x_i] = \frac{k_{\theta^{(t)}}(x_i,\Delta_i=1)}{k_{\theta^{(t)}}(x_i)} \\
&= \frac{I(x_i<C) f_{\theta^{(t)}}(x_i)}{I(x_i<C) f_{\theta^{(t)}}(x_i) + I(x_i>0)\int_0^{x_i/C} f_{\theta^{(t)}}(x_i/\omega)h(\omega)\omega^{-1}d\omega}, \qquad (8)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi_2(x_i,\theta^{(t)},\theta) &= E_{\theta^{(t)}}[(1-\Delta_i)\ln f_\theta(x_i/r_i)|x_i] \\
&= \sum_{\Delta_i=0}^{1}\int_0^\infty (1-\Delta_i)[\ln f_\theta(x_i/r_i)]\frac{k_{\theta^{(t)}}(x_i,r_i,\delta_i)}{k_{\theta^{(t)}}(x_i)}dr_i \\
&= \int_0^\infty [\ln f_\theta(x_i/r_i)]\frac{k_{\theta^{(t)}}(x_i,r_i,\delta_i=0)}{k_{\theta^{(t)}}(x_i)}dr_i \\
&= \frac{\int_0^{x_i/C}[\ln f_\theta(x_i/r_i)] f_{\theta^{(t)}}(x_i/r_i)h(r_i)r_i^{-1}dr_i}{I(x_i<C) f_{\theta^{(t)}}(x_i) + I(x_i>0)\int_0^{x_i/C} f_{\theta^{(t)}}(x_i/\omega)h(\omega)\omega^{-1}d\omega}. \qquad (9)
\end{aligned}
$$

M-step. As usual, if $\theta^{(t)}$ is the current estimate of θ, then the M step updates the current estimate to $\theta^{(t+1)} = \arg\max_\theta Q(\theta|\theta^{(t)})$.

Application of the above computational steps to the exponential, normal and lognormal models is explained in Appendix C of the supplementary material, and expression for the observed Fisher information are provided.

4. DESCRIPTION OF MULTIPLE IMPUTATION METHODS

In the context of statistical disclosure control, MI is used to create public use data that protect confidential records from disclosure, while still yielding valid inferences for population quantities. The MI methodology is generally motivated from a Bayesian perspective, but it has been shown that the inferences obtained are often properly calibrated in the frequentist sense; we refer to the references given above. To implement this method, the data collector, who would have access to the confidential records, formulates an appropriate model for the data. To complete the Bayesian model specification, noninformative prior distributions are usually placed on model parameters. A public use data file is then created by deleting all confidential records (which may include all or part of the data) and imputing the deleted values, say $m > 1$ times, using m independent draws from the posterior predictive distribution. Hence, the public use file appears as m data sets, each of the same structure as the original, but with confidential records replaced by imputed values. Such a public use file is referred to in the literature as a *synthetic data* file. The number of imputations m is chosen to be larger than 1, often in the range of 5-10, which allows the additional variance due to imputation to be properly incorporated into the inferences obtained from the synthetic data. The above is a brief general description of MI for statistical disclosure control which is based on the method of Reiter (2003). Variants of this method have also appeared in the literature, appropriate for certain types of data; these include imputing the population m times, and then sampling from each synthetic population as in Raghunathan, Reiter, and Rubin (2003).

We now briefly review the MI methods which are applicable to the two general scenarios considered in this paper: (1) the case of total MI, which serves as a competitor to the noise perturbation methods developed in Section 2, and (2) the case of MI for extreme values only, which serves as a competitor to the noise perturbation methods developed in Section 3. The original data are assumed to consist of an iid sample $\boldsymbol{y} = (y_1, ..., y_n)$ from a parametric model $f_\theta(y)$, and we place a noninformative prior distribution $p(\theta)$ on the unknown parameter θ. We let $Q = Q(\theta)$ denote the population quantity that we wish to draw inference upon, using the multiply imputed data. For ease of presentation, we assume Q is a scalar. Inferences for Q are obtained from the synthetic data as follows. Let $q = q(\boldsymbol{y})$ denote an estimator of Q that is computed from the original data, and let $v = v(\boldsymbol{y})$ denote an estimator of the variance of q, also computed from the original data. For instance, q may be the maximum likelihood estimator of Q, and v may be the estimated asymptotic variance obtained from the inverse of the observed Fisher information matrix. Let $\boldsymbol{y}^{*(1)}, ..., \boldsymbol{y}^{*(m)}$ denote the m sets of multiply imputed/synthetic data. We discuss how to generate these multiply imputed data sets in the subsections below. Given the synthetic data, one then proceeds to compute $q_j = q(\boldsymbol{y}^{*(j)})$ and $v_j = v(\boldsymbol{y}^{*(j)})$, the analogs of q and v, on the jth synthetic data set, for $j = 1, ..., m$. The appropriate MI estimators for the settings considered here are given by Reiter (2003). The MI estimator of Q is $\bar{q}_m = \frac{1}{m}\sum_{j=1}^m q_j$, and the estimator of the variance of $\bar{q}_m$ is $T_m = b_m/m + \bar{v}_m$, where $b_m = \frac{1}{m-1}\sum_{j=1}^m (q_j - \bar{q}_m)^2$ and $\bar{v}_m = \frac{1}{m}\sum_{j=1}^m v_j$. We next discuss the generation of $\boldsymbol{y}^{*(1)}, ..., \boldsymbol{y}^{*(m)}$ for the cases of total MI, and MI for extreme values.

4.1 Total MI

In the case of total MI, the entire original data set is confidential. Thus, as a method of statistical disclosure control, total MI serves as an alternative to the noise multiplication methods developed in Section 2. The posterior distribution is obtained as usual via Bayes theorem: $p(\theta|\boldsymbol{y}) \propto p(\theta) \times \prod_{i=1}^n f_\theta(y_i)$. The synthetic data are then obtained as follows.

1. Draw θ^* from the posterior distribution $p(\theta|\boldsymbol{y})$.

2. Draw $\boldsymbol{y}^* = (y_1^*, ..., y_n^*)$ as iid from $f_{\theta^*}(y)$.

Steps (1) and (2) above are repeated independently m times to obtain $\boldsymbol{y}^{*(1)}, ..., \boldsymbol{y}^{*(m)}$, the multiply imputed/synthetic data. The notation $f_{\theta^*}(y)$ denotes the *pdf* $f_\theta(y)$ with the unknown parameter θ set equal to θ^*.

4.2 MI for Extreme Values

In the case of MI for extreme values, only those data values above a known threshold C are considered to be confidential. Thus, as a method of statistical disclosure control, MI for extreme values serves as an alternative to the noise multiplication methods developed in Section 3. In this scenario, we consider two MI methods presented by An and Little (2007), namely, parametric MI based on complete data (PMIC) and parametric MI based on deleted data (PMID). We now briefly describe these methods. Although it is only the values above C that are considered sensitive, a cut-point $C_I < C$ is selected, and any value $y_i > C_I$ is imputed. As discussed by An and Little (2007), by choosing the cut-point C_I to be less than C, a mixing of sensitive and non-sensitive values is achieved, which should enhance the level of protection against disclosure. Let $\boldsymbol{y} = (\boldsymbol{y}_{\text{ret}}, \boldsymbol{y}_{\text{del}})$ where $\boldsymbol{y}_{\text{ret}} = \{y_i : y_i \leq C_I\}$ denotes the values in $\boldsymbol{y}$ that will be retained, and $\boldsymbol{y}_{\text{del}} = \{y_i : y_i > C_I\}$ denotes the values in $\boldsymbol{y}$ that will be deleted.

The PMIC method proceeds as follows. One first fits the parametric model to the complete data $\boldsymbol{y}$, and thus obtains the posterior distribution $p(\theta|\boldsymbol{y}) \propto p(\theta) \times \prod_{i=1}^n f_\theta(y_i)$. Then the synthetic data are generated as follows.

1. Draw θ^* from the posterior distribution $p(\theta|\boldsymbol{y})$.

2. Obtain the imputed data $\boldsymbol{y}^*_{\text{del}}$, which is of the same length as $\boldsymbol{y}_{\text{del}}$, by taking *iid* draws from the truncated version of $f_\theta(y)$ defined as $f_{(C_I,\infty)}(y|\theta^*) = \frac{f_{\theta^*}(y) \times I_{(C_I,\infty)}(y)}{\int_{C_I}^\infty f_{\theta^*}(u)du}$.

Steps (1) and (2) above are repeated independently m times to obtain $\boldsymbol{y}^{*(1)}_{\text{del}}, ..., \boldsymbol{y}^{*(m)}_{\text{del}}$. Finally, we obtain the synthetic data as $\boldsymbol{y}^{*(j)} = (\boldsymbol{y}_{\text{ret}}, \boldsymbol{y}^{*(j)}_{\text{del}})$, $j = 1, ..., m$.

Now we describe the PMID method. In the PMID method, one fits the parametric model to the deleted data $\boldsymbol{y}_{\text{del}}$ instead of the complete data. That is, the posterior distribution of θ

is computed as $p(\theta|\boldsymbol{y}_{\rm del}) \propto p(\theta) \times \prod_{\{i: y_i \in \boldsymbol{y}_{\rm del}\}} f_\theta(y_i)$, and synthetic data are generated through the following steps.

1. Draw θ^* from the posterior distribution $p(\theta|\boldsymbol{y}_{\rm del})$.

2. Obtain the imputed data $\boldsymbol{y}^*_{\rm del}$, which again is of the same length as $\boldsymbol{y}_{\rm del}$, by taking *iid* draws from $f_{\theta^*}(y)$. Note here that we do not draw from the truncated distribution as in PMIC.

As usual, steps (1) and (2) above are repeated independently m times to get $\boldsymbol{y}^{*(1)}_{\rm del}, ..., \boldsymbol{y}^{*(m)}_{\rm del}$, and the synthetic data are then obtained as $\boldsymbol{y}^{*(j)} = (\boldsymbol{y}_{\rm ret}, \boldsymbol{y}^{*(j)}_{\rm del})$, $j = 1, ..., m$. A comparison of PMID and PMIC methods will be reported later, based on numerical results.

4.3 Details for the Exponential, Normal and Lognormal Distributions

We now briefly review the forms of the prior and posterior distributions under the exponential, normal, and lognormal models. In the following, we present the form of the posterior assuming that the complete data $\boldsymbol{y}$ are used to fit the model. We also use the notation $\boldsymbol{y}$ to denote the observed data. The necessary adjustments for the PMID method are straightforward.

Exponential distribution. In this case we have $f_\theta(y) = \frac{1}{\theta} e^{-y/\theta}$, $0 < y < \infty$, and we take the improper uniform prior $p(\theta) \propto 1$, $0 < \theta < \infty$. The posterior distribution is readily obtained as

$$p(\theta|\boldsymbol{y}) = \frac{(\sum_{i=1}^n y_i)^{n-1}}{\Gamma(n-1)} \theta^{-(n-1)-1} e^{-(\sum_{i=1}^n y_i)/\theta}, \quad 0 < \theta < \infty,$$

which has the form of an inverse gamma distribution. That is, $(\theta^{-1}|\boldsymbol{y}) \sim \text{Gamma}\left(n-1, \frac{1}{\sum_{i=1}^n y_i}\right)$.

Normal distribution. In this case we have $f_\theta(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-\mu)^2/(2\sigma^2)}$, $-\infty < y < \infty$, $\theta = (\mu, \sigma^2)$, and we specify a standard noninformative prior which places a uniform distribution on $(\mu, \ln\sigma)$ and assumes these two quantities to be independent a priori. Equivalently, the prior on (μ, σ^2) is $p(\mu, \sigma^2) \propto (\sigma^2)^{-1}, -\infty < \mu < \infty$, $0 < \sigma^2 < \infty$. The posterior distribution is then readily obtained as $p(\mu, \sigma^2|\boldsymbol{y}) = p(\mu|\sigma^2, \boldsymbol{y})p(\sigma^2|\boldsymbol{y})$ where

$$(\sigma^2|\boldsymbol{y}) \sim \frac{(n-1)s^2}{\chi^2_{n-1}},\ (\mu|\sigma^2, \boldsymbol{y}) \sim N(\bar{y}, \sigma^2/n),$$

with $\bar{y} = \frac{1}{n}\sum_{i=1}^n y_i$ and $s^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2$ (Gelman et al., 2003).

Lognormal distribution. In this case we have $f_\theta(y) = \frac{1}{y\sigma\sqrt{2\pi}} e^{-(\ln y - \mu)^2/(2\sigma^2)}$, $0 < y < \infty$, $\theta = (\mu, \sigma^2)$. But since $\ln y_i \sim N(\mu, \sigma^2)$, we simply apply the normal distribution results upon replacing y_i by $\ln y_i$, C_I by $\ln C_I$, and C by $\ln C$.

5. ANALYSIS OF TOP CODED DATA

The statistical analysis of top coded data is essentially asymptotic in nature, based on the MLEs of the parameters and the observed Fisher information. The MLE computation along with the expressions for the observed Fisher information matrix for the three parametric models are explained in Appendix D of the supplementary material.

6. NUMERICAL RESULTS

In order to compare multiple imputation with noise multiplication in terms of accuracy of the inference, we performed a rather extensive simulation study in the context of the one parameter exponential distribution, the normal distribution, and the lognormal distribution. Furthermore, under each distribution, we considered the case of full noise perturbation and total multiple imputation, as well as the case of noise perturbation and multiple imputation for the situation of top coded data. Tables resulting from the simulation study are provided in Appendix E of the supplementary material.

The setups considered for the simulation are as follows. For the exponential distribution, the mean was chosen to be one, and the inference problem considered is the point and interval estimation of the mean. For the normal and lognormal distributions, simulations were done under the parameter choices $\mu = 0$ and $\sigma^2 = 1$. For the normal distribution, point and interval estimation were considered for the mean, variance, 84th percentile (i.e., $\mu+\sigma$) and the 95th percentile (i.e., $\mu+1.645\sigma$). For the lognormal distribution, inferences on the same parameters, namely, μ, σ^2, the 84th percentile (i.e., $e^{\mu+\sigma}$) and the 95th percentile (i.e., $e^{\mu+1.645\sigma}$) were considered as well as those for the lognormal mean ($e^{\mu+\frac{\sigma^2}{2}}$) and the lognormal variance ($e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$). A nominal confidence level of 95% was used in all interval estimation problems. For the point estimates of the various parameters, the following quantities were estimated by Monte Carlo simulation based on 5000 iterations: the root mean squared error (RMSE), bias, standard deviation (SD), and expected value of estimated standard deviation ($\widehat{\text{SD}}$). For the confidence interval, the coverage probability was estimated and the expected length of the nominal 95% two-sided confidence interval was calculated relative to that based on the complete data (unmasked). To facilitate the comparison, all results shown for unmasked data are based on MLEs, observed Fisher information, and confidence intervals of the form (MLE) $\pm$ (1.96 $\times$ estimated standard deviation). For top-coding two choices were made for the threshold C: the 90th and 95th percentiles of each distribution.

We shall now explain the notations used in the tables regarding the multiple imputation and noise multiplication scenarios used in the simulations. The notation CD denotes the case of complete data without any masking. MI denotes multiple imputation used to generate fully masked data. MI.C and MI.D are used to denote parametric multiple imputation based on the complete data, and based on the deleted data, respectively. These are explained in Section 4.2. In the case of a top coded sample with a top coding threshold C, values beyond a threshold C_I were also replaced with imputed values, where $C_I < C$, following An and

Little (2007). Two choices were made for C_I following the recipe prescribed by An and Little (2007). If n_S denotes the number of values beyond the top coding threshold C, the two choices of C_I correspond to thresholds for which $2n_S$ values in the sample are larger, and $4n_S$ values are larger. Since these values approximately correspond to the 90th and 80th percentiles of the distribution when C is the 95th percentile, these are denoted (following An and Little, 2007) by MI.C90 and MI.C80, respectively, in the case of multiple imputation based on the complete data, and denoted by MI.D90 and MI.D80, respectively, in the case of multiple imputation based on the deleted data. Throughout we have used 50 sets of imputed values under MI.

The notations used under noise multiplication are as follows. We have used either a Uniform$(1-\epsilon, 1+\epsilon)$ distribution or a customized distribution for generating the noise multipliers. In the case of Uniform$(1-\epsilon, 1+\epsilon)$, the notation NM10U denotes noise multiplication with the above uniform distribution under the choice $\epsilon = 0.10$. NM20U, NM30U etc. are similarly defined. Furthermore, NM10C, NM20C etc. are the corresponding notations under the customized prior whose mean and variance match those of the respective uniform distributions. Additional notation used under the uniform noise distribution Uniform$(1-\epsilon, 1+\epsilon)$ is as follows. For the choice $\epsilon = 0.1$, NM10.1 and NM10.2, respectively, represent the case of observing the (x_i, Δ_i)s, or only the x_is; these are Case (i) and Case (ii) explained in Section 3 for the three different distributions. NM20.1, NM20.2 etc. are similarly defined. In addition to the above notations, entries in the tables corresponding to CENS denote results from the analysis of the censored data, i.e., the top coded data.

The statistical computing software R (R Development Core Team, 2011) was used for all computations and the *integrate* function in R was used to evaluate the required univariate integrals that could not be obtained in closed form. We shall now summarize the conclusions emerging from the extensive numerical results with detailed tables appearing in Appendix E of the supplementary material.

The Exponential Distribution.

For the case of fully masked data, confidence interval coverage probabilities resulting from noise multiplication are not satisfactory for small sample sizes; see the entries in the tables (supplementary material) corresponding to the sample size of 30. However, asymptotic inference based on the unmasked data also yields unsatisfactory coverage probability when $n = 30$. Interestingly, even in this case where the sample is as small as $n = 30$, multiple imputation does provide satisfactory coverage probability. The numerical results corresponding to sample size of 50 or more show that the coverages are quite satisfactory under the various noise multiplication scenarios, and under multiple imputation. As the noise variance increases, the relative length increases, as expected. The RMSE values, SD, and relative length each appear to show an increasing trend as the noise variance goes up, but an increasing trend in the bias is not always present in the numerical results. In terms of RMSE and relative length of confidence intervals, the noise multiplication yields results similar to MI when ϵ is about 0.3 or 0.4. For smaller (larger) values of ϵ, the noise multiplication generally yields smaller (larger) RMSE and narrower (wider) confidence intervals. In terms of bias, a much larger bias has been noted under multiple imputation, compared to all scenarios of noise multiplication. We note that the uniform and customized noise distributions generally give similar results, with the customized distribution providing slightly more accurate inference than uniform when ϵ is large.

In the case of multiple imputation and noise multiplication for top coded data, the performance of MI.C80 and MI.C90 are not satisfactory, even for samples as large as 2000, as the true confidence interval coverage is below the nominal level. However, both MI.D80 and MI.D90 are satisfactory for various sample sizes, even for sample size as small as 100. All the noise multiplication scenarios provide satisfactory coverage probabilities. The entries under CENS show that the top coding method itself provides satisfactory coverage probabilities, but it tends to be less accurate than the noise multiplication methods unless ϵ is quite large.

Interestingly, for $n = 100$ and $n = 200$, the top coded data and noise multiplied data (even with large $\epsilon = 0.90$) yield more accurate inferences than MI.D80 and MI.D90 in terms of RMSE and confidence interval length. For $n = 1000$ and $n = 2000$ the noise multiplied data are generally more or equally accurate than MI.D80 and MI.D90 for $\epsilon \leq 0.5$ or 0.6. In our numerical results, the CENS method (top code data) never provide more accurate inference (in terms of RMSE and confidence interval length) than the case (i) noise multiplied sample, but they do sometimes provide more accurate inference than a case (ii) noise multiplied sample when ϵ is large. At the end of this section, we explain this finding, which holds for all three distributions considered. As one would expect, the case (i) noise multiplied sample $(x_i, \Delta_i), i = 1, \ldots, n$, provides greater accuracy of inference than the case (ii) noise multiplied sample $x_1, \ldots, x_n$. Regarding MI.D80 versus MI.D90, with the larger sample sizes $n = 1000$ or 2000 the two methods are mostly similar. Interestingly, in the smaller sample sizes $n = 100$ or 200, the MI.D90 method, which imputes less data than MI.D80, but also uses less data to fit the imputation model, is less accurate than the MI.D80 method. The remaining conclusions are similar to those noted above for the case of fully masked data; a rather large bias has been noted under multiple imputation.

The Normal Distribution

Under full masking of the data, for estimating the normal mean, the results are comparable under multiple imputation and noise multiplication when ϵ is equal to about 0.4 or 0.5 (depending on the sample size). By choosing ϵ less than or greater than 0.4 or 0.5, the results under noise multiplication can be made more or less accurate than multiple imputation, respectively. However, for estimating the variance under the smaller sample sizes of $n = 30, 50, 100$, multiple imputation appears to have an edge in terms of coverage probability and relative length, and in terms of the other quantities. For estimating the variance, the asymptotic confidence intervals based on unmasked data and noise multiplied data all have coverage probability below nominal, while confidence intervals based on multiply imputed data have closer to nominal coverage probability. This effect diminishes for both the unmasked and noise multiplied data for the larger sample sizes $n = 200, 1000, 2000$. For estimating the percentiles, the results are mostly similar between multiple imputation and noise multiplication and one can locate a value of ϵ (e.g., $\epsilon = 0.2$ or 0.3 here) at which the inferences have nearly equivalent accuracy.

Under noise multiplication or multiple imputation for top coded data, MI.C90 and MI.C80 again have poor coverages even with large samples. With the smaller sample size of $n = 100$ we notice that when the parameter of interest is the variance σ^2 or $\mu + \sigma^2/2$ the MI.D90 method can yield a huge RMSE, bias, SD, and relative confidence interval length, while the coverage probability is maintained at the nominal level. Such inaccurate inference occurs under MI.D90 because when $n = 100$ and C is the 95th percentile, there is a non-negligible probability that the number of data points used to fit the imputation model can be as small as 2 if $n_S = 1$ (refer to Section 4.2); and when this happens there is a non-negligible probability that the posterior draw for σ^2 is huge. Apart from this anomaly in MI.D90, the results between MI.D90, MI.D80 and the noise multiplication methods are generally similar with the MI.D methods often yielding a noticeably high level of accuracy in terms of RMSE, SD, and relative confidence interval length. It is interesting that in the normal case the MI.D methods generally yield highly accurate inference, whereas in the exponential case, we noted that their accuracy was somewhat low, sometimes lower than top coding. In each case one can determine an ϵ at which the noise multiplied data provide a similar level of accuracy as MI. For instance when estimating the normal mean, a value of $\epsilon = 0.4$ generally gives a similar level of accuracy as the MI.D methods.

The Lognormal Distribution

For fully masked data, when estimating the log scale mean μ, the results are very much in line with those noted above for the normal case for estimation of the normal mean. Similarly, when estimating the log scale variance σ^2, the results are in line with the findings noted above for the normal case for estimation of the normal variance. With the smaller sample sizes of $n = 30, 50, 100$, the noise multiplied data and unmasked data also provide less than nominal confidence interval coverage when the parameter of interest is the lognormal mean, lognormal variance, the 84th percentile and the 95th percentile. Multiple imputation provides closer to nominal coverage probability in these cases. For the larger sample sizes $n = 200, 1000, 2000$, this effect diminishes. Regarding the choice of uniform vs. customized noise distributions, the findings are similar to those of the exponential case.

For noise multiplication or multiple imputation for top coded data, we note that MI.C90 and MI.C80, exhibit poor confidence interval coverages, similar to what has been noted earlier. When the parameter of interest is the variance, lognormal mean, lognormal variance, 84th percentile, or 95th percentile, and when $n = 100$ or 200, the MI.D80 and MI.D90 methods can yield huge RMSE, bias, SD and relative confidence interval length. This effect occurs for reasons similar to those noted above for the normal case, namely, there is a non-negligible probability that the number of data points available to fit the model is as small as four (MI.D80) or two (MI.D90), i.e., $n_S = 1$. In these cases the noise multiplication methods can still provide reasonable results. In larger sample sizes, the multiple imputation and noise multiplication methods both provide reasonably accurate results, and one can select an ϵ at which their accuracy is nearly equivalent (in some cases such as when estimating the log scale mean, the ϵ value can be fairly large).

Remarks on Noise Multiplication Versus Top Coding at C

1. If we observe noise multiplied data under case (i) of Section 3, then a top coded sample can be re-constructed from the observed data. However, a case (i) noise multiplied sample cannot be constructed from the top coded data. So in this sense, case (i) noise

multiplied data carry more information than top coded data. The simulation results confirm this statement because the case (i) noise multiplication method generally leads to a shorter confidence interval and smaller SD than top coding for all values of ϵ that we consider. As ϵ increases, the case (i) noise multiplication method begins to yield results which are quite similar to top coding, as one would expect.

2. Suppose we observe noise multiplied data under case (ii) of Section 3 and the noise density is Uniform$(1-\epsilon, 1+\epsilon)$. Then the top coded sample cannot be re-constructed from these observed data because for any x_i such that $(1-\epsilon) \times C < x_i < C$, we do not know whether or not this x_i was noise perturbed. Of course, we also cannot re-construct a case (ii) noise multiplied sample based on only the top coded data. Therefore, depending on the value of ϵ, the case (ii) noise multiplied sample may contain either more or less information than the top coded sample. The simulation results confirm this statement since for small ϵ, the case (ii) noise multiplication scenario leads to a shorter confidence interval and smaller SD than top coding. When ϵ becomes large, this situation is reversed. It is interesting to note that given the noise multiplied data under case (ii), we could construct a top coded sample with a lower top code value of $(1-\epsilon) \times C$.

7. DATA ANALYSIS

In this section we provide an application based on data from the 2000 U.S. Current Population Survey which has been thoroughly and repeatedly analyzed in Reiter (2005c,d; 2009) and Drechsler and Reiter (2010) for illustrating various aspects of multiple imputation methodology. Here we shall use the data on two variables whose distributions can be well approximated by the lognormal distribution. We shall report the analysis based on full noise multiplication and total MI (Tables F1 and F2 in Appendix F of the supplementary material). We shall also top code 10% of the data and then report the analysis based on the full data sets, the censored data sets resulting from top coding, the data sets obtained by multiple imputation of the top coded values, and the data sets resulting from noise multiplication of the top coded values (Tables 1 and 2). This will of course permit us to compare the results of the different analysis for the two data sets.

The entire data comprise of seventeen variables measured on 51,016 heads of households (see Table 1 in Drechsler and Reiter, 2010 for a description of nine variables) including age, race, sex and marital status as key identifiers, and a mix of other categorical and numerical variables with non-Gaussian distributions (with large percentages of values equal to zero). We have selected two such numerical variables with extreme sample sizes, namely, *total household income* with sample size = 50,661 and *household alimony payment* with sample size 206, deleting the 0 values, and have carried out the analysis. It turns out that a lognormal distribution fits both the datasets well.

Results of the analysis are given in the Tables 1 − 2 (and Tables F1 − F2 of the supplementary material). As in the previous section, we used the statistical computing software R (R Development Core Team, 2011) for all computations, and 50 multiple imputations were used in the MI methods. The notations used in the tables are as explained in Section 6. In addition, Est denotes point estimate of the parameter and Rel. Len. denotes the length of a 95% confidence interval, relative to that of the interval based on the complete data. We note that as with simulated data the conclusions about the common parameters of interest are similar based on multiple imputation and noise perturbation (with noise variance not exceeding about 8.33%, $\epsilon = 0.5$ in uniform case) and both almost reproduce the estimates based on the original unmasked data. It is interesting to note from Table 1 (total household income) that the MI.C methods yield considerably wider confidence intervals than the MI.D and noise multiplication methods with small to moderate ϵ. In these cases the MI.C methods also yield larger point estimates of the lognormal mean, lognormal variance, 84th percentile and 95th percentile in comparison with the other methods. The results of Table 2 are based on a much smaller sample size and here we do not observe such wide confidence intervals under the MI.C methods.

8. DISCUSSION

How to alter the data before releasing it to the public continues to be a vital concern of statistical agencies in order to minimize the risk of disclosure. However, this goal has to be balanced with the interest of preserving the utility of the released data. A number of strategies exist for achieving the goal of disclosure limitation, and for accurately analyzing the released data. Perhaps not much effort has been devoted to compare the different disclosure limitation strategies in terms of accuracy of the inference resulting from the released data. If a particular data masking approach can provide more accurate inference compared to other competing approaches, this should be of obvious interest to statistical agencies, survey organizers, and the public. This work is an attempt in this direction, and our goal has been two-fold: (a) to develop rigorous data analysis methodologies for noise multiplied data under noise multiplication of the entire data, or under noise multiplication of the top coded data, and (b) to compare noise multiplication and multiple imputation in terms of accuracy of the resulting inference. We have developed all the necessary theoretical results for analyzing noise multiplied data, when the original (unmasked) sample arises from three parametric distributions: exponential, normal and lognormal distributions. Masking the entire data, and masking data above a threshold, are treated separately. As is well known, the latter situation is very common in the situation of top coding, where sample values above a threshold are suppressed for privacy protection. We have also reported extensive numerical results to assess the accuracy of the inference concerning a variety of parameters. The numerical results have brought out the similarities, differences and advantages of the two data masking techniques we have investigated.

In the top code scenario where only values above a threshold $C > 0$ require protection, we have considered noise multiplication under two data release scenarios: case (i) in which each released value includes an indicator of whether or not it was noise perturbed, and case (ii) in which no such indicator is provided. We developed data analysis methods under both cases, and argued that case (i) should always provide more (or equally) accurate inference than top coding at C, while case (ii) can provide either more or less accurate inference than top coding at C, depending on the magnitude of the noise variance. Our empirical results show that both cases provide almost equally accurate inferences when the noise variance is small, and the difference in accuracy increases, but perhaps is not too substantial, when the noise variance is moderately large. The case (ii) data release may sometimes be more desirable for statistical agencies than the case (i) data release, as case (ii) would appear to provide an enhanced level of protection against disclosure. The results of this article show how to obtain valid inferences in both cases.

In summary, parametric statistical procedures for fully masked data based on multiple imputation and noise multiplication can provide comparable inferences in many cases that we considered. The inferences obtained from these two different methods can usually be made nearly equivalent by setting the noise variance appropriately. When a large noise variance providing a high degree of confidentiality is used, inference is typically less accurate but still reasonable. In the case of top coded data, similar conclusions hold. We note however that the procedures MI.C90 and MI.C80 do not perform well in most cases as they tend to produce confidence intervals whose true coverage probability is below the nominal level even when the sample size is quite large. On the other hand, the methods MI.D90 and MI.D80 generally do yield satisfactory results, but as noted in Section 6, for smaller sample sizes, there is a potential for a large RMSE, SD, and wide confidence interval under these methods. Compared to multiple imputation, an appealing feature of noise multiplication is its flexibility; the noise variance acts as a tuning parameter, and its choice allows one to balance data quality with confidentiality protection. The results of this article provide guidance regarding the effect of noise variance on data quality. Our results show, in particular situations, precisely the value of the noise variance at which noise multiplied data will provide inferences with nearly equivalent accuracy as synthetic (multiply imputed) data. By setting the noise variance smaller than this value, we have shown that one can obtain inferences that are more accurate than synthetic data, but perhaps at the expense of a higher disclosure risk. Similarly, one can increase the noise variance, decreasing accuracy of inferences, but perhaps enhancing the level of protection against disclosure. Indeed the impact of the noise variance on disclosure risk requires further investigation. While inferences become less accurate when the noise variance is large, we note that the inferences generally are still valid, i.e., confidence interval coverage probability is generally maintained at the nominal level and bias is small.

Table 1. Analysis of *household total income* with protection for values above C = 90th empirical percentile

	μ		σ^2		$e^{\mu+\sigma^2/2}$		$e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$		$e^{\mu+\sigma}$		$e^{\mu+1.645\sigma}$	
	Est	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.	Est $\times 10^{-6}$	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.
CD	10.495	1.000	0.981	1.000	58996	1.000	5805.083	1.000	97263	1.000	184262	1.000
MI.C90	10.526	1.049	1.078	1.101	63865	1.156	7903.475	1.447	105220	1.137	205534	1.173
MI.C80	10.551	1.075	1.130	1.159	67245	1.262	9478.622	1.798	110648	1.229	219649	1.289
MI.D90	10.494	1.000	0.981	1.001	58965	1.001	5793.529	1.000	97212	1.001	184129	1.001
MI.D80	10.494	1.001	0.981	1.002	58967	1.003	5795.687	1.002	97215	1.003	184146	1.003
CENS	10.510	1.040	1.037	1.140	61592	1.150	6906.667	1.338	101531	1.139	195816	1.178
NM10.1	10.495	1.001	0.982	1.002	59048	1.002	5824.788	1.005	97349	1.002	184486	1.003
NM10.2	10.495	1.001	0.982	1.002	59051	1.002	5826.076	1.005	97355	1.002	184501	1.003
NM20.1	10.496	1.002	0.985	1.006	59169	1.008	5872.738	1.017	97551	1.007	185022	1.009
NM20.2	10.496	1.003	0.985	1.007	59200	1.009	5884.083	1.020	97602	1.008	185152	1.010
NM30.1	10.497	1.004	0.987	1.012	59293	1.014	5921.903	1.031	97756	1.013	185567	1.017
NM30.2	10.497	1.006	0.989	1.015	59384	1.018	5955.307	1.039	97907	1.017	185951	1.021
NM40.1	10.497	1.006	0.990	1.019	59420	1.021	5971.468	1.046	97965	1.020	186120	1.025
NM40.2	10.499	1.009	0.994	1.025	59639	1.031	6052.017	1.067	98327	1.029	187037	1.036
NM50.1	10.499	1.009	0.995	1.028	59650	1.033	6065.054	1.071	98345	1.030	187139	1.038
NM50.2	10.502	1.016	1.002	1.042	60084	1.053	6226.125	1.114	99061	1.050	188952	1.061
NM60.1	10.499	1.011	0.996	1.034	59685	1.038	6080.100	1.080	98403	1.035	187299	1.044
NM60.2	10.505	1.023	1.008	1.058	60404	1.075	6349.797	1.154	99589	1.070	190308	1.085
NM70.1	10.501	1.014	1.001	1.045	59942	1.050	6186.262	1.108	98828	1.047	188442	1.059
NM70.2	10.510	1.035	1.021	1.086	61117	1.114	6635.825	1.238	100760	1.106	193357	1.129
NM80.1	10.502	1.017	1.005	1.053	60128	1.060	6263.484	1.130	99133	1.056	189264	1.070
NM80.2	10.515	1.050	1.031	1.114	61707	1.156	6871.419	1.318	101725	1.146	195836	1.175
NM90.1	10.502	1.018	1.007	1.060	60221	1.066	6303.106	1.144	99286	1.061	189680	1.078
NM90.2	10.518	1.070	1.039	1.141	62177	1.201	7064.943	1.396	102492	1.189	197822	1.223

Table 2. Analysis of *household alimony payments* with protection for values above C = 90th empirical percentile

	μ		σ^2		$e^{\mu+\sigma^2/2}$		$e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$		$e^{\mu+\sigma}$		$e^{\mu+1.645\sigma}$	
	Est	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.	Est $\times 10^{-6}$	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.
CD	8.715	1.000	1.168	1.000	10930	1.000	264.812	1.000	17962	1.000	36069	1.000
MI.C90	8.711	1.007	1.183	1.018	10976	1.022	277.204	1.098	18016	1.016	36370	1.025
MI.C80	8.725	1.022	1.215	1.049	11337	1.085	314.648	1.320	18574	1.069	37884	1.092
MI.D90	8.710	0.994	1.154	0.991	10804	0.985	255.589	0.978	17759	0.986	35529	0.984
MI.D80	8.718	1.006	1.179	1.013	11038	1.026	278.497	1.096	18124	1.021	36543	1.028
CENS	8.740	1.055	1.277	1.181	11829	1.224	361.734	1.574	19338	1.192	40082	1.251
NM10.1	8.715	1.000	1.168	1.000	10922	0.999	264.096	0.998	17949	0.999	36034	0.999
NM10.2	8.715	1.000	1.169	1.001	10934	1.001	265.245	1.003	17969	1.001	36088	1.002
NM20.1	8.715	1.000	1.168	1.002	10930	1.002	264.725	1.002	17962	1.002	36066	1.002
NM20.2	8.715	1.000	1.168	1.003	10930	1.002	264.783	1.003	17962	1.002	36069	1.003
NM30.1	8.719	1.007	1.182	1.017	11046	1.023	275.907	1.056	18142	1.020	36581	1.025
NM30.2	8.719	1.007	1.182	1.018	11042	1.023	275.624	1.056	18137	1.021	36567	1.026
NM40.1	8.719	1.007	1.182	1.020	11047	1.026	276.037	1.060	18143	1.023	36586	1.029
NM40.2	8.717	1.005	1.176	1.016	10990	1.019	270.739	1.038	18055	1.017	36340	1.021
NM50.1	8.724	1.017	1.203	1.043	11219	1.059	293.152	1.145	18411	1.051	37349	1.065
NM50.2	8.721	1.014	1.193	1.039	11130	1.048	284.737	1.111	18272	1.043	36967	1.054
NM60.1	8.724	1.019	1.207	1.051	11246	1.067	296.198	1.165	18451	1.058	37474	1.075
NM60.2	8.720	1.018	1.200	1.054	11165	1.064	289.284	1.145	18324	1.056	37144	1.072
NM70.1	8.725	1.019	1.206	1.055	11246	1.070	296.116	1.169	18452	1.061	37472	1.078
NM70.2	8.714	1.013	1.181	1.047	10985	1.045	272.534	1.080	18043	1.041	36371	1.053
NM80.1	8.725	1.021	1.207	1.063	11252	1.078	296.798	1.182	18461	1.068	37501	1.087
NM80.2	8.726	1.032	1.215	1.093	11315	1.117	303.461	1.249	18558	1.104	37783	1.130
NM90.1	8.724	1.019	1.203	1.059	11217	1.071	293.298	1.164	18407	1.063	37348	1.080
NM90.2	8.687	1.003	1.134	1.034	10449	0.999	230.225	0.922	17191	1.004	34168	1.005

SUPPLEMENTARY MATERIALS

The supplementary material consists of five appendices as follows.

Appendix B: MLEs and Fisher information under full noise multiplication.

Appendix C: MLEs and Fisher information under noise multiplication of extreme values.

Appendix D: MLEs and Fisher information under top coding.

Appendix E: Simulation results.

Appendix F: Analysis of fully masked data on *total household income* and *household alimony payment* based on data from the 2000 U.S. Current Population Survey.

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APPENDIX A: PROOF OF PROPOSITION 1 AND SOME RELATED RESULTS

We let $Y \sim f_\theta(y)$, independent of $R \sim h(r)$, as defined in Section 2 and we let $\Delta = I(Y \leq C)$, $X = Y$ if $Y \leq C$ and $= R \times Y$ if $Y > C$, as defined in Section 3, where $C > 0$ is a constant. Then letting $F_\theta(y) = \int_{-\infty}^{y} f_\theta(t)dt$ and $H(r) = \int_0^r h(t)dt$, we have, for $r > 0$,

$$\begin{aligned}
\Pr(X \leq x, R \leq r, \Delta = 1) &= \Pr(X \leq x, R \leq r \,|\, \Delta = 1)\Pr(\Delta = 1) \\
&= \Pr(Y \leq x, R \leq r \,|\, Y \leq C) P(Y \leq C) \\
&= \Pr(Y \leq x, Y \leq C, R \leq r) \\
&= \begin{cases} F_\theta(x)H(r), & \text{if } x \leq C, \\ F_\theta(C)H(r), & \text{if } x > C, \end{cases}
\end{aligned} \tag{10}$$

and, for $r > 0$, $C > 0$,

$$\begin{aligned}
\Pr(X \leq x, R \leq r, \Delta = 0) &= \Pr(X \leq x, R \leq r \,|\, \Delta = 0)\Pr(\Delta = 0) \\
&= \Pr(YR \leq x, R \leq r \,|\, Y > C)\Pr(Y > C) \\
&= \Pr(YR \leq x, Y > C, R \leq r) \\
&= \Pr(C < Y \leq \frac{x}{R}, R \leq r) \\
&= \begin{cases} \int_0^r \int_C^{x/\omega} f_\theta(y)h(\omega)dyd\omega, & \text{if } r \leq \frac{x}{C}, \\ \int_0^{x/C} \int_C^{x/\omega} f_\theta(y)h(\omega)dyd\omega, & \text{if } r > \frac{x}{C} > 0, \\ 0, & \text{if } x \leq 0, \end{cases} \\
&= \begin{cases} \int_0^r [F_\theta(x/\omega) - F_\theta(C)]h(\omega)d\omega, & \text{if } r \leq \frac{x}{C}, \\ \int_0^{x/C} [F_\theta(x/\omega) - F_\theta(C)]h(\omega)d\omega, & \text{if } 0 < \frac{x}{C} < r, \\ 0, & \text{if } x \leq 0. \end{cases}
\end{aligned} \tag{11}$$

To obtain the joint *pdf* of (X, R, Δ), we differentiate (10) and (11) with respect to x and r to obtain:

$$\frac{\partial^2}{\partial x \partial r} \Pr(X \leq x, R \leq r, \Delta = 1) = \begin{cases} f_\theta(x)h(r), & \text{if } x < C, \\ 0, & \text{if } x > C, \end{cases} \tag{12}$$

$$\frac{\partial^2}{\partial x \partial r} \Pr(X \leq x, R \leq r, \Delta = 0) = \begin{cases} f_\theta(x/r)h(r)(1/r), & \text{if } r < \frac{x}{C}, \\ 0, & \text{if } 0 < \frac{x}{C} < r, \\ 0, & \text{if } x < 0, \end{cases} \tag{13}$$

which completes the proof of Proposition 1. Letting $(x_1, \Delta_1), ..., (x_n, \Delta_n)$ be *iid* as (X, Δ), the following results then follow from Proposition 1.

Result 1. The joint *pdf* of (X, Δ) is given by

$$k_\theta(x, \delta) = \begin{cases} f_\theta(x), & \text{if } x < C, \ \delta = 1, \\ 0, & \text{if } x_i > C, \ \delta = 1, \\ \int_0^{x/C} f_\theta(x/r)h(r)(1/r)dr, & \text{if } x > 0, \ \delta = 0, \\ 0, & \text{if } x < 0, \ \delta = 0. \end{cases} \tag{14}$$

Result 2. The likelihood function for θ based on $(x_1, \Delta_1), ..., (x_n, \Delta_n)$ is given by

$$\begin{aligned} L(\theta | x_1, ..., x_n, \Delta_1, ..., \Delta_n) &= \prod_{i=1}^{n} k_\theta(x_i, \Delta_i) \\ &= \prod_{i=1}^{n} \{[f_\theta(x_i)]^{\Delta_i} \times [\int_0^{x_i/C} f_\theta(x_i/r)h(r)(1/r)dr]^{1-\Delta_i}\}. \end{aligned} \tag{15}$$

Result 3. The marginal *pdf* of X is given by

$$k_\theta(x) = f_\theta(x)I(x < C) + \int_0^{x/C} f_\theta(x/r)h(r)(1/r)dr I(x > 0) \tag{16}$$

$$= \begin{cases} f_\theta(x) + \int_0^{x/C} f_\theta(x/r)h(r)(1/r)dr, & \text{if } 0 < x < C, \\ f_\theta(x), & \text{if } x < 0, \\ \int_0^{x/C} f_\theta(x/r)h(r)(1/r)dr, & \text{if } x > C. \end{cases}$$

Result 4. The likelihood function for θ based on $x_1, ..., x_n$ is given by

$$L(\theta|x_1, ..., x_n) = \prod_{i=1}^{n} k_\theta(x_i)$$

$$= \prod_{i=1}^{n} \{f_\theta(x_i)I(x_i < C) + \int_0^{x_i/C} f_\theta(x_i/r)h(r)(1/r)dr I(x_i > 0)\}. \tag{17}$$

SUPPLEMENTARY MATERIAL

A Comparison of Statistical Disclosure Control Methods: Multiple Imputation Versus Noise Multiplication

Martin Klein, Thomas Mathew, and Bimal Sinha

APPENDIX B: MLEs AND FISHER INFORMATION UNDER FULL NOISE MULTIPLICATION

B.1 Exponential distribution

We assume $Y \sim f_\theta(y) = \frac{1}{\theta}e^{-\frac{y}{\theta}}$, $0 < y < \infty$. Then the joint *pdf* $g(z, r)$ of (Z, R), the marginal *pdf* $g(z)$ of Z and the conditional *pdf* $g(r|z)$ of R, given $Z = z$, are given by

$$g_\theta(z,r) = \frac{1}{\theta}e^{-\frac{z}{r\theta}}\frac{h(r)}{r},\;\; g_\theta(z) = \int_0^\infty \frac{1}{\theta}e^{-\frac{z}{r\theta}}\frac{h(r)}{r}dr,\;\; g_\theta(r|z) = \frac{e^{-\frac{z}{r\theta}}\frac{h(r)}{r}}{\int_0^\infty e^{-\frac{z}{w\theta}}\frac{h(w)}{w}dw}, \tag{1}$$

for $0 < r < \infty$ and $0 < z < \infty$, and hence the complete data likelihood $L(\theta|\boldsymbol{u}_\mathrm{c})$ and loglikelihood $\ell(\theta|\boldsymbol{u}_\mathrm{c})$ can be expressed as

$$L(\theta|\boldsymbol{u}_\mathrm{c}) \sim \frac{1}{\theta^n}e^{-\frac{1}{\theta}\sum_{i=1}^n \frac{z_i}{r_i}},\;\; \ell(\theta|\boldsymbol{u}_\mathrm{c}) = -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^n \frac{z_i}{r_i}. \tag{2}$$

The E and M-steps for computing the MLE of θ based on $\boldsymbol{z}$ are as follows.
E-step.

$$Q(\theta|\theta^{(t)}) = E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{u}_\mathrm{c})|\boldsymbol{u}_\mathrm{obs}] = -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^n z_i E[\frac{1}{r_i}|z_i] \tag{3}$$

M-step. Obviously $Q(\theta|\theta^{(t)})$ is maximized with respect to θ at $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n z_i\psi(\theta^{(t)}, z_i)$ where $\psi(\theta^{(t)}, z_i) = E_{\theta^{(t)}}[\frac{1}{r_i}|z_i]$.
Thus the EM iterations are defined by

$$\theta^{(t+1)} = \frac{1}{n}\sum_{i=1}^n z_i\psi(\theta^{(t)}, z_i). \tag{4}$$

The computation of $\psi(\theta^{(t)}, z)$ for any z follows directly from the conditional *pdf* of r, given z, which is given by (1).

In the special case when $R \sim \text{Uniform}(1-\epsilon, 1+\epsilon)$, upon direct integration, $\psi(\theta^{(t)}, z)$ simplifies to

$$\psi_\text{uniform}(\theta^{(t)}, z) = \frac{\frac{\theta^{(t)}}{z}\left[e^{-\frac{z}{\theta^{(t)}(1+\epsilon)}} - e^{-\frac{z}{\theta^{(t)}(1-\epsilon)}}\right]}{\int_{1-\epsilon}^{1+\epsilon} e^{-\frac{z}{w\theta^{(t)}}}\frac{dw}{w}}. \tag{5}$$

Lastly, the observed Fisher information is computed by adding component-wise terms based on $z_1, \ldots, z_n$, with a generic term as follows. Since the marginal *pdf* of z is given by (1), we get the first two derivatives of $\ln[g_\theta(z)]$ as

$$\frac{\partial \ln[g_\theta(z)]}{\partial\theta} = -\frac{1}{\theta} + \frac{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{zh(r)}{r^2\theta^2} dr}{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{h(r)}{r} dr}, \tag{6}$$

$$\frac{\partial^2 \ln[g_\theta(z)]}{\partial\theta^2} = \frac{1}{\theta^2} - \frac{2z}{\theta^3} \frac{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{h(r)}{r^2} dr}{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{h(r)}{r} dr} + \frac{z^2}{\theta^4} \frac{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{h(r)}{r^3} dr}{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{h(r)}{r} dr} - \frac{z^2}{\theta^4} \left[\frac{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{h(r)}{r^2} dr}{\int_0^\infty e^{-\frac{z}{r\theta}} \frac{h(r)}{r} dr} \right]^2. \tag{7}$$

A remark is now in order. Since the choice of the noise distribution mostly relies on the data analyst, a special case leading to simplified inferential methods warrants some attention. We refer to this as a *customized* or *conjugate* noise distribution. We choose

$$R \sim h_\delta(r) = \frac{\delta^{\delta+1}}{\Gamma(\delta+1)} r^{-\delta-2} e^{-\frac{\delta}{r}}, \quad \delta > 1. \tag{8}$$

It is easy to verify that $E(R) = 1$, as desired, and $\mathrm{var}(R) = \sigma_r^2 = \frac{1}{\delta-1}$. We choose $\delta > 1$ for a desirable level of noise variation. A direct computation shows that the *pdf* of $Z = Y \times R$ is obtained as

$$g_\theta(z) = \frac{\delta+1}{\theta} \times \frac{\delta^{\delta+1}}{(\frac{z}{\theta} + \delta)^{\delta+2}}. \tag{9}$$

This readily leads to the following likelihood based on $z_1, ..., z_n$:

$$L(\theta|\boldsymbol{u}_{\text{obs}}) \sim \theta^{-n} \times \prod_{i=1}^n (\frac{z_i}{\theta} + \delta)^{-\delta-2}. \tag{10}$$

The maximum likelihood estimate of θ in this case can be directly computed by solving the equation:

$$\frac{d\ell(\theta|\boldsymbol{u}_{\text{obs}})}{d\theta} = -\frac{n}{\theta} + \frac{\delta+2}{\theta^2} \sum_{i=1}^n \frac{z_i}{(\frac{z_i}{\theta} + \delta)} = 0, \tag{11}$$

which can be simplified as

$$\sum_{i=1}^n \frac{z_i}{z_i + \theta\delta} = \frac{n}{\delta+2}. \tag{12}$$

To compute the observed Fisher information about θ contained in the data $\boldsymbol{u}_{\text{obs}}$, we note that upon simplification,

$$-\frac{d^2\ell(\theta|\boldsymbol{u}_{\text{obs}})}{d\theta^2} = \frac{n(\delta+1)}{\theta^2} - \delta^2(\delta+2) \sum_{i=1}^n \frac{1}{(z_i + \theta\delta)^2}. \tag{13}$$

B.2 Normal distribution

We assume $Y \sim f_\theta(y) \sim \frac{1}{\sigma}e^{-\frac{(y-\mu)^2}{2\sigma^2}}$, $\theta = (\mu, \sigma^2)$. Then the joint *pdf* $g(z,r)$ of (Z,R), the marginal *pdf* $g(z)$ of Z and the conditional *pdf* $g(r|z)$ of R, given $Z = z$, are given by

$$g_\theta(z,r) \sim \frac{1}{\sigma}e^{-\frac{(\frac{z}{r}-\mu)^2}{2\sigma^2}}\frac{h(r)}{r},\ g_\theta(z) \sim \frac{1}{\sigma}\int_0^\infty e^{-\frac{(\frac{z}{r}-\mu)^2}{2\sigma^2}}\frac{h(r)}{r}dr,\ g_\theta(r|z) = \frac{e^{-\frac{(\frac{z}{r}-\mu)^2}{2\sigma^2}}\frac{h(r)}{r}}{\int_0^\infty e^{-\frac{(\frac{z}{r}-\mu)^2}{2\sigma^2}}\frac{h(r)}{r}dr}, \quad (14)$$

for $0 < r < \infty$ and $0 < z < \infty$, and hence the complete data likelihood $L(\theta|\boldsymbol{u}_{\rm c})$ and loglikelihood $\ell(\theta|\boldsymbol{u}_{\rm c})$ are now expressed as

$$L(\theta|\boldsymbol{u}_{\rm c}) \sim \frac{1}{\sigma^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n(\frac{z_i}{r_i}-\mu)^2},\ \ell(\theta|\boldsymbol{u}_{\rm c}) = -n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^n(\frac{z_i}{r_i}-\mu)^2. \quad (15)$$

The EM steps for computing the MLE of θ based on $\boldsymbol{z}$ are as follows.
E-step. We compute

$$\begin{aligned} Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{u}_{\rm c})|\boldsymbol{u}_{\rm obs}], \\ &= -n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^n[z_i^2\psi_2(\theta^{(t)}, z_i) - 2\mu z_i\psi_1(\theta^{(t)}, z_i) + \mu^2], \end{aligned} \quad (16)$$

where $\psi_1(\theta^{(t)}, z_i) = E_{\theta^{(t)}}[\frac{1}{r_i}|z_i]$ and $\psi_2(\theta^{(t)}, z_i) = E_{\theta^{(t)}}[\frac{1}{r_i^2}|z_i]$, and these two quantities can be readily computed based on $g(r|z)$ given in (14).
M-step. Note that

$$\begin{aligned} \frac{\partial Q(\theta|\theta^{(t)})}{\partial\mu} &= \frac{1}{\sigma^2}\sum_{i=1}^n z_i\psi_1(\theta^{(t)}, z_i) - \frac{n\mu}{\sigma^2}, \\ \frac{\partial Q(\theta|\theta^{(t)})}{\partial\sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^n\left[z_i^2\psi_2(\theta^{(t)}, z_i) - 2\mu z_i\psi_1(\theta^{(t)}, z_i) + \mu^2\right]. \end{aligned} \quad (17)$$

Hence $Q(\theta|\theta^{(t)})$ is maximized at

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^n z_i\psi_1(\theta^{(t)}, z_i),\ \ \hat{\sigma^2} = \frac{1}{n}\sum_{i=1}^n z_i^2\psi_2(\theta^{(t)}, z_i) - (\hat{\mu})^2. \quad (18)$$

Thus the EM iterations are defined by

$$\mu^{(t+1)} = \frac{1}{n}\sum_{i=1}^n z_i\psi_1(\theta^{(t)}, z_i),\ \ [\sigma^{(t+1)}]^2 = \frac{1}{n}\sum_{i=1}^n z_i^2\psi_2(\theta^{(t)}, z_i) - [\mu^{(t+1)}]^2. \quad (19)$$

The observed Fisher information is computed by adding component-wise terms based on $z_1, ..., z_n$. A generic term can be obtained as follows. We note that the marginal *pdf* of Z is given by

$$g_\theta(z) \sim \frac{1}{\sigma}\int_0^\infty e^{-\frac{(\frac{z}{r}-\mu)^2}{2\sigma^2}}\frac{h(r)}{r}dr. \quad (20)$$

Writing the integrand above as $g_\theta(z,r)$, we get

$$\frac{\partial g_\theta(z)}{\partial \mu} = \int_0^\infty g_\theta(z,r)[\frac{(\frac{z}{r}-\mu)}{\sigma^2}]dr = \frac{1}{\sigma^2}\left[\int_0^\infty g_\theta(z,r)\frac{z}{r}dr - \mu g_\theta(z)\right], \tag{21}$$

resulting in

$$\frac{\partial \ln g_\theta(z)}{\partial \mu} = \frac{1}{\sigma^2}\left[\frac{\int_0^\infty g_\theta(z,r)\frac{z}{r}dr}{g_\theta(z)} - \mu g(z)\right], \tag{22}$$

and hence in

$$\frac{\partial^2 \ln g_\theta(z)}{\partial \mu^2} = \frac{1}{\sigma^2}\frac{\partial}{\partial \mu}\left[\frac{\int_0^\infty g_\theta(z,r)\frac{z}{r}dr}{g_\theta(z)}\right] - \frac{1}{\sigma^2}. \tag{23}$$

Note that

$$\frac{\partial}{\partial \mu}\left[\frac{\int_0^\infty g_\theta(z,r)\frac{z}{r}dr}{g_\theta(z)}\right] = \frac{\frac{\partial}{\partial \mu}\int_0^\infty g_\theta(z,r)\frac{z}{r}dr}{g_\theta(z)} - \frac{[\int_0^\infty g_\theta(z,r)\frac{z}{r}dr][\frac{\partial g_\theta(z)}{\partial \mu}]}{g_\theta^2(z)}. \tag{24}$$

Since

$$\begin{aligned}\frac{\partial}{\partial \mu}\int_0^\infty g_\theta(z,r)\frac{z}{r}dr &= \int_0^\infty g_\theta(z,r)\frac{z}{r}(\frac{z}{r}-\mu)\frac{1}{\sigma^2}dr \\ &= \frac{z^2}{\sigma^2}\int_0^\infty g_\theta(z,r)\frac{dr}{r^2} - \frac{z\mu}{\sigma^2}\int_0^\infty g_\theta(z,r)\frac{dr}{r},\end{aligned} \tag{25}$$

upon using (23) and simplifying, we get

$$\frac{\partial^2 \ln g_\theta(z)}{\partial \mu^2} = \frac{z^2}{\sigma^4 g_\theta(z)}\left[\int_0^\infty g_\theta(z,r)\frac{dr}{r^2} - \frac{(\int_0^\infty g_\theta(z,r)\frac{dr}{r})^2}{g_\theta(z)}\right] - \frac{1}{\sigma^2}. \tag{26}$$

We next compute $\frac{\partial g_\theta(z)}{\partial \sigma^2}$ which easily simplifies to

$$\frac{\partial g_\theta(z)}{\partial \sigma^2} = -\frac{1}{2\sigma^2}g_\theta(z) + \frac{1}{2\sigma^4}\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr, \tag{27}$$

and readily leads to

$$\frac{\partial \ln g_\theta(z)}{\partial \sigma^2} = \frac{1}{2\sigma^4}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr}{g_\theta(z)} - \frac{1}{2\sigma^2}. \tag{28}$$

Keeping aside the last term, the partial derivative of the first term again with respect to σ^2 is obtained as

$$\begin{aligned}\frac{\partial}{\partial \sigma^2}\left[\frac{1}{2\sigma^4}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr}{g_\theta(z)}\right] &= -\frac{1}{\sigma^6}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr}{g_\theta(z)} \\ &\quad + \frac{1}{2\sigma^4}\frac{\int_0^\infty \frac{\partial}{\partial \sigma^2}g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr}{g_\theta(z)} \\ &\quad - \frac{1}{2\sigma^4}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr \times \frac{\partial}{\partial \sigma^2}g_\theta(z)}{g_\theta^2(z)}.\end{aligned} \tag{29}$$

Using the expressions for $\frac{\partial}{\partial\sigma^2} g_\theta(z,r)$ and $\frac{\partial}{\partial\sigma^2} g_\theta(z)$ given above in (27) and simplifying, we get

$$\begin{aligned}
\frac{\partial^2}{(\partial\sigma^2)^2}\ln g_\theta(z) &= -\frac{1}{\sigma^6}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr}{g_\theta(z)} + \frac{1}{4\sigma^8}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^4 dr}{g_\theta(z)} \\
&\quad -\frac{1}{4\sigma^8}[\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr}{g_\theta(z)}]^2 + \frac{1}{2\sigma^4}. \qquad (30)
\end{aligned}$$

Finally, we compute $\frac{\partial^2}{\partial\mu\partial\sigma^2}[\ln g_\theta(z)]$. Recalling (28), we can write

$$\begin{aligned}
\frac{\partial^2}{\partial\mu\partial\sigma^2}[\ln g_\theta(z)] &= \frac{\partial}{\partial\mu}\left[\frac{1}{2\sigma^4}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr}{g_\theta(z)}\right] \\
&= \frac{1}{2\sigma^4 g_\theta(z)}\left[-2\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)dr + \frac{1}{\sigma^2}\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^3 dr\right] \\
&\quad -\frac{1}{2\sigma^6}\frac{\left[\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)dr\right]\left[\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr\right]}{g_\theta^2(z)} \\
&= -\frac{1}{\sigma^4}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)dr}{g_\theta(z)} + \frac{1}{2\sigma^6}\frac{\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^3 dr}{g_\theta(z)} \\
&\quad -\frac{1}{2\sigma^6}\frac{[\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)dr][\int_0^\infty g_\theta(z,r)(\frac{z}{r}-\mu)^2 dr]}{g_\theta^2(z)}. \qquad (31)
\end{aligned}$$

B.3 Lognormal distribution

We assume $Y \sim f_\theta(y) \sim \frac{1}{y\sigma}e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}$, $0 < y < \infty$, and $\theta = (\mu, \sigma^2)$. Then the joint *pdf* $g(z,r)$ of (Z,R), the marginal *pdf* $g(z)$ of Z and the conditional *pdf* $g(r|z)$ of R, given $Z = z$, are given by

$$\begin{aligned}
g_\theta(z,r) &\sim \frac{1}{z\sigma}e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r), \\
g_\theta(z) &\sim \frac{1}{z\sigma}\int_0^\infty e^{-\frac{\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr, \qquad (32)
\end{aligned}$$

$$g_\theta(r|z) = \frac{e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)}{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}, \qquad (33)$$

for $0 < r < \infty$ and $0 < z < \infty$, and hence the complete data likelihood and log-likelihood can be expressed as

$$L(\theta|\boldsymbol{z}_c) \sim \frac{1}{\sigma^n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n(\ln\frac{z_i}{r_i}-\mu)^2}, \qquad (34)$$

$$\begin{aligned}
\ell(\theta|\boldsymbol{u}_c) &= -n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^n(\ln\frac{z_i}{r_i}-\mu)^2 \qquad (35) \\
&= -n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^n(\ln\frac{z_i}{r_i})^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^n\ln\frac{z_i}{r_i} - \frac{n\mu^2}{2\sigma^2}.
\end{aligned}$$

The EM steps for computing the MLE of θ based on $\boldsymbol{z}$ are as follows.
E-step. We compute

$$\begin{aligned} Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{u}_{\mathrm{c}})|\boldsymbol{u}_{\mathrm{obs}}] \\ &= -n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}[\psi_2(\theta^{(t)}, z_i) + \frac{\mu}{\sigma^2}\psi_1(\theta^{(t)}, z_i)] - \frac{n\mu^2}{2\sigma^2}, \end{aligned} \tag{36}$$

where $\psi_1(\theta^{(t)}, z_i) = E_{\theta^{(t)}}[\ln \frac{z_i}{r_i}|z_i]$ and $\psi_2(\theta^{(t)}, z_i) = E_{\theta^{(t)}}[(\ln \frac{z_i}{r_i})^2|z_i]$, and these two quantities can be readily computed based on the conditional *pdf* of R, given z, mentioned in (33).
M-step. We note that

$$\frac{\partial Q(\theta|\theta^{(t)})}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\psi_1(\theta^{(t)}, z_i) - \frac{n\mu}{\sigma^2} = 0$$

yields $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\psi_1(\theta^{(t)}, z_i)$, and,

$$\frac{\partial Q(\theta|\theta^{(t)})}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}\psi_2(\theta^{(t)}, z_i) - \frac{\mu}{\sigma^4}\sum_{i=1}^{n}\psi_1(\theta^{(t)}, z_i) + \frac{n\mu^2}{2\sigma^4} = 0$$

yields $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\psi_2(\theta^{(t)}, z_i) - \hat{\mu}^2$. Thus the EM iterations are defined by

$$\mu^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n}\psi_1(\theta^{(t)}, z_i), \;\; (\sigma^{(t+1)})^2 = \frac{1}{n}\sum_{i=1}^{n}\psi_2(\theta^{(t)}, z_i) - (\mu^{(t+1)})^2. \tag{37}$$

The observed Fisher information is computed by adding component-wise terms based on $z_1, ..., z_n$, with a generic term as follows. Since the marginal *pdf* of Z is given by (32), we obtain,

$$\frac{\partial \ln g_\theta(z)}{\partial \mu} = \frac{\int_0^\infty e^{-\frac{(\ln z - \ln r - \mu)^2}{2\sigma^2}} \frac{\ln z - \ln r - \mu}{\sigma^2} h(r)dr}{\int_0^\infty e^{-\frac{(\ln z - \ln r - \mu)^2}{2\sigma^2}} h(r)dr}, \tag{38}$$

$$\begin{aligned} \frac{\partial^2 \ln g_\theta(z)}{\partial \mu^2} &= -\frac{1}{\sigma^2} + \frac{\int_0^\infty e^{-\frac{(\ln z - \ln r - \mu)^2}{2\sigma^2}} (\frac{\ln z - \ln r - \mu}{\sigma^2})^2 h(r)dr}{\int_0^\infty e^{-\frac{(\ln z - \ln r - \mu)^2}{2\sigma^2}} h(r)dr} \\ &\quad - \left[\frac{\int_0^\infty e^{-\frac{(\ln z - \ln r - \mu)^2}{2\sigma^2}} \frac{\ln z - \ln r - \mu}{\sigma^2} h(r)dr}{\int_0^\infty e^{-\frac{(\ln z - \ln r - \mu)^2}{2\sigma^2}} h(r)dr}\right]^2, \end{aligned} \tag{39}$$

and

$$\frac{\partial^2 \ln g_\theta(z)}{\partial\mu\partial\sigma^2} = -\frac{1}{\sigma^4} \times \frac{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln z - \ln r - \mu)h(r)dr}{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}$$
$$+\frac{1}{2\sigma^6} \times \frac{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln z - \ln r - \mu)^3 h(r)dr}{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}$$
$$-\frac{1}{2\sigma^6}\frac{\left[\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln z - \ln r - \mu)h(r)dr \times \int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln(z/r) - \mu)^2 h(r)dr\right]}{\left[\int_0^\infty e^{-\frac{(\ln(z/r)-\mu)^2}{2\sigma^2}}h(r)dr\right]^2}. \tag{40}$$

Finally, we get

$$\begin{aligned}
\frac{\partial \ln g_\theta(z)}{\partial\sigma^2} &= -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}\frac{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln z - \ln r - \mu)^2 h(r)dr}{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}, \\
\frac{\partial^2 \ln g_\theta(z)}{(\partial\sigma^2)^2} &= \frac{1}{2\sigma^4} + \frac{1}{\sigma^6}\frac{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln z - \ln r - \mu)^2 h(r)dr}{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr} \\
&\quad + \frac{1}{4\sigma^8}\frac{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln z - \ln r - \mu)^4 h(r)dr}{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr} \\
&\quad - \frac{1}{4\sigma^8}\left[\frac{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}(\ln z - \ln r - \mu)^2 h(r)dr}{\int_0^\infty e^{-\frac{(\ln z-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}\right]^2.
\end{aligned} \tag{41}$$

A customized or conjugate noise distribution is obtained as follows. Since Y follows a lognormal distribution, implying $\ln Y \sim N(\mu, \sigma^2)$ and since $Z = Y \times R$, we choose $R \sim$ lognormal with $\ln R \sim N(-\frac{\psi^2}{2}, \psi^2)$. Then $E(R) = 1$ and $\text{var}(R) = \sigma_r^2 = e^{\psi^2} - 1$. This readily yields

$$\ln Z \sim N(\mu - \frac{\psi^2}{2} = \mu_z, \sigma^2 + \psi^2 = \sigma_z^2) \tag{42}$$

Since $\hat{\mu}_z = \frac{1}{n}\sum_{i=1}^n \ln z_i$ and $\hat{\sigma}_z^2 = \frac{1}{n}\sum_{i=1}^n(\ln z_i - \hat{\mu}_z)^2$, we obtain $\hat{\mu} = \hat{\mu}_z + \frac{\psi^2}{2}$ and $\hat{\sigma}^2 = \hat{\sigma}_z^2 - \psi^2$. Since the estimated variance-covariance matrix of $(\hat{\mu}_z, \hat{\sigma}_z^2)$ readily follows from the observed Fisher information matrix for (μ_z, σ_z^2) given by

$$I_{obs}(\hat{\mu}_z, \hat{\sigma}_z^2) = \begin{pmatrix} \frac{n}{\sigma_z^2} & 0 \\ 0 & \frac{n}{2\sigma_z^4} \end{pmatrix}. \tag{43}$$

The estimated variances of meaningful and useful functions $g(.,.)$ of μ_z and σ_z^2 can be easily obtained by computing the required partial derivatives. We provide below examples of such functions.

1. $g(\mu_z, \sigma_z^2) = e^{\mu + \frac{\sigma^2}{2}} = e^{\mu_z + \frac{\psi^2}{2}}$.
2. $g(\mu_z, \sigma_z^2) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu_z + 2\sigma_z^2 - \psi^2} - e^{2\mu_z + \sigma_z^2}$.
3. $g(\mu_z, \sigma_z^2) = e^{\mu + c\sigma} = e^{\mu_z + \frac{\psi^2}{2} + c(\sigma_z^2 - \psi^2)^{\frac{1}{2}}}$.

Note that the first and second functions are, respectively, the mean and variance of the original lognormal random variable Y, expressed in terms of the mean and variance of the random variable $\ln Z$, specified in (42). Similarly, the third function is a percentile of Y. Recall once again that data are not available on Y, but observations are available on the perturbed quantity Z. In other words, inference on the above functions related to the distribution of Y can now be carried out using the data on Z.

APPENDIX C: MLEs AND FISHER INFORMATION UNDER NOISE MULTIPLICATION OF EXTREME VALUES

Here we present details concerning the computation of the MLEs, and derivation of the Fisher information matrices in the context of the exponential, normal and lognormal distributions. These are required for the data analysis presented in Section 3 of the paper, dealing with noise multiplication of the extreme values in the sample. In what follows, the notations used are as given in Section 3 of the paper.

C1. Exponential distribution

We consider the case of $Y \sim f_\theta(y) = \frac{1}{\theta}e^{-y/\theta}$, $0 < y < \infty$. Then the complete data log-likelihood function can be expressed as

$$\ell(\theta|\boldsymbol{y}) = -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n}[\Delta_i x_i + (1-\Delta_i)\frac{x_i}{r_i}]. \tag{44}$$

Case (i). We now describe the EM algorithm for computing the MLE of θ based on the data $\boldsymbol{v}_{\text{i,obs}}$.

E-step.

$$\begin{aligned}Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}\left[\ell(\theta|\boldsymbol{v}_{\text{c}})|\boldsymbol{v}_{\text{i,obs}}\right]\\ &= -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n}\left\{\Delta_i x_i + (1-\Delta_i)x_i E_{\theta^{(t)}}\left[r_i^{-1}|x_i,\Delta_i=0\right]\right\}\end{aligned}$$

where

$$E_{\theta^{(t)}}\left[r_i^{-1}|x_i,\Delta_i=0\right] = \frac{\int_0^{x_i/C}\exp\left(-\frac{1}{\theta^{(t)}}\frac{x_i}{r}\right)h(r)r^{-2}dr}{\int_0^{x_i/C}\exp\left(-\frac{1}{\theta^{(t)}}\frac{x_i}{\omega}\right)h(\omega)\omega^{-1}d\omega}.$$

M-step. The value of θ that maximizes $Q(\theta|\theta^{(t)})$ is readily computed, thus the following equation defines the sequence of EM iterations:

$$\theta^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n}\left\{\Delta_i x_i + (1-\Delta_i)x_i E_{\theta^{(t)}}\left[r_i^{-1}|x_i,\Delta_i=0\right]\right\} \tag{45}$$

with the expression for $E_{\theta^{(t)}}[r_i^{-1}|x_i,\Delta_i=0]$ as given above.

From Result 2 in Appendix A of the paper, the loglikelihood is

$$\ell(\theta|x_1,\ldots,x_n,\Delta_1,\ldots,\Delta_n) = -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n}x_i\Delta_i + \sum_{i=1}^{n}(1-\Delta_i)\ln[\int_0^{x/C}\exp(-\frac{x_i}{r\theta})h(r)r^{-1}dr],$$

and hence the observed Fisher information is readily obtained based on the following:

$$\begin{aligned}\frac{\partial^2 l(\theta|x_1,\ldots,x_n,\Delta_1,\ldots,\Delta_n)}{\partial\theta^2} &= \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n}x_i\Delta_i - \frac{2}{\theta^3}\sum_{i=1}^{n}(1-\Delta_i)x_i\frac{\int_0^{x_i/C}e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r^2}dr}{\int_0^{x_i/C}e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r}dr}\\ &+\frac{1}{\theta^4}\sum_{i=1}^{n}(1-\Delta_i)x_i^2\frac{\int_0^{x_i/C}e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r^3}dr}{\int_0^{x_i/C}e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r}dr} - \frac{1}{\theta^4}\sum_{i=1}^{n}(1-\Delta_i)x_i^2[\frac{\int_0^{x_i/C}e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r^2}dr}{\int_0^{x_i/C}e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r}dr}]^2.\end{aligned} \tag{46}$$

Case (ii). We now describe the EM algorithm for computing the MLE of θ based on the data $\boldsymbol{v}_{\text{ii,obs}}$.
E-step.

$$
\begin{aligned}
Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}\left[\ell(\theta|\boldsymbol{v}_{\text{c}})|\boldsymbol{v}_{\text{ii,obs}}\right] \\
&= -n\ln\theta - \frac{1}{\theta}\sum_{i=1}^{n}\left\{x_i E_{\theta^{(t)}}[\Delta_i|x_i] + x_i E_{\theta^{(t)}}[(1-\Delta_i)r_i^{-1}|x_i]\right\},
\end{aligned}
$$

where

$$
E_{\theta^{(t)}}[\Delta_i|x_i] = \frac{I(x_i<C)\exp\left[-x_i/\theta^{(t)}\right]}{I(x_i<C)\exp\left[-x_i/\theta^{(t)}\right] + \int_0^{x_i/C}\exp\left[-\frac{1}{\theta^{(t)}}\frac{x_i}{\omega}\right]h(\omega)\omega^{-1}d\omega},
$$

and

$$
E_{\theta^{(t)}}[(1-\Delta_i)r_i^{-1}|x_i] = \frac{\int_0^{x_i/C}\exp\left[-\frac{1}{\theta^{(t)}}\frac{x_i}{r_i}\right]h(r_i)r_i^{-2}dr_i}{I(x_i<C)\exp\left[-x_i/\theta^{(t)}\right] + \int_0^{x_i/C}\exp\left[-\frac{1}{\theta^{(t)}}\frac{x_i}{\omega}\right]h(\omega)\omega^{-1}d\omega}.
$$

M-step. Maximization of $Q(\theta|\theta^{(t)})$ with respect to θ is readily accomplished, and hence we obtain the following equation which defines the sequence of EM iterations:

$$
\theta^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n}\left\{x_i E_{\theta^{(t)}}[\Delta_i|x_i] + x_i E_{\theta^{(t)}}[(1-\Delta_i)r_i^{-1}|x_i]\right\}, \tag{47}
$$

where the expressions for $E_{\theta^{(t)}}[\Delta_i|x_i]$ and $E_{\theta^{(t)}}[(1-\Delta_i)r_i^{-1}|x_i]$ are given above.

Recalling Result 4 given in Appendix A of the paper, the observed Fisher information in this case is based on the loglikelihood

$$
\ell(\theta|x_1,\ldots,x_n) = -n\ln\theta + \sum_{i=1}^{n}\ln[e^{-\frac{x_i}{\theta}}I(x_i<C) + \int_0^{\frac{x_i}{C}} e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r}dr] \tag{48}
$$

This gives

$$
\frac{\partial l(\theta|x_1,\ldots,x_n)}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}x_i[\frac{e^{-\frac{x_i}{\theta}}I(x_i<C) + \int_0^{\frac{x_i}{C}} e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r^2}dr}{k_\theta(x_i)}]. \tag{49}
$$

Hence we get

$$
\begin{aligned}
\frac{\partial^2 l(\theta|x_1,\ldots,x_n)}{\partial\theta^2} \quad = \quad & \frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{n}x_i[\frac{e^{-\frac{x_i}{\theta}}I(x_i<C) + \int_0^{\frac{x_i}{C}} e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r^2}dr}{k_\theta(x_i)}] \\
& + \frac{1}{\theta^4}\sum_{i=1}^{n}x_i^2[\frac{e^{-\frac{x_i}{\theta}}I(x_i<C) + \int_0^{\frac{x_i}{C}} e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r^3}dr}{k_\theta(x_i)}] \\
& - \frac{1}{\theta^4}\sum_{i=1}^{n}x_i^2[\frac{e^{-\frac{x_i}{\theta}}I(x_i<C) + \int_0^{\frac{x_i}{C}} e^{-\frac{x_i}{r\theta}}\frac{h(r)}{r^2}dr}{k_\theta(x_i)}]^2.
\end{aligned} \tag{50}
$$

C2. Normal distribution

We next consider the case $Y \sim f_\theta(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right]$, $-\infty < y < \infty$, where $\theta = (\mu, \sigma^2)$. Then the complete data likelihood function is

$$\begin{aligned} L(\theta|\boldsymbol{v}_\text{c}) &= \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right] \right\}^{\Delta_i} \left\{ \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(\frac{x_i}{r_i}-\mu)^2}{2\sigma^2}\right] \right\}^{1-\Delta_i} \\ &\propto \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\sigma^2}} \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right] \right\}^{\Delta_i} \left\{ \frac{1}{\sqrt{\sigma^2}} \exp\left[-\frac{(\frac{x_i}{r_i}-\mu)^2}{2\sigma^2}\right] \right\}^{1-\Delta_i} \\ &= (\sigma^2)^{-n/2} \prod_{i=1}^{n} \left\{ \exp\left[-\frac{(x_i-\mu)^2}{2\sigma^2}\right] \right\}^{\Delta_i} \left\{ \exp\left[-\frac{(\frac{x_i}{r_i}-\mu)^2}{2\sigma^2}\right] \right\}^{1-\Delta_i}, \end{aligned}$$

and hence we get the complete data log-likelihood function as

$$\begin{aligned} \ell(\theta|\boldsymbol{v}_\text{c}) &= -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} \left\{ \Delta_i (x_i-\mu)^2 + (1-\Delta_i)\left(\frac{x_i}{r_i}-\mu\right)^2 \right\} \\ &= -\frac{n}{2}\ln(\sigma^2) - \frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n} \left\{ \Delta_i \left[(x_i)^2 - 2\mu x_i\right] + (1-\Delta_i)\left[(\frac{x_i}{r_i})^2 - 2\mu\frac{x_i}{r_i}\right] \right\}. \end{aligned}$$

Case (i). The EM algorithm for computing the MLE of θ based on $(x_1, \Delta_1), \ldots, (x_n, \Delta_n)$ is as follows.

E-step.

$$\begin{aligned} Q(\theta|\theta^{(t)}) = E_{\theta^{(t)}}\left[\ell(\theta|\boldsymbol{v}_\text{c})|\boldsymbol{v}_\text{i,obs}\right] &= -\frac{n}{2}\ln(\sigma^2) - \frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{\Delta_i[(x_i)^2 - 2\mu x_i]\right\} \\ &- \frac{1}{2\sigma^2}\sum_{i=1}^{n}\{(1-\Delta_i)(E_{\theta^{(t)}}[(\frac{x_i}{r_i})^2|x_i, \Delta_i = 0] - 2\mu E_{\theta^{(t)}}[(\frac{x_i}{r_i})|x_i, \Delta_i = 0])\} \end{aligned}$$

where

$$\begin{aligned} E_{\theta^{(t)}}\left[(\frac{x_i}{r_i})^2|x_i, \Delta_i = 0\right] &= x_i^2 \frac{\int_0^{x_i/C} \exp\left[-\frac{(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(r)}{r^3}dr}{\int_0^{x_i/C} \exp\left[-\frac{(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(\omega)}{\omega}d\omega} \equiv x_i^2\psi_1(\theta^{(t)}, x_i), \\ E_{\theta^{(t)}}\left[(\frac{x_i}{r_i})|x_i, \Delta_i = 0\right] &= x_i \frac{\int_0^{x_i/C} \exp\left[-\frac{(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(r)}{r^2}dr}{\int_0^{x_i/C} \exp\left[-\frac{(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(\omega)}{\omega}d\omega} \equiv x_i\psi_2(\theta^{(t)}, x_i). \end{aligned}$$

M-step. By maximizing $Q(\theta|\theta^{(t)})$ with respect to θ, we obtain the following equations which define the sequence of EM iterations:

$$\begin{aligned} \mu^{(t+1)} &= \frac{1}{n}\sum_{i=1}^{n}\left\{\Delta_i x_i + (1-\Delta_i)x_i\psi_2(\theta^{(t)}, x_i)\right\}, \\ (\sigma^{(t+1)})^2 &= \frac{1}{n}\left(\sum_{i=1}^{n}\left\{\Delta_i(x_i)^2 + (1-\Delta_i)x_i^2\psi_1(\theta^{(t)}, x_i)\right\}\right) - (\mu^{(t+1)})^2, \end{aligned}$$

where the expressions for $\psi_1(\theta^{(t)}, x_i)$ and $\psi_2(\theta^{(t)}, x_i)$ are given above. When $\Delta_i = 1$, $\psi_1(\theta^{(t)}, x_i)$ and $\psi_2(\theta^{(t)}, x_i)$ are defined as equal to 0.

To compute the observed Fisher information, note that the loglikelihood based on one observation x can be expressed as

$$\ell(\theta|x,\Delta) = \ln L(\theta|x,\Delta) \sim -\frac{1}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\Delta(x-\mu)^2 + (1-\Delta)\ln\left[\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr\right]. \quad (51)$$

This yields

$$\begin{aligned}\frac{\partial \ell(\theta|x,\Delta)}{\partial \mu} &= \frac{\Delta(x-\mu)}{\sigma^2} + \frac{(1-\Delta)}{\sigma^2}\left[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr}\right] \\ &= \frac{\Delta(x-\mu)}{\sigma^2} - \frac{\mu(1-\Delta)}{\sigma^2} + \frac{(1-\Delta)}{\sigma^2}\left[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r})\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr}\right]. \quad (52)\end{aligned}$$

Hence we get

$$\begin{aligned}\frac{\partial^2 \ell(\theta|x,\Delta)}{\partial \mu^2} &= -\frac{1}{\sigma^2} + \frac{(1-\Delta)}{\sigma^4}\left[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)(\frac{x}{r})\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr}\right. \\ &\quad \left. -\frac{(\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)\frac{h(r)}{r}dr)(\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r})\frac{h(r)}{r}dr)}{(\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr)^2}\right]. \quad (53)\end{aligned}$$

From (52), we get

$$\begin{aligned}\frac{\partial^2 \ell(\theta|x,\Delta)}{\partial \mu \partial \sigma^2} &= -\frac{x\Delta-\mu}{\sigma^4} - \frac{(1-\Delta)}{\sigma^4}[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r})\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr}] \\ &\quad + \frac{(1-\Delta)}{2\sigma^6}\left[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)^2(\frac{x}{r})\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr}\right. \\ &\quad \left. -\frac{(\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)^2\frac{h(r)}{r}dr)(\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r})\frac{h(r)}{r}dr)}{(\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr)^2}\right]. \quad (54)\end{aligned}$$

Lastly, we note that

$$\frac{\partial \ell(\theta|x,\Delta)}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{\Delta(x-\mu)^2}{2\sigma^4} + \frac{(1-\Delta)}{2\sigma^4}\left[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)^2\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr}\right]. \quad (55)$$

Hence we get

$$\begin{aligned}\frac{\partial^2 \ell(\theta|x,\Delta)}{\partial (\sigma^2)^2} &= \frac{1}{2\sigma^4} - \frac{\Delta(x-\mu)^2}{\sigma^6} - \frac{(1-\Delta)}{\sigma^6}\left[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)^2\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr}\right] \\ &\quad + \frac{(1-\Delta)}{4\sigma^8}\left[\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)^4\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr} - (\frac{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}(\frac{x}{r}-\mu)^2\frac{h(r)}{r}dr}{\int_0^{\frac{x}{C}} e^{-\frac{1}{2\sigma^2}(\frac{x}{r}-\mu)^2}\frac{h(r)}{r}dr})^2\right]. \quad (56)\end{aligned}$$

Case (ii). The EM algorithm for computing the MLE of θ based on $(x_1, \ldots, x_n)$ is as follows.

E-step.

$$
\begin{aligned}
Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}\left[\ell(\theta|\boldsymbol{v}_{\mathrm{c}})|\boldsymbol{v}_{\mathrm{ii,obs}}\right] \\
&= -\frac{n}{2}\ln(\sigma^2) - \frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{(x_i)^2 - 2\mu x_i\right\}E_{\theta^{(t)}}[\Delta_i|x_i]\} \\
&- \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{E_{\theta^{(t)}}[(1-\Delta_i)(\frac{x_i}{r_i})^2|x_i] - 2\mu E_{\theta^{(t)}}[(1-\Delta_i)(\frac{x_i}{r_i})|x_i]\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
E_{\theta^{(t)}}[\Delta_i|x_i] &= \frac{I(x_i < C)\exp\left[-\frac{(x_i-\mu^{(t)})^2}{2(\sigma^{(t)})^2}\right]}{I(x_i < C)\exp\left[-\frac{(x_i-\mu^{(t)})^2}{2(\sigma^{(t)})^2}\right] + \int_0^{x_i/C}\exp\left[-\frac{\{(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(\omega)}{\omega}d\omega} \\
&\equiv \psi_1(\theta^{(t)}, x_i).
\end{aligned}
$$

$$
\begin{aligned}
E_{\theta^{(t)}}[(1-\Delta_i)(\frac{x_i}{r_i})^2|x_i] &= x_i^2\frac{\int_0^{x_i/C}\exp\left[-\frac{\{\ln(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(r)}{r^3}dr}{I(x_i < C)\exp\left[-\frac{\{x_i-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right] + \int_0^{x_i/C}\exp\left[-\frac{\{(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(\omega)}{\omega}d\omega} \\
&\equiv x_i^2\psi_2(\theta^{(t)}, x_i).
\end{aligned}
$$

$$
\begin{aligned}
E_{\theta^{(t)}}[(1-\Delta_i)(\frac{x_i}{r_i})|x_i] &= x_i\frac{\int_0^{x_i/C}(\frac{x_i}{r})\exp\left[-\frac{\{(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(r)}{r^2}dr}{I(x_i < C)\exp\left[-\frac{\{x_i-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right] + \int_0^{x_i/C}\exp\left[-\frac{\{(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]\frac{h(\omega)}{\omega}d\omega} \\
&\equiv x_i\psi_3(\theta^{(t)}, x_i).
\end{aligned}
$$

Hence we can write

$$
\begin{aligned}
Q(\theta|\theta^{(t)}) = -\frac{n}{2}\ln(\sigma^2) - \frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{(x_i)^2 - 2\mu x_i\right\}\psi_1(\theta^{(t)}, x_i) \\
- \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i^2\psi_2(\theta^{(t)}, x_i) - 2\mu x_i\psi_3(\theta^{(t)}, x_i).
\end{aligned}
$$

M-step. By maximizing $Q(\theta|\theta^{(t)})$ with respect to θ, we obtain the following equations which define the sequence of EM iterations:

$$
\begin{aligned}
\mu^{(t+1)} &= \frac{1}{n}\sum_{i=1}^{n}\left[x_i(\psi_1(\theta^{(t)}, x_i) + \psi_3(\theta^{(t)}, x_i))\right], \\
(\sigma^{(t+1)})^2 &= \frac{1}{n}\sum_{i=1}^{n}\left[x_i^2(\psi_1(\theta^{(t)}, x_i) + \psi_2(\theta^{(t)}, x_i))\right] - (\mu^{(t+1)})^2
\end{aligned}
$$

where the expressions for $\psi_1(\theta^{(t)}, x_i)$, $\psi_1(\theta^{(t)}, x_i)$ and $\psi_1(\theta^{(t)}, x_i)$ are given above.

To compute the observed Fisher information in this case, we can express the *pdf* of a single observation x as

$$k_\theta(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})I(x<C) + \int_0^{\frac{x}{C}} \phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\frac{h(r)}{r}dr, \tag{57}$$

where $\phi(.)$ is the standard normal *pdf*. This yields

$$\frac{\partial k_\theta(x)}{\partial \mu} = \frac{x-\mu}{\sigma^3}\phi\left(\frac{x-\mu}{\sigma}\right) I(x<C) + \int_0^{\frac{x}{C}} \frac{\frac{x}{r}-\mu}{\sigma^3}\phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\frac{h(r)}{r}dr, \tag{58}$$

and hence

$$\begin{aligned}\frac{\partial^2 k_\theta(x)}{\partial \mu^2} &= -\frac{k_\theta(x)}{\sigma^2} + \frac{(x-\mu)^2}{\sigma^4}\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) I(x<C) \\ &\quad + \int_0^{\frac{x}{C}} \frac{(\frac{x}{r}-\mu)^2}{\sigma^4}\frac{1}{\sigma}\phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\frac{h(r)}{r}dr. \end{aligned}\tag{59}$$

We also have

$$\begin{aligned}\frac{\partial^2 k_\theta(x)}{\partial \mu \partial \sigma^2} &= -\frac{3(x-\mu)}{2\sigma^5}\phi\left(\frac{x-\mu}{\sigma}\right) I(x<C) \\ &\quad -\frac{3}{2\sigma^5}\int_0^{\frac{x}{C}} (\frac{x}{r}-\mu)\phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\frac{h(r)}{r}dr \\ &\quad +\frac{x-\mu}{\sigma^3}[-\frac{1}{2\sigma^2}+\frac{(x-\mu)^2}{2\sigma^4}]\phi\left(\frac{x-\mu}{\sigma}\right) I(x<C) \\ &\quad +\int_0^{\frac{x}{C}} \frac{\frac{x}{r}-\mu}{\sigma^3}\left[-\frac{1}{2\sigma^2}+\frac{(\frac{x}{r}-\mu)^2}{2\sigma^4}\right]\phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\frac{h(r)}{r}dr. \end{aligned}\tag{60}$$

Lastly, we can write

$$\begin{aligned}\frac{\partial k_\theta(x)}{\partial \sigma^2} &= -\frac{1}{2\sigma^2}k_\theta(x) + \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left[-\frac{1}{2\sigma^2}+\frac{(x-\mu)^2}{2\sigma^4}\right] I(x<C) \\ &\quad +\int_0^{\frac{x}{C}} \phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\left[-\frac{1}{2\sigma^2}+\frac{(\frac{x}{r}-\mu)^2}{2\sigma^4}\right]\frac{h(r)}{r}dr, \end{aligned}\tag{61}$$

so that

$$\begin{aligned}\frac{\partial^2 k_\theta(x)}{\partial (\sigma^2)^2} &= \frac{1}{2\sigma^4}k_\theta(x) - \frac{1}{2\sigma^2}\frac{\partial k_\theta(x)}{\partial \sigma^2} + \phi\left(\frac{x-\mu}{\sigma}\right)\left[\frac{3}{4\sigma^5}-\frac{5(x-\mu)^2}{4\sigma^7}\right] I(x<C) \\ &\quad + \int_0^{\frac{x}{C}} \phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\left[\frac{3}{4\sigma^5}-\frac{5(\frac{x}{r}-\mu)^2}{4\sigma^7}\right]\frac{h(r)}{r}dr \\ &\quad + \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\left[-\frac{1}{2\sigma^2}+\frac{(x-\mu)^2}{2\sigma^4}\right]^2 I(x<C) \\ &\quad + \int_0^{\frac{x}{C}} \phi\left(\frac{\frac{x}{r}-\mu}{\sigma}\right) I(x>0)\left[-\frac{1}{2\sigma^2}+\frac{(\frac{x}{r}-\mu)^2}{2\sigma^4}\right]^2\frac{h(r)}{r}dr. \end{aligned}\tag{62}$$

We can use the above expressions to compute the elements of the observed Fisher information matrix.

C3. Lognormal distribution

We now assume that $Y \sim f_\theta(y) = \frac{1}{y\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\ln y-\mu)^2}{2\sigma^2}\right]$, $0 < y < \infty$, where $\theta = (\mu, \sigma^2)$. Then the complete data likelihood function is

$$\begin{aligned}
L(\theta|\boldsymbol{v}_\mathrm{c}) &= \prod_{i=1}^{n} \left\{ \frac{1}{x_i\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\ln x_i - \mu)^2}{2\sigma^2}\right]\right\}^{\Delta_i} \left\{ \frac{r_i}{x_i\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\ln(x_i/r_i) - \mu)^2}{2\sigma^2}\right]\right\}^{1-\Delta_i} \\
&\propto \prod_{i=1}^{n} \left\{ \frac{1}{\sqrt{\sigma^2}} \exp\left[-\frac{(\ln x_i - \mu)^2}{2\sigma^2}\right]\right\}^{\Delta_i} \left\{ \frac{1}{\sqrt{\sigma^2}} \exp\left[-\frac{(\ln(x_i/r_i) - \mu)^2}{2\sigma^2}\right]\right\}^{1-\Delta_i} \\
&= (\sigma^2)^{-n/2} \prod_{i=1}^{n} \left\{ \exp\left[-\frac{(\ln x_i - \mu)^2}{2\sigma^2}\right]\right\}^{\Delta_i} \left\{ \exp\left[-\frac{(\ln(x_i/r_i) - \mu)^2}{2\sigma^2}\right]\right\}^{1-\Delta_i},
\end{aligned}$$

and we get the complete data log-likelihood function as

$$\begin{aligned}
\ell(\theta|\boldsymbol{v}_\mathrm{c}) &= -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{\Delta_i(\ln x_i - \mu)^2 + (1-\Delta_i)(\ln\frac{x_i}{r_i} - \mu)^2\right\} \\
&= -\frac{n}{2}\ln(\sigma^2) - \frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{\Delta_i[(\ln x_i)^2 - 2\mu\ln x_i] + (1-\Delta_i)[(\ln\frac{x_i}{r_i})^2 - 2\mu\ln\frac{x_i}{r_i}]\right\}.
\end{aligned}$$

Case (i). The EM algorithm for computing the MLE of θ based on $(x_1, \Delta_1), \ldots, (x_n, \Delta_n)$ is as follows.

E-step.

$$\begin{aligned}
Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{v}_\mathrm{c})|\boldsymbol{v}_\mathrm{i,obs}] \\
&= -\frac{n}{2}\ln(\sigma^2) - \frac{n\mu^2}{2\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\{\Delta_i[(\ln x_i)^2 - 2\mu\ln x_i]\} \\
&- \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left\{(1-\Delta_i)\left(E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})^2|x_i, \Delta_i = 0] - 2\mu E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})|x_i, \Delta_i = 0]\right)\right\}
\end{aligned}$$

where

$$\begin{aligned}
E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})^2|x_i, \Delta_i = 0] &= \frac{\int_0^{x_i/C}(\ln\frac{x_i}{r})^2\exp\left[-\frac{\{\ln(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(r)dr}{\int_0^{x_i/C}\exp\left[-\frac{\{\ln(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(\omega)d\omega}, \\
E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})|x_i, \Delta_i = 0] &= \frac{\int_0^{x_i/C}(\ln\frac{x_i}{r})\exp\left[-\frac{\{\ln(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(r)dr}{\int_0^{x_i/C}\exp\left[-\frac{\{\ln(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(\omega)d\omega}.
\end{aligned}$$

M-step. By maximizing $Q(\theta|\theta^{(t)})$ with respect to θ, we obtain the following equations which define the sequence of EM iterations:

$$\mu^{(t+1)} = \frac{1}{n}\sum_{i=1}^{n}\left\{\Delta_i \ln x_i + (1-\Delta_i)E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})|x_i,\Delta_i=0]\right\},$$

$$(\sigma^{(t+1)})^2 = \frac{1}{n}\left(\sum_{i=1}^{n}\left\{\Delta_i(\ln x_i)^2 + (1-\Delta_i)E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})^2|x_i,\Delta_i=0]\right\} - n(\mu^{(t+1)})^2\right),$$

where the expressions for $E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})^2|x_i,\Delta_i=0]$ and $E_{\theta^{(t)}}[(\ln\frac{x_i}{r_i})|x_i,\Delta_i=0]$ are given above.

To compute the observed Fisher information in this case, recalling Result 2 given in Appendix A of the paper, a direct computation using the above formulas shows that for a generic term with x and Δ,

$$\frac{\partial l(\theta)}{\partial \mu} = \frac{(\ln x-\mu)\Delta}{\sigma^2} + (1-\Delta)\frac{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}\frac{(\ln x-\ln r-\mu)^2}{\sigma^2}h(r)dr}{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr} \tag{63}$$

leading to

$$\begin{aligned}\frac{\partial^2 l(\theta)}{\partial \mu^2} &= -\frac{1}{\sigma^2} + (1-\Delta)[\frac{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\frac{(\ln x-\ln r-\mu)^2}{\sigma^2})^2 h(r)dr}{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr} \\ &\quad -\{\frac{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}\frac{(\ln x-\ln r-\mu)^2}{\sigma^2}h(r)dr}{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}\}^2].\end{aligned} \tag{64}$$

Likewise,

$$\begin{aligned}\frac{\partial^2 l(\theta)}{\partial \mu \partial \sigma^2} &= -\frac{(\ln x-\mu)\Delta}{\sigma^4} + (1-\Delta)[-\frac{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\frac{(\ln x-\ln r-\mu)^2}{\sigma^4})h(r)dr}{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr} \\ &+\frac{1}{2\sigma^6}\frac{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\ln x-\ln r-\mu)^3 h(r)dr}{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr} \\ &-\frac{1}{2\sigma^6}\frac{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\ln x-\ln r-\mu)h(r)dr \int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\ln x-\ln r-\mu)^2 h(r)dr}{\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr^2}]\end{aligned} \tag{65}$$

Finally,

$$\begin{aligned}\frac{\partial^2 l(\theta)}{(\partial\sigma^2)^2} &= \frac{1}{2\sigma^4}-\frac{(\ln x-\mu)^2\Delta}{\sigma^6}+(1-\Delta)[-\frac{1}{\sigma^6}\frac{\int_0^{\frac{x}{C}}e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\ln x-\ln r-\mu)^2h(r)dr}{\int_0^{\frac{x}{C}}e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}\\ &\quad+\frac{1}{4\sigma^8}\frac{1}{2\sigma^6}\frac{\int_0^{\frac{x}{C}}e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\ln x-\ln r-\mu)^4h(r)dr}{\int_0^{\frac{x}{C}}e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}\\ &\quad-\frac{1}{4\sigma^8}\{\frac{\int_0^{\frac{x}{C}}e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}(\ln x-\ln r-\mu)^2h(r)dr}{\int_0^{\frac{x}{C}}e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}}h(r)dr}\}^2]. \qquad (66)\end{aligned}$$

Case (ii). The EM algorithm for computing the MLE of θ based on $(x_1,\ldots,x_n)$ is as follows.

E-step.

$$\begin{aligned}Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}[\ell(\theta|\boldsymbol{v}_{\mathrm{c}})|\boldsymbol{v}_{\mathrm{ii,obs}}]\\ &= -\frac{n}{2}\ln(\sigma^2)-\frac{n\mu^2}{2\sigma^2}-\frac{1}{2\sigma^2}\sum_{i=1}^n\{\{(\ln x_i)^2-2\mu\ln x_i\}E_{\theta^{(t)}}[\Delta_i|x_i]\}\\ &\quad-\frac{1}{2\sigma^2}\sum_{i=1}^n\left\{E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})^2|x_i]-2\mu E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})|x_i]\right\}\end{aligned}$$

where

$$E_{\theta^{(t)}}[\Delta_i|x_i]=\frac{I(x_i<C)\exp\left[-\frac{(\ln x_i-\mu^{(t)})^2}{2(\sigma^{(t)})^2}\right]}{I(x_i<C)\exp\left[-\frac{(\ln x_i-\mu^{(t)})^2}{2(\sigma^{(t)})^2}\right]+\int_0^{x_i/C}\exp\left[-\frac{(\ln(x_i/\omega)-\mu^{(t)})^2}{2(\sigma^{(t)})^2}\right]h(\omega)d\omega}.$$

$$\begin{aligned}&E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})^2|x_i]\\ &\quad=\frac{\int_0^{x_i/C}(\ln\frac{x_i}{r})^2\exp\left[-\frac{\{\ln(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(r)dr}{I(x_i<C)\exp\left[-\frac{\{\ln x_i-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]+\int_0^{x_i/C}\exp\left[-\frac{\{\ln(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(\omega)d\omega},\end{aligned}$$

$$E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})|x_i] = \frac{\int_0^{x_i/C}(\ln\frac{x_i}{r})\exp\left[-\frac{\{\ln(x_i/r)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(r)dr}{I(x_i<C)\exp\left[-\frac{\{\ln x_i-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]+\int_0^{x_i/C}\exp\left[-\frac{\{\ln(x_i/\omega)-\mu^{(t)}\}^2}{2(\sigma^{(t)})^2}\right]h(\omega)d\omega}.$$

M-step. By maximizing $Q(\theta|\theta^{(t)})$ with respect to θ, we obtain the following equations which define the sequence of EM iterations:

$$\begin{aligned}\mu^{(t+1)} &= \frac{1}{n}\sum_{i=1}^n\left\{(\ln x_i)E_{\theta^{(t)}}[\Delta_i|x_i]+E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})|x_i]\right\},\\ (\sigma^{(t+1)})^2 &= \frac{1}{n}\left(\sum_{i=1}^n\left\{(\ln x_i)^2E_{\theta^{(t)}}[\Delta_i|x_i]+E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})^2|x_i]\right\}-n(\mu^{(t+1)})^2\right),\end{aligned}$$

where formulas for $E_{\theta^{(t)}}[\Delta_i|x_i]$, $E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})^2|x_i]$, and $E_{\theta^{(t)}}[(1-\Delta_i)(\ln\frac{x_i}{r_i})|x_i]$ are given above.

To compute the observed Fisher information in this case, note that (by Result 3 of Appendix A of the paper) the loglikelihood based on a generic x is given by $\ln k(x,\theta)$ where

$$k(x,\theta) \sim \frac{1}{\sigma}[e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr]. \tag{67}$$

Hence we get

$$\begin{aligned}
\frac{\partial \ln k(x,\theta)}{\partial \mu} &= \frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}\frac{\ln x-\mu}{\sigma^2}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)\frac{\ln x-\ln r-\mu}{\sigma^2}h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr} \\
&= -\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x)I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)(\ln x-\ln r)h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right]
\end{aligned} \tag{68}$$

This readily yields

$$\begin{aligned}
&\frac{\partial^2 \ln k(x,\theta)}{\partial \mu^2} = -\frac{1}{\sigma^2} \\
&+\frac{1}{\sigma^4}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x)(\ln x-\mu)I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)\ln(x/r)(\ln(x/r)-\mu)h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right] \\
&-\frac{1}{\sigma^4}\left\{\left[e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x)I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)\ln(x/r)h(r)dr\right]\right. \\
&\times\left[e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x-\mu)I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)(\ln(x/r)-\mu)h(r)dr\right] \\
&\times\left.\left[e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr\right]^{-2}\right\}.
\end{aligned} \tag{69}$$

From (68), we get

$$\begin{aligned}
&\frac{\partial^2 \ln k(x,\theta)}{\partial \mu \partial \sigma^2} = \frac{\mu}{\sigma^4} - \frac{1}{\sigma^4}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x)I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)\ln(x/r)h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right] \\
&+\frac{1}{2\sigma^6}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x)(\ln x-\mu)^2I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)\ln(x/r)(\ln(x/r)-\mu)^2h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right] \\
&-\frac{1}{2\sigma^6}\left\{\left[e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x)I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)\ln(x/r)h(r)dr\right]\right. \\
&\times\left[e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x-\mu)^2I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)(\ln(x/r)-\mu)^2h(r)dr\right] \\
&\times\left.\left[e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}I(x<C) + \int_0^{\frac{x}{C}} e^{-\frac{(\ln(x/r)-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr\right]^{-2}\right\}.
\end{aligned} \tag{70}$$

On the other hand, we get

$$\frac{\partial \ln k(x,\theta)}{\partial \sigma^2} = -\frac{1}{2\sigma^2}$$
$$+\frac{1}{2\sigma^4}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x-\mu)^2 I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)(\ln x-\ln r-\mu)^2 h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right],$$

and hence,

$$\frac{\partial^2 \ln k(x,\theta)}{\partial (\sigma^2)^2} = \frac{1}{2\sigma^4}$$
$$-\frac{1}{\sigma^6}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x-\mu)^2 I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)(\ln x-\ln r-\mu)^2 h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right]$$
$$+\frac{1}{4\sigma^8}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x-\mu)^4 I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)(\ln x-\ln r-\mu)^4 h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right]$$
$$-\frac{1}{4\sigma^8}\left[\frac{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}(\ln x-\mu)^2 I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)(\ln x-\ln r-\mu)^2 h(r)dr}{e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}} I(x<C)+\int_0^{\frac{x}{C}} e^{-\frac{(\ln x-\ln r-\mu)^2}{2\sigma^2}} I(x>0)h(r)dr}\right]^2.$$

In this Appendix, we present the MLEs, along with the expressions for the observed Fisher information matrix for the three parametric models: exponential, normal and lognormal.

D1. Exponential distribution

Assume $y_1, \ldots, y_n$ is a random sample from the exponential distribution $f_\theta(y) = \frac{1}{\theta} e^{-y/\theta}$, $y > 0$. Let $x_i = \min\{y_i, C\}$, and $\Delta_i = I(y_i \leq C)$, for $i = 1, \ldots, n$. The likelihood for θ based on the observed data $\boldsymbol{w}_{\text{obs}} = (x_1, \ldots, x_n, \Delta_1, \ldots, \Delta_n)$ is given by

$$L(\theta|\boldsymbol{w}_{\text{obs}}) = (\frac{1}{\theta})^{\sum_{i=1}^n \Delta_i} \times e^{-\frac{\sum_{i=1}^n x_i}{\theta}}.$$

It is easily seen that the MLE of θ is

$$\hat{\theta}_{\text{MLE}} = \frac{\sum_{i=1}^n x_i}{\sum_{i=1}^n \Delta_i},$$

and the observed Fisher information is

$$\frac{(\sum_{i=1}^n \Delta_i)^2}{\sum_{i=1}^n x_i}.$$

D2. Normal distribution

Let C denote the known top code threshold. Assume $y_1, \ldots, y_n \sim iid\ N(\mu, \sigma^2)$ and let $x_i = \min\{y_i, C\}$ and $\Delta_i = I(y_i \leq C)$, $i = 1, \ldots, n$. In this case we can denote the complete and observed data, respectively, as

$$\boldsymbol{w}_{\text{c}} = (y_1, \ldots, y_n),\ \boldsymbol{w}_{\text{obs}} = (x_1, \ldots, x_n, \Delta_1, \ldots, \Delta_n).$$

Obviously the complete data loglikelihood is given by

$$\ell(\mu, \sigma^2|\boldsymbol{w}_{\text{c}}) \sim -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n y_i^2 + \frac{\mu}{\sigma^2}\sum_{i=1}^n y_i - \frac{n\mu^2}{2\sigma^2}. \tag{71}$$

The MLE of (μ, σ^2) based on the observed data $\boldsymbol{w}_{\text{obs}}$ can be computed via the EM algorithm. The E and M steps are as follows.
E-step. We have

$$\begin{aligned} Q(\theta|\theta^{(t)}) &= E_{\theta^{(t)}}[\ell(\mu, \sigma^2|\boldsymbol{w}_{\text{c}})|\boldsymbol{w}_{\text{obs}}] \\ &= -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n E_{\theta^{(t)}}[y_i^2|\boldsymbol{w}_{\text{obs}}] + \frac{\mu}{\sigma^2}\sum_{i=1}^n E_{\theta^{(t)}}[y_i|\boldsymbol{w}_{\text{obs}}] - \frac{n\mu^2}{2\sigma^2} \\ &= -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n E_{\theta^{(t)}}[y_i^2|x_i, \Delta_i] + \frac{\mu}{\sigma^2}\sum_{i=1}^n E_{\theta^{(t)}}[y_i|x_i, \Delta_i] - \frac{n\mu^2}{2\sigma^2}. \end{aligned} \tag{72}$$

where

$$E_{\theta^{(t)}}[y_i|x_i,\Delta_i] = x_i\Delta_i + (1-\Delta_i)\frac{\int_C^\infty \frac{y}{\sigma^{(t)}}\phi\left(\frac{y-\mu^{(t)}}{\sigma^{(t)}}\right)dy}{1-\Phi\left(\frac{C-\mu^{(t)}}{\sigma^{(t)}}\right)}, \tag{73}$$

$$E_{\theta^{(t)}}[y_i^2|x_i,\Delta_i] = x_i^2\Delta_i + (1-\Delta_i)\frac{\int_C^\infty \frac{y^2}{\sigma^{(t)}}\phi\left(\frac{y-\mu^{(t)}}{\sigma^{(t)}}\right)dy}{1-\Phi\left(\frac{C-\mu^{(t)}}{\sigma^{(t)}}\right)}. \tag{74}$$

It is easy to verify by direct computation that

$$\psi_1(\theta) \equiv \frac{\int_C^\infty \frac{y}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)dy}{1-\Phi\left(\frac{C-\mu}{\sigma}\right)} = \mu + \sigma\frac{\phi\left(\frac{C-\mu}{\sigma}\right)}{1-\Phi\left(\frac{C-\mu}{\sigma}\right)} \tag{75}$$

$$\psi_2(\theta) \equiv \frac{\int_C^\infty \frac{y^2}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right)dy}{1-\Phi\left(\frac{C-\mu}{\sigma}\right)} = \mu^2 + \sigma^2 + \sigma(\mu + C)\frac{\phi\left(\frac{C-\mu}{\sigma}\right)}{1-\Phi\left(\frac{C-\mu}{\sigma}\right)}. \tag{76}$$

We can therefore simplify $Q(\theta|\theta^{(t)})$ as

$$\begin{aligned} Q(\theta|\theta^{(t)}) &= -\frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2\Delta_i - \frac{1}{2\sigma^2}\psi_2(\theta^{(t)})\sum_{i=1}^n(1-\Delta_i) \\ &\quad + \frac{\mu}{\sigma^2}\sum_{i=1}^n x_i\Delta_i + \frac{\mu}{\sigma^2}\psi_1(\theta^{(t)})\sum_{i=1}^n(1-\Delta_i) - \frac{n\mu^2}{2\sigma^2}. \end{aligned} \tag{77}$$

M-step. Using (77) we can simplify $\frac{\partial Q(\theta|\theta^{(t)}}{\partial\mu} = 0$ and $\frac{\partial Q(\theta|\theta^{(t)})}{\partial\sigma^2} = 0$, resulting in the solution

$$\begin{aligned} \hat{\mu} &= \frac{1}{n}[\sum_{i=1}^n x_i\Delta_i + \psi_1(\theta^{(t)})\sum_{i=1}^n(1-\Delta_i)], \\ \hat{\sigma}^2 &= \frac{1}{n}[\sum_{i=1}^n x_i^2\Delta_i + \psi_2(\hat{\theta})\sum_{i=1}^n(1-\Delta_i)] - \hat{\mu}^2. \end{aligned} \tag{78}$$

Thus the EM algorithm in this case is defined by the following iterations:

$$\begin{aligned} \mu^{(t+1)} &= \frac{1}{n}[\sum_{i=1}^n x_i\Delta_i + \psi_1(\theta^{(t)})\sum_{i=1}^n(1-\Delta_i)], \\ (\sigma^{(t+1)})^2 &= \frac{1}{n}[\sum_{i=1}^n x_i^2\Delta_i + \psi_2(\theta^{(t)})\sum_{i=1}^n(1-\Delta_i)] - [\mu^{(t+1)}]^2. \end{aligned} \tag{79}$$

We now discuss the computation of observed Fisher information. Since the likelihood function for θ based on $\boldsymbol{w}_{\text{obs}}$ is given by

$$L(\theta|\boldsymbol{w}_{\text{obs}}) \sim \prod_{i=1}^n\left(\left[\frac{1}{\sigma}\phi\left(\frac{x_i-\mu}{\sigma}\right)\right]^{\Delta_i}\left[\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)\right]^{1-\Delta_i}\right),$$

the loglikelihood $\ell(\theta)$ is obtained as

$$\ell(\theta|\boldsymbol{w}_{\text{obs}}) \sim -\ln(\sigma)\left(\sum_{i=1}^{n}\Delta_i\right) + \sum_{i=1}^{n}\Delta_i \ln\phi\left(\frac{x_i-\mu}{\sigma}\right) + \sum_{i=1}^{n}(1-\Delta_i)\ln\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right). \tag{80}$$

In order to derive the elements of the Fisher information matrix, we shall use the following properties: $\phi'(u) = -u\phi(u)$ and

$$\frac{\partial}{\partial\sigma^2}\left[\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}(\frac{x_i-\mu}{\sigma})}\right] = \frac{1}{2\sigma^2}\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)}\left[\left(\frac{x_i-\mu}{\sigma}\right) - \frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)}\right],$$

where $\phi(u)$ is the standard normal *pdf*, and $\bar{\Phi}(u) = 1 - \Phi(u)$, where $\Phi(u)$ is the standard normal *cdf*. By straightforward algebra, we get

$$\begin{aligned}
\frac{\partial\ell(\theta|\boldsymbol{w}_{\text{obs}})}{\partial\mu} &= -\frac{1}{\sigma}\sum_{i=1}^{n}\Delta_i\frac{\phi'\left(\frac{x_i-\mu}{\sigma}\right)}{\phi(\frac{x_i-\mu}{\sigma})} + \frac{1}{\sigma}\sum_{i=1}^{n}(1-\Delta_i)\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)} \\
&= \frac{1}{\sigma^2}\sum_{i=1}^{n}\Delta_i(x_i-\mu) + \frac{1}{\sigma}\sum_{i=1}^{n}(1-\Delta_i)\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)}, \\
\frac{\partial\ell(\theta|\boldsymbol{w}_{\text{obs}})}{\partial\sigma^2} &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\Delta_i + \frac{1}{2\sigma^4}\sum_{i=1}^{n}\Delta_i(x_i-\mu)^2 + \frac{1}{2\sigma^3}\sum_{i=1}^{n}(1-\Delta_i)(x_i-\mu)\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)},
\end{aligned} \tag{81}$$

and hence,

$$\begin{aligned}
\frac{\partial^2\ell(\theta|\boldsymbol{w}_{\text{obs}})}{\partial\mu^2} &= -\frac{1}{\sigma^2}\sum_{i=1}^{n}\Delta_i + \frac{1}{\sigma^2}\sum_{i=1}^{n}(1-\Delta_i)\left(\frac{x_i-\mu}{\sigma}\right)\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)} - \frac{1}{\sigma^2}\sum_{i=1}^{n}(1-\Delta_i)\left[\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)}\right]^2, \\
\frac{\partial^2\ell(\theta|\boldsymbol{w}_{\text{obs}})}{\partial\mu\partial\sigma^2} &= -\frac{1}{\sigma^4}\sum_{i=1}^{n}\Delta_i(x_i-\mu) - \frac{1}{2\sigma^3}\sum_{i=1}^{n}(1-\Delta_i)\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)} + \frac{1}{\sigma}\sum_{i=1}^{n}(1-\Delta_i)\frac{\partial}{\sigma^2}\left[\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)}\right], \\
\frac{\partial^2\ell(\theta|\boldsymbol{w}_{\text{obs}})}{\partial(\sigma^2)^2} &= \frac{1}{2\sigma^4}\sum_{i=1}^{n}\Delta_i - \frac{1}{\sigma^6}\sum_{i=1}^{n}\Delta_i(x_i-\mu)^2 - \frac{3}{4\sigma^5}\sum_{i=1}^{n}(1-\Delta_i)\frac{(x_i-\mu)\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)} \\
&\quad + \frac{1}{4\sigma^4}\sum_{i=1}^{n}(1-\Delta_i)u_i^3\left[\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)}\right] - \frac{1}{4\sigma^4}\sum_{i=1}^{n}(1-\Delta_i)u_i^2\left[\frac{\phi\left(\frac{x_i-\mu}{\sigma}\right)}{\bar{\Phi}\left(\frac{x_i-\mu}{\sigma}\right)}\right]^2.
\end{aligned} \tag{82}$$

D3. Lognormal distribution

We once again use the property that if $y_1, \ldots, y_n$ is a random sample from a lognormal distribution, then $\ln y_1$, …, $\ln y_n$ is a sample from a normal distribution. We can now apply the normal theory results given in the previous section upon replacing y_i by $\ln y_i$ and C by $\ln C$.

APPENDIX E: SIMULATION RESULTS

In this Appendix, we present extensive numerical results for comparing multiple imputation and noise multiplication for the exponential, normal and lognormal models. We consider the case of fully masked samples (where all values are replaced with multiply imputed or noise multiplied data), as well as partially masked samples (where the top coded data are replaced with multiply imputed or noise multiplied data).

The notations used in the tables are as given in Section 6 of the paper. The notation CD denotes the case of complete data without any masking. MI denotes multiple imputation used to generate fully masked data. MI.C and MI.D are used to denote parametric multiple imputation based on the complete data, and based on the deleted data, respectively. These are explained in Section 4.2. In the case of a top coded sample with a top coding threshold C, values beyond a threshold C_I were also replaced with imputed values, where $C_I < C$, following An and Little (2007). Two choices were made for C_I following the recipe prescribed by An and Little (2007). If n_S denotes the number of values beyond the top coding threshold C, the two choices of C_I correspond to thresholds for which $2n_S$ values in the sample are larger, and $4n_S$ values are larger. Since these values approximately correspond to the 90th and 80th percentiles of the distribution when C is the 95th percentile, these are denoted (following An and Little, 2007) by MI.C90 and MI.C80, respectively, in the case of multiple imputation based on the complete data, and denoted by MI.D90 and MI.D80, respectively, in the case of multiple imputation based on the deleted data. Throughout we have used 50 sets of imputed values under MI.

The notations used under noise multiplication are as follows. We have used either a Uniform$(1-\epsilon, 1+\epsilon)$ distribution or a customized prior distribution for the noise multiplication. In the case of Uniform$(1-\epsilon, 1+\epsilon)$, the notation NM10U denotes noise multiplication with the above uniform distribution under the choice $\epsilon = 0.10$. NM20U, NM30U etc. are similarly defined. Furthermore, NM10C, NM20C etc. are the corresponding notations under the customized prior whose mean and variance match those of the respective uniform distributions. Additional notation used under the uniform noise distribution Uniform$(1-\epsilon, 1+\epsilon)$ is as follows. For the choice $\epsilon = 0.1$, NM10.1 and NM10.2 represent the case of observing the (x_i, Δ_i)s, or only the x_is; these are Case (i) and Case (ii) explained in Section 3 for the three different distributions. NM20.1, NM20.2 etc. are similarly defined. In addition to the above notations, entries in the tables corresponding to CENS denote results from the analysis of the censored data, i.e., the top coded data.

Table E1 − Table E36 are for comparing fully masked samples where all values have been protected, and are as follows:

Table E1 − Table E3: Exponential distribution

Table E4 − Table E21: Lognormal distribution

Table E22 − Table E36: Normal distribution

Table E37 − Table E84 are for comparing partially masked samples where the extreme values have been protected, and are as follows:

Table E37 − Table E40: Exponential distribution

Table E41 − Table E64: Lognormal distribution

Table E65 − Table E84: Normal distribution

Table E1: Inference for the mean based on fully masked Exponential(mean = 1) data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	183.74	-3.17	183.73	181.99	93.18	1.000	140.09	-0.43	140.10	141.36	94.28	1.000
MI	211.85	68.36	200.54	205.74	96.08	1.130	154.34	40.56	148.93	153.12	96.10	1.083
NM10U	184.29	-3.23	184.28	182.57	93.14	1.003	140.65	-0.12	140.67	141.86	94.34	1.004
NM10C	184.14	-2.91	184.14	182.63	92.98	1.003	140.78	-0.24	140.79	141.85	94.20	1.003
NM20U	186.07	-2.85	186.06	184.36	93.02	1.013	141.58	0.14	141.60	143.25	93.98	1.013
NM20C	186.24	-2.12	186.25	184.46	92.98	1.014	141.62	0.03	141.64	143.22	94.18	1.013
NM30U	191.17	-0.31	191.19	187.62	92.96	1.031	144.86	0.46	144.88	145.47	94.42	1.029
NM30C	188.96	-0.80	188.97	187.37	93.08	1.030	144.41	0.62	144.42	145.40	94.14	1.029
NM40U	193.55	0.40	193.57	191.51	93.02	1.052	147.19	1.28	147.20	148.49	94.10	1.050
NU40C	193.01	0.23	193.03	190.98	92.84	1.049	147.03	0.40	147.04	148.06	94.38	1.047
NM50U	198.28	-0.96	198.30	195.90	92.72	1.076	152.38	0.59	152.40	151.97	94.22	1.075
NM50C	199.11	1.18	199.13	195.10	92.94	1.072	150.62	2.77	150.61	151.49	94.14	1.072
NM60U	202.40	-0.74	202.42	201.48	92.60	1.107	155.62	2.54	155.61	156.51	94.40	1.107
NM60C	201.02	2.92	201.02	199.70	92.94	1.097	155.03	4.03	154.99	155.00	93.94	1.096
NM70U	210.80	4.65	210.77	209.01	92.80	1.148	161.33	4.94	161.27	161.82	94.60	1.145
NM70C	205.91	3.21	205.90	204.04	93.18	1.121	157.70	3.59	157.68	158.30	93.94	1.120
NM80U	216.04	-0.50	216.06	215.32	92.88	1.183	164.93	1.93	164.94	166.99	94.30	1.181
NM80C	211.10	7.18	211.00	209.10	93.14	1.149	160.19	4.91	160.13	161.82	94.20	1.145
NM90U	222.79	2.17	222.80	224.54	93.04	1.234	172.99	4.65	172.94	174.13	94.48	1.232
NM90C	215.40	9.49	215.22	213.77	93.34	1.175	166.88	5.43	166.81	165.08	93.74	1.168

Table E2: Inference for the mean based on fully masked Exponential(mean = 1) data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	100.44	-1.62	100.44	99.84	94.16	1.000	71.39	0.41	71.40	70.74	94.22	1.000
MI	106.44	18.91	104.75	104.94	95.34	1.051	74.32	10.71	73.55	73.24	95.00	1.035
NM10U	100.67	-1.78	100.66	100.15	94.34	1.003	71.60	0.39	71.61	70.97	94.02	1.003
NM10C	100.93	-1.47	100.93	100.18	94.26	1.003	71.66	0.36	71.66	70.97	94.38	1.003
NM20U	101.54	-1.31	101.54	101.15	94.32	1.013	72.23	0.45	72.24	71.66	94.64	1.013
NM20C	101.68	-1.17	101.69	101.16	94.56	1.013	72.18	0.60	72.19	71.66	94.40	1.013
NM30U	103.20	-1.55	103.20	102.66	94.78	1.028	73.04	0.45	73.05	72.74	94.66	1.028
NM30C	103.41	-0.76	103.42	102.69	94.52	1.029	72.98	0.59	72.99	72.72	94.50	1.028
NM40U	106.26	-0.71	106.27	104.79	94.28	1.050	75.00	0.93	75.00	74.23	94.66	1.049
NU40C	104.83	-0.43	104.84	104.62	94.36	1.048	74.60	0.45	74.61	74.06	94.54	1.047
NM50U	108.85	-1.29	108.85	107.26	94.24	1.074	75.67	0.54	75.67	75.98	94.76	1.074
NM50C	107.64	-0.68	107.65	106.78	93.88	1.070	76.17	1.38	76.17	75.68	94.56	1.070
NM60U	111.01	-0.02	111.02	110.38	94.42	1.106	78.39	1.13	78.39	78.13	94.70	1.104
NM60C	108.43	-0.35	108.44	109.15	94.26	1.093	77.84	1.83	77.83	77.39	94.68	1.094
NM70U	114.37	-0.62	114.38	113.76	94.08	1.139	81.22	1.36	81.21	80.58	94.58	1.139
NM70C	110.82	1.09	110.83	111.69	94.90	1.119	80.03	1.64	80.02	79.05	94.48	1.118
NM80U	118.16	0.12	118.17	117.84	94.26	1.180	82.96	1.43	82.96	83.40	94.82	1.179
NM80C	113.60	1.39	113.60	114.08	94.76	1.143	81.89	1.78	81.88	80.72	94.04	1.141
NM90U	124.63	-0.08	124.64	122.47	94.14	1.227	87.05	2.20	87.03	86.76	94.76	1.226
NM90C	117.32	1.87	117.32	116.35	94.46	1.165	83.89	1.90	83.88	82.31	94.60	1.164

Table E3: Inference for the mean based on fully masked Exponential(mean = 1) data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.05	-1.13	31.03	31.59	95.12	1.000	22.61	-0.15	22.61	22.36	94.76	1.000
MI	31.69	0.78	31.69	32.31	95.50	1.023	23.12	0.93	23.10	22.84	94.92	1.021
NM10U	31.15	-1.08	31.13	31.69	95.06	1.003	22.69	-0.17	22.69	22.43	94.68	1.003
NM10C	31.17	-1.14	31.15	31.69	95.14	1.003	22.75	-0.15	22.75	22.43	94.74	1.003
NM20U	31.40	-1.16	31.39	31.99	95.24	1.013	22.89	-0.06	22.89	22.65	94.86	1.013
NM20C	31.40	-0.98	31.39	32.00	95.08	1.013	22.84	-0.15	22.84	22.65	94.82	1.013
NM30U	31.91	-1.21	31.89	32.48	95.22	1.028	23.28	-0.28	23.28	22.99	94.54	1.028
NM30C	32.03	-0.97	32.02	32.47	95.22	1.028	23.18	-0.16	23.18	22.98	94.66	1.028
NM40U	32.81	-1.09	32.79	33.13	94.86	1.049	23.65	-0.09	23.66	23.45	94.90	1.049
NU40C	32.60	-1.19	32.58	33.07	95.48	1.047	23.61	-0.10	23.61	23.41	94.74	1.047
NM50U	33.34	-1.18	33.32	33.92	95.36	1.074	24.27	-0.41	24.27	24.00	94.74	1.074
NM50C	32.82	-1.20	32.80	33.76	95.26	1.069	24.17	-0.06	24.17	23.90	94.62	1.069
NM60U	34.58	-0.85	34.57	34.87	94.98	1.104	24.78	-0.05	24.78	24.67	94.54	1.104
NM60C	34.10	-0.82	34.10	34.52	95.18	1.093	24.66	-0.28	24.66	24.42	94.52	1.092
NM70U	35.87	-1.04	35.86	35.95	95.02	1.138	25.55	-0.24	25.55	25.44	94.76	1.138
NM70C	34.43	-1.38	34.40	35.25	95.26	1.116	24.95	-0.26	24.95	24.96	95.12	1.116
NM80U	36.81	-0.70	36.81	37.21	95.12	1.178	26.32	-0.21	26.33	26.33	95.00	1.178
NM80C	35.61	-0.93	35.60	36.01	94.88	1.140	25.84	0.11	25.84	25.49	94.66	1.140
NM90U	38.03	-1.59	38.00	38.65	95.22	1.223	27.59	0.07	27.59	27.37	94.78	1.224
NM90C	36.41	-1.21	36.39	36.71	95.48	1.162	26.23	0.17	26.23	25.99	94.60	1.163

Table E4: Inference for the log scale mean μ based fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	180.08	-0.10	180.09	178.24	94.36	1.000	142.92	0.95	142.93	139.29	94.06	1.000
MI	184.65	-0.71	184.67	188.34	94.92	1.057	145.79	1.19	145.80	145.04	94.64	1.041
NM10U	180.29	-0.32	180.30	178.55	94.18	1.002	143.21	0.90	143.22	139.51	94.06	1.002
NM10C	180.46	-0.20	180.48	178.58	94.32	1.002	143.27	1.02	143.28	139.52	94.14	1.002
NM20U	180.99	-0.37	181.00	179.34	93.98	1.006	143.30	1.07	143.31	140.24	94.10	1.007
NM20C	181.34	-0.13	181.36	179.35	94.10	1.006	143.90	0.86	143.91	140.26	93.90	1.007
NM30U	183.34	0.36	183.36	180.93	94.20	1.015	144.94	1.68	144.95	141.52	94.22	1.016
NM30C	182.12	0.77	182.13	180.89	94.42	1.015	145.26	1.37	145.27	141.35	93.88	1.015
NM40U	184.28	0.46	184.30	183.26	94.32	1.028	145.78	1.85	145.78	143.30	94.42	1.029
NM40C	184.37	-0.38	184.39	182.93	94.16	1.026	146.27	1.28	146.28	142.78	94.24	1.025
NM50U	189.38	-0.24	189.40	186.52	94.10	1.046	149.82	0.85	149.83	145.60	93.96	1.045
NM50C	188.88	-0.51	188.90	185.02	94.40	1.038	149.54	1.31	149.55	144.87	93.80	1.040
NM60U	193.14	-0.34	193.15	190.76	94.24	1.070	152.49	1.37	152.50	149.14	93.82	1.071
NM60C	189.61	0.43	189.63	188.21	94.10	1.056	151.02	0.31	151.03	147.18	93.96	1.057
NM70U	199.34	1.13	199.36	196.17	93.90	1.101	159.06	-0.00	159.07	153.78	93.44	1.104
NM70C	195.07	-1.62	195.08	191.19	93.88	1.073	153.84	1.00	153.85	149.67	93.90	1.075
NM80U	207.71	2.08	207.72	204.14	93.90	1.145	165.26	0.68	165.28	159.88	94.06	1.148
NM80C	199.22	0.09	199.24	194.36	93.42	1.090	157.14	0.27	157.16	152.23	93.92	1.093
NM90U	223.68	3.98	223.67	217.41	93.98	1.220	172.53	2.11	172.54	170.03	94.06	1.221
NM90C	201.57	1.48	201.58	198.56	94.04	1.114	159.67	1.63	159.68	154.99	93.56	1.113

Table E5: Inference for the log scale mean μ based fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	101.60	0.84	101.60	99.27	94.00	1.000	70.05	0.68	70.06	70.51	95.10	1.000
MI	104.05	0.75	104.05	102.27	94.26	1.030	71.57	0.79	71.57	72.30	95.24	1.025
NM10U	101.71	0.92	101.71	99.45	94.22	1.002	70.15	0.61	70.15	70.63	95.22	1.002
NM10C	101.80	1.05	101.81	99.43	94.22	1.002	70.21	0.66	70.21	70.64	95.08	1.002
NM20U	102.54	0.98	102.55	99.96	94.32	1.007	70.43	0.62	70.44	70.98	95.14	1.007
NM20C	102.23	1.04	102.23	99.95	94.20	1.007	70.54	0.62	70.54	70.99	95.18	1.007
NM30U	102.90	0.55	102.91	100.77	94.24	1.015	71.00	0.42	71.01	71.61	95.24	1.016
NM30C	102.83	0.90	102.83	100.71	94.34	1.014	71.07	0.82	71.07	71.55	95.30	1.015
NM40U	104.33	1.00	104.34	102.07	94.32	1.028	72.49	0.57	72.50	72.53	95.24	1.029
NM40C	103.45	0.78	103.46	101.80	94.12	1.025	71.89	0.24	71.90	72.36	95.42	1.026
NM50U	106.55	1.28	106.56	103.87	94.12	1.046	73.22	0.62	73.22	73.76	95.16	1.046
NM50C	105.12	0.59	105.13	103.13	94.08	1.039	73.05	0.65	73.06	73.31	95.12	1.040
NM60U	108.20	2.03	108.19	106.27	94.24	1.070	75.41	0.55	75.41	75.44	95.16	1.070
NM60C	107.52	0.50	107.53	104.81	94.08	1.056	74.51	0.08	74.51	74.38	94.84	1.055
NM70U	111.48	-0.52	111.49	109.50	94.06	1.103	78.32	1.17	78.32	77.79	94.66	1.103
NM70C	109.21	0.79	109.22	106.49	94.14	1.073	75.00	0.53	75.01	75.63	95.28	1.073
NM80U	116.57	0.63	116.58	114.05	93.90	1.149	81.71	0.26	81.72	81.00	94.88	1.149
NM80C	110.15	0.46	110.16	108.44	94.50	1.092	76.57	1.14	76.57	76.95	94.72	1.091
NM90U	123.52	1.24	123.52	121.14	94.56	1.220	85.86	-0.16	85.87	86.05	95.00	1.220
NM90C	112.41	0.31	112.42	110.63	94.26	1.114	77.98	0.74	77.98	78.44	95.00	1.112

Table E6: Inference for the log scale mean μ based fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.58	0.59	31.58	31.60	95.20	1.000	22.55	0.24	22.55	22.36	94.84	1.000
MI	32.02	0.54	32.02	32.26	95.10	1.021	23.03	0.27	23.03	22.81	94.94	1.020
NM10U	31.63	0.61	31.63	31.66	95.08	1.002	22.60	0.26	22.60	22.40	94.82	1.002
NM10C	31.63	0.60	31.63	31.66	95.02	1.002	22.56	0.26	22.56	22.40	94.96	1.002
NM20U	31.87	0.51	31.87	31.82	95.20	1.007	22.70	0.21	22.70	22.51	94.98	1.007
NM20C	31.91	0.64	31.91	31.81	95.00	1.007	22.69	0.20	22.69	22.51	94.90	1.007
NM30U	32.13	0.64	32.12	32.09	95.02	1.015	22.84	0.17	22.84	22.71	95.04	1.016
NM30C	32.10	0.80	32.09	32.07	95.28	1.015	22.86	0.28	22.86	22.69	94.84	1.015
NM40U	32.56	0.61	32.56	32.50	94.96	1.028	23.10	0.16	23.11	23.00	94.90	1.028
NM40C	32.31	0.69	32.30	32.41	95.14	1.026	23.06	0.30	23.06	22.93	94.92	1.026
NM50U	32.96	0.56	32.95	33.08	95.26	1.047	23.53	0.27	23.53	23.40	94.54	1.046
NM50C	32.82	0.60	32.82	32.84	95.00	1.039	23.49	0.21	23.49	23.24	95.22	1.039
NM60U	33.75	0.70	33.75	33.83	95.00	1.070	24.00	0.20	24.00	23.93	95.12	1.070
NM60C	33.23	0.44	33.24	33.35	95.10	1.055	23.65	0.12	23.65	23.59	94.96	1.055
NM70U	34.54	0.36	34.54	34.86	95.34	1.103	24.94	0.04	24.94	24.66	95.04	1.103
NM70C	34.15	0.60	34.15	33.92	95.00	1.073	24.31	-0.01	24.31	23.99	94.68	1.073
NM80U	36.62	0.80	36.61	36.32	94.52	1.149	25.52	0.42	25.52	25.69	95.28	1.149
NM80C	34.80	0.53	34.80	34.53	94.50	1.093	24.59	0.09	24.60	24.42	95.02	1.092
NM90U	38.52	0.49	38.52	38.55	94.94	1.220	27.60	0.09	27.60	27.27	94.68	1.220
NM90C	35.06	0.49	35.06	35.18	94.96	1.113	24.94	0.43	24.94	24.89	94.68	1.113

Table E7: Inference for the log scale variance σ^2 based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	257.50	-30.14	255.76	250.42	89.60	1.000	199.30	-19.91	198.32	196.02	91.48	1.000
MI	283.05	40.68	280.14	293.92	94.14	1.174	211.95	21.45	210.89	216.95	94.24	1.107
NM10U	258.02	-30.13	256.28	251.28	89.54	1.003	200.32	-20.09	199.33	196.65	91.34	1.003
NM10C	257.82	-29.83	256.12	251.36	89.48	1.004	200.29	-19.97	199.31	196.67	91.32	1.003
NM20U	260.03	-31.82	258.10	253.48	89.20	1.012	202.43	-20.00	201.46	198.71	91.46	1.014
NM20C	261.97	-31.11	260.15	253.59	89.72	1.013	201.96	-19.44	201.04	198.76	91.58	1.014
NM30U	264.21	-32.18	262.27	257.93	89.64	1.030	206.21	-19.50	205.30	202.32	91.58	1.032
NM30C	266.40	-30.53	264.68	257.95	89.56	1.030	204.69	-20.33	203.70	201.85	91.56	1.030
NM40U	272.10	-32.54	270.17	264.54	89.38	1.056	210.89	-20.43	209.92	207.30	91.52	1.058
NM40C	272.07	-30.22	270.41	263.81	89.48	1.053	209.70	-22.07	208.55	205.98	91.48	1.051
NM50U	279.67	-32.89	277.75	273.75	90.14	1.093	215.93	-24.10	214.61	213.73	91.60	1.090
NM50C	279.85	-34.70	277.72	269.91	89.10	1.078	213.91	-20.04	213.00	212.00	91.88	1.082
NM60U	299.21	-34.27	297.27	285.98	88.96	1.142	227.60	-22.26	226.54	223.82	91.62	1.142
NM60C	286.55	-32.00	284.78	279.20	89.24	1.115	221.25	-19.18	220.44	218.83	91.76	1.116
NM70U	309.44	-42.62	306.53	300.97	89.40	1.202	238.43	-22.68	237.37	237.02	91.52	1.209
NM70C	298.14	-35.21	296.09	288.17	89.60	1.151	230.41	-19.67	229.59	226.32	91.50	1.155
NM80U	334.66	-49.35	331.03	322.94	88.10	1.290	260.19	-30.19	258.46	253.81	90.72	1.295
NM80C	305.85	-40.30	303.22	297.72	89.56	1.189	238.91	-22.74	237.85	234.13	91.32	1.194
NM90U	366.82	-52.33	363.10	356.45	88.38	1.423	286.05	-30.25	284.48	280.63	91.30	1.432
NM90C	317.95	-35.60	315.98	310.72	89.22	1.241	247.29	-25.57	245.98	242.69	91.48	1.238

Table E8: Inference for the log scale variance σ^2 based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	138.01	-9.72	137.68	140.05	94.04	1.000	102.64	-2.90	102.61	99.71	93.36	1.000
MI	144.13	10.64	143.75	148.71	95.28	1.062	106.04	7.62	105.78	103.76	94.08	1.041
NM10U	138.46	-9.59	138.15	140.54	93.74	1.004	102.85	-2.87	102.82	100.05	93.42	1.003
NM10C	138.56	-9.92	138.22	140.49	94.06	1.003	102.97	-2.79	102.94	100.05	93.36	1.003
NM20U	140.11	-9.53	139.80	141.99	93.84	1.014	104.11	-3.25	104.07	101.03	93.18	1.013
NM20C	139.62	-9.47	139.31	141.95	93.90	1.014	103.74	-2.78	103.71	101.05	93.58	1.013
NM30U	143.65	-10.86	143.26	144.28	93.50	1.030	106.28	-2.87	106.26	102.82	93.36	1.031
NM30C	142.46	-10.38	142.10	144.13	93.44	1.029	106.02	-2.82	105.99	102.67	93.20	1.030
NM40U	146.98	-10.83	146.59	147.94	93.68	1.056	108.47	-2.93	108.44	105.40	93.62	1.057
NM40C	145.02	-10.60	144.65	147.27	93.76	1.052	107.08	-1.90	107.07	105.01	93.66	1.053
NM50U	151.89	-10.51	151.54	153.05	93.60	1.093	111.76	-3.92	111.71	108.87	93.58	1.092
NM50C	149.72	-11.34	149.31	151.14	93.62	1.079	111.15	-2.41	111.14	107.76	94.04	1.081
NM60U	159.34	-10.91	158.98	159.85	93.38	1.141	117.71	-4.52	117.63	113.65	93.50	1.140
NM60C	155.28	-9.43	155.01	156.11	93.80	1.115	113.63	-3.82	113.57	110.95	93.58	1.113
NM70U	169.59	-11.88	169.19	169.05	93.00	1.207	123.41	-3.63	123.37	120.33	93.84	1.207
NM70C	163.02	-11.60	162.62	161.18	93.28	1.151	117.71	-4.22	117.64	114.71	93.28	1.150
NM80U	182.80	-13.72	182.31	181.81	93.10	1.298	131.21	-5.25	131.11	129.34	93.40	1.297
NM80C	166.68	-11.54	166.29	167.14	93.24	1.193	121.30	-6.16	121.16	118.72	93.28	1.191
NM90U	201.43	-15.06	200.89	201.05	92.90	1.436	148.31	-3.72	148.27	143.20	92.94	1.436
NM90C	170.98	-9.24	170.75	173.92	93.86	1.242	126.36	-5.13	126.27	123.39	94.04	1.237

Table E9: Inference for the log scale variance σ^2 based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	45.04	-0.63	45.04	44.69	94.64	1.000	32.17	0.20	32.17	31.63	94.86	1.000
MI	46.08	1.46	46.06	45.77	94.72	1.024	33.06	1.20	33.04	32.32	94.70	1.022
NM10U	45.23	-0.62	45.23	44.84	94.64	1.003	32.28	0.21	32.28	31.74	94.74	1.003
NM10C	45.21	-0.54	45.21	44.85	94.66	1.003	32.27	0.20	32.28	31.73	94.76	1.003
NM20U	45.45	-0.51	45.45	45.30	94.66	1.014	32.39	0.21	32.40	32.06	95.02	1.014
NM20C	45.59	-0.63	45.59	45.29	94.90	1.013	32.61	0.26	32.61	32.05	94.82	1.013
NM30U	46.32	-0.87	46.32	46.07	94.88	1.031	32.95	0.12	32.96	32.61	94.74	1.031
NM30C	46.37	-0.55	46.37	46.02	94.82	1.030	32.92	0.30	32.92	32.57	94.62	1.030
NM40U	47.78	-0.79	47.78	47.23	94.18	1.057	33.82	0.09	33.82	33.42	94.74	1.057
NM40C	47.20	-0.75	47.20	47.01	94.62	1.052	33.67	0.07	33.67	33.27	94.98	1.052
NM50U	49.29	-0.06	49.30	48.86	94.40	1.093	34.83	0.49	34.83	34.57	94.88	1.093
NM50C	48.38	-0.99	48.38	48.26	94.46	1.080	34.66	0.67	34.66	34.18	94.82	1.080
NM60U	51.75	-0.95	51.74	50.98	94.10	1.141	36.52	0.25	36.52	36.09	94.68	1.141
NM60C	50.00	-0.82	50.00	49.75	94.84	1.113	35.81	-0.03	35.81	35.21	95.14	1.113
NM70U	54.52	-1.42	54.50	53.90	94.38	1.206	38.46	-0.06	38.47	38.16	94.78	1.206
NM70C	51.53	-0.29	51.54	51.47	95.20	1.152	36.90	-0.02	36.91	36.41	94.34	1.151
NM80U	58.50	-0.96	58.49	58.04	94.90	1.299	41.76	-0.25	41.77	41.06	94.58	1.298
NM80C	53.99	-0.31	54.00	53.36	94.42	1.194	37.84	0.05	37.84	37.74	94.88	1.193
NM90U	64.16	-1.05	64.16	64.17	94.70	1.436	45.98	0.28	45.98	45.41	94.86	1.436
NM90C	55.78	-0.53	55.78	55.39	94.70	1.239	40.07	0.31	40.07	39.19	94.24	1.239

Table E10: Inference for the mean $e^{\mu+\sigma^2/2}$ based on fully masked $LN(\mu=0,\sigma^2=1)$ data

	Results for $n=30$						Results for $n=50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	378.18	15.85	377.88	371.76	91.00	1.000	297.25	10.78	297.08	286.92	92.32	1.000
MI	527.67	196.00	489.97	600.18	96.18	1.614	353.91	106.84	337.43	362.65	96.04	1.264
NM10U	378.87	15.63	378.59	372.58	91.02	1.002	298.14	10.71	297.98	287.56	92.32	1.002
NM10C	379.55	16.18	379.25	372.79	91.00	1.003	298.35	11.03	298.18	287.63	92.38	1.002
NM20U	380.28	14.53	380.04	374.54	91.08	1.007	299.54	11.30	299.36	289.70	92.32	1.010
NM20C	384.03	16.13	383.73	375.23	90.86	1.009	300.26	11.50	300.07	289.78	92.34	1.010
NM30U	386.01	16.58	385.69	379.45	90.80	1.021	304.90	13.54	304.62	293.65	92.42	1.023
NM30C	384.52	18.22	384.13	380.01	91.24	1.022	303.33	12.14	303.11	292.95	92.32	1.021
NM40U	391.60	17.42	391.25	386.11	91.04	1.039	305.86	13.22	305.61	298.20	92.86	1.039
NM40C	392.68	18.22	392.30	386.10	91.00	1.039	307.51	11.30	307.34	296.95	92.20	1.035
NM50U	403.94	18.74	403.55	395.03	90.64	1.063	315.78	10.44	315.64	303.57	91.94	1.058
NM50C	403.76	16.76	403.45	392.13	90.86	1.055	310.96	13.63	310.69	303.14	92.66	1.057
NM60U	416.58	20.13	416.13	406.48	90.48	1.093	323.94	14.09	323.67	312.95	92.06	1.091
NM60C	408.74	21.47	408.22	402.53	91.16	1.083	322.09	14.44	321.79	310.22	92.40	1.081
NM70U	431.59	18.93	431.22	417.67	90.32	1.123	333.40	13.32	333.16	322.92	91.66	1.125
NM70C	425.20	19.30	424.81	411.56	90.22	1.107	332.22	16.97	331.82	318.19	92.12	1.109
NM80U	440.41	17.28	440.11	431.57	89.84	1.161	349.04	11.19	348.90	333.76	91.36	1.163
NM80C	429.35	19.09	428.97	420.70	90.08	1.132	340.80	14.88	340.51	325.47	91.90	1.134
NM90U	466.61	24.08	466.04	451.88	89.26	1.216	357.67	14.97	357.39	348.83	91.58	1.216
NM90C	448.06	28.09	447.22	436.24	91.18	1.173	349.10	16.19	348.76	334.07	91.78	1.164

Table E11: Inference for the mean $e^{\mu+\sigma^2/2}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	204.51	5.86	204.44	202.41	93.66	1.000	142.03	4.77	141.96	143.32	94.90	1.000
MI	224.19	49.82	218.61	226.79	95.70	1.120	150.98	26.74	148.61	152.96	95.86	1.067
NM10U	204.86	6.14	204.79	202.92	93.76	1.003	142.30	4.70	142.24	143.64	95.02	1.002
NM10C	204.95	6.09	204.88	202.87	93.64	1.002	142.57	4.88	142.50	143.66	94.80	1.002
NM20U	206.88	6.52	206.80	204.37	93.64	1.010	143.59	4.52	143.54	144.56	94.90	1.009
NM20C	206.05	6.58	205.96	204.35	93.84	1.010	143.09	4.87	143.02	144.61	94.72	1.009
NM30U	208.41	4.92	208.37	206.36	93.44	1.020	145.38	4.65	145.32	146.26	94.60	1.021
NM30C	207.64	5.77	207.59	206.36	93.62	1.020	145.10	5.32	145.02	146.22	94.86	1.020
NM40U	212.88	6.21	212.81	209.92	93.56	1.037	147.66	5.06	147.59	148.65	94.86	1.037
NM40C	210.65	5.78	210.59	209.41	93.78	1.035	146.45	5.24	146.37	148.43	94.90	1.036
NM50U	216.87	7.45	216.76	214.60	93.56	1.060	149.63	4.49	149.58	151.65	94.58	1.058
NM50C	215.28	5.42	215.23	213.11	93.74	1.053	151.46	5.94	151.36	151.16	94.66	1.055
NM60U	221.71	8.93	221.55	220.49	93.72	1.089	155.35	4.40	155.31	155.61	94.68	1.086
NM60C	220.44	7.48	220.34	218.15	93.80	1.078	152.95	4.00	152.91	153.99	94.76	1.074
NM70U	230.47	5.16	230.43	227.05	93.40	1.122	161.69	6.76	161.57	160.92	94.68	1.123
NM70C	228.04	7.15	227.95	223.04	93.50	1.102	156.69	4.72	156.64	157.62	94.52	1.100
NM80U	241.33	7.05	241.26	235.77	93.06	1.165	167.20	4.46	167.16	166.59	94.30	1.162
NM80C	230.25	6.99	230.16	228.65	93.30	1.130	160.87	4.55	160.83	161.39	94.38	1.126
NM90U	249.83	8.01	249.72	245.70	93.44	1.214	173.55	5.71	173.47	173.94	94.36	1.214
NM90C	235.48	9.38	235.32	235.38	93.56	1.163	163.84	5.02	163.78	165.80	94.44	1.157

Table E12: Inference for the mean $e^{\mu+\sigma^2/2}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	63.72	1.69	63.70	63.93	95.28	1.000	46.10	1.20	46.09	45.20	94.52	1.000
MI	65.32	5.82	65.06	65.77	95.60	1.029	47.32	3.33	47.20	46.30	94.46	1.024
NM10U	63.88	1.73	63.87	64.07	95.30	1.002	46.22	1.24	46.21	45.31	94.44	1.002
NM10C	63.86	1.79	63.85	64.08	95.30	1.002	46.11	1.24	46.10	45.31	94.54	1.002
NM20U	64.21	1.66	64.20	64.51	95.22	1.009	46.33	1.17	46.32	45.61	94.72	1.009
NM20C	64.33	1.79	64.31	64.50	95.12	1.009	46.56	1.20	46.55	45.61	94.58	1.009
NM30U	64.93	1.61	64.92	65.22	95.36	1.020	46.88	1.04	46.88	46.12	94.40	1.020
NM30C	65.20	2.14	65.18	65.21	95.10	1.020	46.89	1.38	46.87	46.10	94.38	1.020
NM40U	66.25	1.69	66.24	66.27	95.14	1.037	47.72	1.03	47.71	46.86	94.34	1.037
NM40C	65.92	1.84	65.90	66.14	95.32	1.035	47.70	1.24	47.69	46.76	94.40	1.034
NM50U	67.45	2.26	67.42	67.72	95.12	1.059	48.52	1.56	48.50	47.87	94.52	1.059
NM50C	66.58	1.52	66.57	67.30	95.42	1.053	48.59	1.61	48.57	47.63	94.56	1.054
NM60U	69.31	1.82	69.29	69.44	95.04	1.086	50.04	1.29	50.03	49.11	94.12	1.086
NM60C	67.84	1.44	67.84	68.71	95.20	1.075	49.30	0.91	49.29	48.58	94.30	1.075
NM70U	70.67	0.93	70.67	71.60	94.98	1.120	51.51	0.82	51.51	50.65	94.58	1.120
NM70C	70.36	2.24	70.33	70.35	94.86	1.100	50.79	0.75	50.79	49.69	94.64	1.099
NM80U	73.99	2.17	73.96	74.36	94.98	1.163	52.87	1.33	52.86	52.56	94.70	1.163
NM80C	72.42	2.20	72.40	72.10	95.06	1.128	51.78	1.00	51.77	50.94	94.56	1.127
NM90U	77.33	1.75	77.32	77.46	94.86	1.212	55.72	1.32	55.71	54.78	94.50	1.212
NM90C	74.01	2.03	73.99	73.98	95.36	1.157	53.21	1.82	53.19	52.31	94.38	1.157

Table E13: Inference for the variance $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	5464.06	990.04	5374.16	4724.94	82.26	1.000	3616.26	562.15	3572.65	3250.99	85.66	1.000
MI	2×10^{10}	3×10^8	2×10^{10}	2×10^{10}	97.82	3×10^6	22211.99	5685.50	21474.17	27444.25	96.88	8.442
NM10U	5484.67	994.02	5394.38	4743.14	82.32	1.004	3635.98	566.53	3591.93	3264.36	85.64	1.004
NM10C	5489.35	1000.55	5397.93	4747.80	82.10	1.005	3646.02	570.21	3601.51	3267.23	85.70	1.005
NM20U	5495.04	985.60	5406.47	4774.18	81.48	1.010	3685.57	584.45	3639.30	3307.95	85.80	1.018
NM20C	5648.27	1032.96	5553.57	4827.64	81.70	1.022	3704.54	589.24	3657.74	3310.79	85.78	1.018
NM30U	5723.72	1041.11	5628.80	4906.50	81.82	1.038	3836.28	633.01	3784.08	3396.00	85.58	1.045
NM30C	5765.71	1074.67	5665.23	4941.18	82.04	1.046	3744.20	602.28	3695.81	3366.48	85.86	1.036
NM40U	6015.49	1110.90	5912.61	5091.84	81.50	1.078	3851.57	638.69	3798.62	3473.16	85.42	1.068
NM40C	6062.50	1146.39	5953.73	5116.33	81.60	1.083	3843.97	616.21	3794.64	3443.65	85.32	1.059
NM50U	6268.31	1191.14	6154.71	5317.02	81.68	1.125	3994.12	638.87	3943.09	3570.82	84.76	1.098
NM50C	6301.05	1180.16	6190.16	5274.80	80.80	1.116	3929.81	662.51	3873.95	3560.72	85.60	1.095
NM60U	7353.51	1398.10	7220.10	5794.84	80.92	1.226	4282.00	747.65	4216.64	3797.12	84.86	1.168
NM60C	6682.96	1302.50	6555.46	5555.36	81.52	1.176	4253.76	751.44	4187.28	3736.56	85.46	1.149
NM70U	7634.49	1435.35	7499.10	6088.80	80.18	1.289	4607.57	800.36	4537.98	4018.59	84.62	1.236
NM70C	7082.98	1411.92	6941.52	5855.78	80.70	1.239	4641.12	840.43	4564.84	3928.95	84.92	1.209
NM80U	8395.05	1522.42	8256.68	6565.23	79.70	1.389	5146.55	884.66	5070.46	4341.52	83.48	1.335
NM80C	7470.51	1415.93	7335.83	6046.78	81.14	1.280	4956.78	869.69	4880.38	4086.03	84.62	1.257
NM90U	9161.28	1830.45	8977.45	7467.59	77.96	1.580	5660.24	1041.28	5564.19	4826.78	82.72	1.485
NM90C	8440.41	1707.33	8266.75	6610.96	81.54	1.399	5105.79	917.40	5023.20	4264.82	83.60	1.312

Table E14: Inference for the variance $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	2159.09	254.04	2144.30	2101.53	90.12	1.000	1487.10	157.22	1478.91	1442.37	91.88	1.000
MI	3536.33	1627.36	3139.95	4214.81	96.94	2.006	1927.71	739.45	1780.42	1976.81	96.64	1.371
NM10U	2166.99	258.30	2151.75	2109.92	89.88	1.004	1492.85	157.85	1484.63	1446.91	92.12	1.003
NM10C	2167.72	255.71	2152.80	2108.46	90.08	1.003	1494.89	160.03	1486.45	1447.67	92.12	1.004
NM20U	2196.00	267.78	2179.83	2133.84	90.14	1.015	1509.26	157.86	1501.13	1459.85	91.84	1.012
NM20C	2176.31	265.75	2160.24	2131.94	90.30	1.014	1500.24	161.57	1491.67	1460.83	91.94	1.013
NM30U	2240.86	260.87	2225.85	2163.70	89.66	1.030	1543.65	167.66	1534.68	1485.80	92.18	1.030
NM30C	2232.94	264.55	2217.44	2162.65	89.62	1.029	1535.95	170.39	1526.62	1484.99	91.96	1.030
NM40U	2305.33	284.43	2287.94	2222.59	89.84	1.058	1574.44	174.99	1564.84	1520.34	91.98	1.054
NM40C	2277.00	275.55	2260.49	2210.26	90.08	1.052	1559.49	180.13	1549.21	1517.48	92.50	1.052
NM50U	2389.94	312.17	2369.70	2301.67	88.96	1.095	1604.96	172.15	1595.86	1562.36	91.56	1.083
NM50C	2370.68	289.06	2353.23	2271.32	89.16	1.081	1645.86	196.24	1634.28	1559.90	91.76	1.081
NM60U	2507.79	346.20	2484.03	2405.16	89.24	1.144	1697.23	185.75	1687.20	1624.82	91.38	1.126
NM60C	2482.45	338.15	2459.56	2362.17	89.58	1.124	1640.05	177.89	1630.54	1595.23	92.00	1.106
NM70U	2659.77	361.82	2635.31	2526.98	87.64	1.202	1784.13	224.40	1770.14	1713.76	91.44	1.188
NM70C	2621.48	359.44	2596.98	2447.88	88.60	1.165	1715.66	193.63	1704.87	1650.10	91.62	1.144
NM80U	2905.02	419.25	2874.89	2707.91	87.70	1.289	1881.03	219.67	1868.34	1808.42	91.30	1.254
NM80C	2644.62	371.44	2618.67	2533.97	88.44	1.206	1777.77	191.51	1767.61	1702.87	91.34	1.181
NM90U	3159.05	476.65	3123.19	2943.67	87.28	1.401	2062.15	281.54	2043.05	1967.90	90.22	1.364
NM90C	2735.17	424.22	2702.34	2651.43	88.78	1.262	1826.14	214.60	1813.67	1771.42	91.18	1.228

Table E15: Inference for the variance $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	624.78	34.30	623.90	621.18	94.60	1.000	448.30	23.69	447.72	437.80	94.38	1.000
MI	669.08	134.94	655.40	668.20	95.92	1.076	471.55	73.37	465.85	458.41	94.82	1.047
NM10U	626.85	34.78	625.95	623.09	94.58	1.003	449.83	24.11	449.23	439.14	94.46	1.003
NM10C	627.00	35.76	626.04	623.25	94.60	1.003	449.10	23.90	448.51	439.11	94.20	1.003
NM20U	629.93	35.50	628.99	628.79	94.64	1.012	450.92	23.71	450.35	443.02	94.62	1.012
NM20C	631.71	35.47	630.78	628.63	94.44	1.012	454.11	24.54	453.49	443.03	94.64	1.012
NM30U	639.38	33.49	638.57	637.90	94.38	1.027	457.83	22.83	457.31	449.59	94.62	1.027
NM30C	643.47	39.21	642.34	638.18	94.56	1.027	457.76	26.08	457.06	449.51	94.60	1.027
NM40U	658.36	36.42	657.42	652.12	94.36	1.050	468.94	23.42	468.40	459.32	93.94	1.049
NM40C	654.20	37.16	653.21	650.17	94.76	1.047	468.27	24.33	467.69	457.97	94.30	1.046
NM50U	678.57	46.95	677.01	672.48	94.58	1.083	481.17	30.13	480.27	473.15	94.38	1.081
NM50C	662.79	34.43	661.96	665.15	94.80	1.071	480.98	31.74	479.98	469.74	94.50	1.073
NM60U	704.48	40.71	703.38	695.62	94.26	1.120	501.26	28.43	500.50	490.02	94.46	1.119
NM60C	682.73	37.28	681.78	683.97	94.52	1.101	491.66	23.57	491.14	481.64	94.14	1.100
NM70U	726.78	34.95	726.01	725.96	93.96	1.169	519.30	25.01	518.75	511.64	94.58	1.169
NM70C	711.48	48.50	709.90	706.76	95.12	1.138	507.76	23.97	507.25	496.39	94.16	1.134
NM80U	773.61	50.04	772.07	769.36	94.48	1.239	550.33	29.01	549.62	540.72	94.30	1.235
NM80C	741.41	51.57	739.69	730.52	94.62	1.176	520.18	26.95	519.54	513.00	94.66	1.172
NM90U	827.03	53.58	825.37	823.99	93.86	1.326	590.69	36.25	589.63	579.55	94.06	1.324
NM90C	766.01	51.88	764.33	755.69	94.22	1.217	545.69	35.58	544.59	531.74	94.32	1.215

Table E16: Inference for the 84th percentile $e^{\mu+\sigma}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	614.81	2.66	614.87	604.37	90.88	1.000	486.00	3.89	486.04	469.27	92.24	1.000
MI	747.06	214.82	715.58	801.30	95.42	1.326	545.81	124.34	531.51	553.13	95.34	1.179
NM10U	615.89	2.22	615.95	605.69	90.80	1.002	487.47	3.63	487.51	470.28	92.08	1.002
NM10C	617.05	3.14	617.10	606.00	90.90	1.003	487.74	4.17	487.77	470.39	92.20	1.002
NM20U	618.50	0.00	618.56	608.95	90.98	1.008	489.57	4.31	489.60	473.69	92.08	1.009
NM20C	624.09	2.25	624.15	609.76	90.68	1.009	490.65	4.70	490.67	473.81	92.04	1.010
NM30U	627.19	2.58	627.25	616.69	90.62	1.020	497.85	7.43	497.84	479.94	92.22	1.023
NM30C	624.25	4.86	624.30	617.26	91.02	1.021	495.70	5.36	495.72	478.97	92.34	1.021
NM40U	635.66	2.44	635.72	627.02	90.94	1.037	499.66	6.23	499.68	487.33	92.76	1.038
NM40C	636.71	3.69	636.77	626.69	90.80	1.037	502.63	3.22	502.67	485.45	92.14	1.034
NM50U	656.08	3.06	656.14	641.05	90.48	1.061	516.10	0.87	516.15	496.19	91.68	1.057
NM50C	655.16	-0.26	655.23	636.45	90.80	1.053	507.88	6.44	507.89	495.32	92.54	1.056
NM60U	673.70	1.10	673.77	658.00	89.98	1.089	528.67	5.03	528.70	510.81	91.86	1.089
NM60C	662.17	6.09	662.21	652.66	91.00	1.080	525.11	6.61	525.12	506.41	92.22	1.079
NM70U	697.77	-3.54	697.83	676.24	90.04	1.119	544.05	1.90	544.10	526.66	91.64	1.122
NM70C	687.86	-0.04	687.92	666.63	90.02	1.103	540.65	9.29	540.62	519.02	91.94	1.106
NM80U	713.70	-12.34	713.67	697.44	89.42	1.154	568.36	-5.53	568.39	543.64	91.18	1.158
NM80C	695.16	-2.11	695.23	681.91	90.00	1.128	554.49	4.42	554.52	530.83	91.72	1.131
NM90U	755.25	-10.35	755.25	726.01	88.42	1.201	581.85	-4.40	581.89	566.27	91.18	1.207
NM90C	722.20	9.53	722.21	705.48	91.06	1.167	568.22	5.07	568.26	544.83	91.68	1.161

Table E17: Inference for the 84th percentile $e^{\mu+\sigma}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	335.86	3.11	335.88	332.47	93.74	1.000	233.51	4.27	233.49	235.72	94.94	1.000
MI	358.76	59.64	353.81	362.73	95.20	1.091	244.77	33.12	242.55	248.59	95.48	1.055
NM10U	336.43	3.53	336.45	333.30	93.70	1.002	233.93	4.13	233.92	236.24	95.00	1.002
NM10C	336.57	3.43	336.59	333.23	93.62	1.002	234.38	4.43	234.36	236.28	94.66	1.002
NM20U	339.74	4.00	339.75	335.65	93.64	1.010	236.11	3.74	236.11	237.77	94.94	1.009
NM20C	338.43	4.15	338.43	335.62	93.82	1.009	235.26	4.34	235.24	237.83	94.68	1.009
NM30U	342.17	1.00	342.20	338.92	93.28	1.019	238.98	3.80	238.98	240.52	94.56	1.020
NM30C	340.92	2.53	340.94	338.93	93.64	1.019	238.56	4.93	238.53	240.46	94.84	1.020
NM40U	349.51	2.80	349.53	344.69	93.48	1.037	242.67	4.31	242.66	244.43	94.88	1.037
NM40C	345.87	2.30	345.90	343.90	93.66	1.034	240.63	4.72	240.61	244.05	94.90	1.035
NM50U	355.82	4.34	355.83	352.28	93.64	1.060	245.99	3.13	246.00	249.37	94.58	1.058
NM50C	353.19	1.21	353.22	349.92	93.66	1.052	248.68	5.56	248.64	248.50	94.68	1.054
NM60U	363.64	5.95	363.62	361.81	93.66	1.088	255.29	2.51	255.30	255.84	94.68	1.085
NM60C	361.43	4.01	361.44	357.98	93.68	1.077	251.49	2.18	251.50	253.20	94.74	1.074
NM70U	378.23	-1.42	378.27	372.36	93.56	1.120	265.64	5.92	265.60	264.45	94.62	1.122
NM70C	373.74	2.61	373.77	365.94	93.46	1.101	257.52	3.04	257.53	259.15	94.62	1.099
NM80U	395.49	0.04	395.53	386.36	92.86	1.162	274.67	1.44	274.70	273.74	94.34	1.161
NM80C	377.69	1.96	377.72	375.10	93.30	1.128	264.31	2.45	264.33	265.40	94.40	1.126
NM90U	409.12	-1.01	409.16	402.08	93.26	1.209	284.98	1.90	285.00	285.40	94.36	1.211
NM90C	386.25	5.39	386.25	385.91	93.54	1.161	269.29	2.80	269.30	272.55	94.46	1.156

Table E18: Inference for the 84th percentile $e^{\mu+\sigma}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	104.98	2.09	104.97	105.35	95.26	1.000	75.98	1.63	75.97	74.51	94.54	1.000
MI	107.32	7.51	107.07	108.14	95.50	1.026	77.86	4.44	77.75	76.23	94.40	1.023
NM10U	105.27	2.15	105.25	105.59	95.26	1.002	76.18	1.70	76.17	74.67	94.38	1.002
NM10C	105.22	2.25	105.21	105.60	95.28	1.002	75.99	1.69	75.98	74.67	94.52	1.002
NM20U	105.80	2.04	105.79	106.30	95.12	1.009	76.35	1.57	76.34	75.17	94.78	1.009
NM20C	105.99	2.24	105.98	106.29	95.08	1.009	76.73	1.62	76.73	75.16	94.64	1.009
NM30U	107.00	1.92	107.00	107.48	95.38	1.020	77.28	1.34	77.27	76.01	94.36	1.020
NM30C	107.43	2.80	107.40	107.46	95.08	1.020	77.28	1.90	77.27	75.98	94.42	1.020
NM40U	109.16	2.00	109.16	109.21	95.06	1.037	78.65	1.31	78.65	77.23	94.32	1.037
NM40C	108.60	2.28	108.58	108.99	95.28	1.035	78.62	1.66	78.61	77.07	94.44	1.034
NM50U	111.10	2.89	111.08	111.58	95.08	1.059	79.95	2.16	79.92	78.89	94.50	1.059
NM50C	109.71	1.70	109.71	110.91	95.40	1.053	80.06	2.25	80.04	78.49	94.52	1.053
NM60U	114.19	2.09	114.19	114.43	95.06	1.086	82.47	1.67	82.46	80.93	94.06	1.086
NM60C	111.78	1.52	111.78	113.22	95.16	1.075	81.26	1.06	81.26	80.06	94.26	1.075
NM70U	116.47	0.53	116.48	117.98	94.98	1.120	84.90	0.86	84.91	83.47	94.54	1.120
NM70C	115.88	2.79	115.85	115.92	94.88	1.100	83.71	0.77	83.71	81.90	94.62	1.099
NM80U	121.86	2.41	121.84	122.51	95.00	1.163	87.13	1.60	87.13	86.62	94.70	1.163
NM80C	119.28	2.63	119.27	118.80	95.00	1.128	85.34	1.17	85.34	83.96	94.56	1.127
NM90U	127.44	1.48	127.44	127.61	94.86	1.211	91.82	1.46	91.81	90.26	94.44	1.212
NM90C	121.90	2.29	121.89	121.89	95.34	1.157	87.69	2.45	87.67	86.21	94.38	1.157

Table E19: Inference for the 95th percentile $e^{\mu+1.645\sigma}$ based on fully masked $LN(\mu=0,\sigma^2=1)$ data

	Results for $n=30$						Results for $n=50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	1490.76	1.20	1490.91	1457.64	89.12	1.000	1164.61	2.54	1164.73	1126.52	91.12	1.000
MI	2015.39	691.68	1893.17	2268.36	95.54	1.556	1386.66	380.01	1333.71	1440.48	95.24	1.279
NM10U	1493.45	0.56	1493.60	1461.45	89.12	1.003	1169.03	2.06	1169.15	1129.44	90.98	1.003
NM10C	1495.99	2.94	1496.14	1462.36	88.88	1.003	1169.54	3.28	1169.65	1129.76	91.06	1.003
NM20U	1500.72	-5.78	1500.86	1470.58	89.04	1.009	1177.00	4.13	1177.11	1139.26	90.98	1.011
NM20C	1517.95	0.97	1518.10	1473.46	88.90	1.011	1178.28	5.63	1178.39	1139.64	91.22	1.012
NM30U	1524.25	-0.01	1524.41	1492.78	88.98	1.024	1200.65	12.23	1200.70	1157.25	90.94	1.027
NM30C	1523.79	7.35	1523.92	1495.05	89.16	1.026	1191.17	6.15	1191.27	1154.18	91.10	1.025
NM40U	1554.80	1.65	1554.95	1523.30	88.90	1.045	1207.94	8.81	1208.03	1178.38	91.12	1.046
NM40C	1560.18	8.45	1560.31	1523.11	88.78	1.045	1212.95	0.79	1213.07	1172.74	90.56	1.041
NM50U	1603.22	5.74	1603.37	1564.65	88.54	1.073	1246.62	-5.08	1246.74	1203.75	90.58	1.069
NM50C	1604.96	-3.24	1605.12	1550.96	88.32	1.064	1225.10	10.40	1225.18	1201.25	91.18	1.066
NM60U	1673.14	5.98	1673.30	1617.31	88.44	1.110	1288.74	8.57	1288.84	1247.36	90.48	1.107
NM60C	1631.97	15.78	1632.05	1598.46	88.78	1.097	1277.33	15.82	1277.36	1234.09	90.90	1.095
NM70U	1733.67	-12.55	1733.80	1670.96	88.12	1.146	1328.07	3.66	1328.20	1295.26	90.26	1.150
NM70C	1704.64	5.11	1704.80	1640.41	88.70	1.125	1324.13	23.39	1324.06	1270.89	90.62	1.128
NM80U	1784.15	-37.01	1783.94	1738.89	87.16	1.193	1407.13	-17.51	1407.16	1348.29	89.90	1.197
NM80C	1720.47	-6.53	1720.63	1683.01	88.32	1.155	1362.23	11.69	1362.32	1305.15	90.80	1.159
NM90U	1901.64	-30.66	1901.59	1838.02	86.14	1.261	1458.75	-12.93	1458.83	1423.93	89.62	1.264
NM90C	1806.23	29.62	1806.17	1754.07	88.92	1.203	1401.73	10.85	1401.83	1345.38	90.52	1.194

Table E20: Inference for the 95th percentile $e^{\mu+1.645\sigma}$ based on fully masked $LN(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	795.67	2.81	795.75	795.66	93.28	1.000	563.07	10.15	563.03	563.89	94.16	1.000
MI	873.60	177.22	855.52	900.05	95.78	1.131	598.97	97.82	590.99	604.98	95.56	1.073
NM10U	797.46	3.90	797.53	798.01	93.44	1.003	564.33	9.97	564.30	565.37	94.16	1.003
NM10C	797.74	3.16	797.81	797.76	93.24	1.003	565.35	10.72	565.30	565.47	94.16	1.003
NM20U	805.92	5.20	805.98	804.69	93.34	1.011	570.56	8.83	570.55	569.68	94.20	1.010
NM20C	802.57	5.48	802.63	804.60	93.46	1.011	567.82	10.64	567.78	569.89	94.18	1.011
NM30U	816.30	-2.14	816.38	814.02	93.16	1.023	579.63	9.87	579.61	577.58	94.14	1.024
NM30C	811.74	1.26	811.82	813.98	93.24	1.023	577.78	12.01	577.71	577.34	93.96	1.024
NM40U	835.34	2.15	835.42	830.50	93.32	1.044	588.54	10.93	588.49	588.69	94.26	1.044
NM40C	825.83	1.19	825.91	828.16	93.44	1.041	582.97	13.24	582.88	587.68	94.26	1.042
NM50U	853.26	6.24	853.32	852.33	93.02	1.071	599.17	7.38	599.18	602.78	93.94	1.069
NM50C	847.22	-1.17	847.30	845.41	93.20	1.063	606.45	14.85	606.32	600.34	94.44	1.065
NM60U	879.22	9.88	879.26	880.05	93.34	1.106	625.60	6.07	625.64	621.56	94.00	1.102
NM60C	870.48	8.23	870.52	868.82	92.86	1.092	612.51	6.18	612.54	613.71	94.10	1.088
NM70U	921.83	-3.56	921.92	911.93	92.58	1.146	652.70	14.90	652.60	646.95	94.24	1.147
NM70C	908.20	4.01	908.28	891.70	92.84	1.121	632.09	7.92	632.10	630.73	93.78	1.119
NM80U	972.78	-1.46	972.88	954.64	92.08	1.200	678.56	4.37	678.61	675.19	93.74	1.197
NM80C	919.13	3.52	919.21	918.13	92.88	1.154	650.41	4.24	650.46	648.48	93.62	1.150
NM90U	1018.37	-4.36	1018.47	1006.19	92.50	1.265	718.55	9.30	718.56	713.22	93.40	1.265
NM90C	941.64	14.97	941.62	949.73	92.94	1.194	666.30	7.18	666.33	669.32	93.98	1.187

Table E21: Inference for the 95th percentile $e^{\mu+1.645\sigma}$ based on fully masked $LN(\mu=0,\sigma^2=1)$ data

	Results for $n=1000$						Results for $n=2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	251.23	4.31	251.22	251.59	95.18	1.000	181.90	4.18	181.87	177.93	94.32	1.000
MI	257.99	20.96	257.16	259.12	95.44	1.030	186.94	12.61	186.53	182.35	94.70	1.025
NM10U	252.04	4.44	252.03	252.26	95.18	1.003	182.43	4.33	182.40	178.40	94.32	1.003
NM10C	251.93	4.77	251.91	252.29	95.14	1.003	182.04	4.28	182.01	178.39	94.24	1.003
NM20U	253.16	4.42	253.15	254.28	95.10	1.011	182.81	4.09	182.78	179.81	94.54	1.011
NM20C	253.73	4.60	253.71	254.23	94.98	1.010	184.03	4.29	184.00	179.79	94.36	1.010
NM30U	256.67	3.64	256.67	257.60	95.26	1.024	185.48	3.53	185.47	182.18	94.68	1.024
NM30C	257.89	5.87	257.85	257.54	94.80	1.024	185.40	4.90	185.35	182.10	94.30	1.023
NM40U	263.05	4.03	263.04	262.53	94.86	1.043	189.45	3.47	189.44	185.64	94.32	1.043
NM40C	261.49	4.62	261.48	261.89	95.14	1.041	189.25	4.08	189.23	185.18	94.52	1.041
NM50U	269.08	7.03	269.01	269.32	95.20	1.070	193.14	5.78	193.07	190.38	94.44	1.070
NM50C	264.57	3.13	264.58	267.33	95.20	1.063	193.29	6.25	193.20	189.23	94.48	1.064
NM60U	278.13	4.15	278.12	277.52	94.72	1.103	200.49	4.53	200.46	196.28	94.32	1.103
NM60C	271.02	3.17	271.03	273.93	95.26	1.089	197.27	2.88	197.27	193.69	94.16	1.089
NM70U	285.68	0.50	285.71	287.93	95.04	1.144	207.00	2.53	207.01	203.72	94.58	1.145
NM70C	281.23	6.62	281.18	281.66	95.30	1.120	203.38	2.41	203.39	198.96	94.52	1.118
NM80U	300.08	5.02	300.07	301.58	95.06	1.199	215.73	3.73	215.72	213.18	94.60	1.198
NM80C	291.57	6.48	291.52	289.94	95.12	1.152	208.02	3.34	208.01	204.85	94.62	1.151
NM90U	317.42	3.36	317.44	317.87	94.72	1.263	228.64	4.46	228.62	224.84	94.56	1.264
NM90C	299.76	5.62	299.74	298.84	94.94	1.188	216.01	6.32	215.94	211.33	94.36	1.188

Table E22: Inference for the mean μ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	181.90	0.46	181.91	177.41	93.56	1.000	139.82	4.71	139.76	139.02	94.48	1.000
MI	185.63	-0.02	185.65	187.70	94.82	1.058	142.57	4.94	142.50	144.76	95.12	1.041
NM10	182.27	0.71	182.28	177.79	93.74	1.002	140.04	4.81	139.97	139.29	94.36	1.002
NM20	183.18	0.22	183.20	178.89	93.72	1.008	140.74	4.63	140.68	140.02	94.44	1.007
NM30	184.20	0.63	184.22	180.16	93.82	1.015	141.70	4.73	141.63	141.07	94.62	1.015
NM40	185.47	1.24	185.48	181.95	94.00	1.026	142.37	4.84	142.30	142.45	94.88	1.025
NM50	187.62	0.18	187.64	183.96	93.96	1.037	144.45	4.51	144.39	143.80	94.48	1.034
NM60	187.32	1.53	187.34	185.50	93.94	1.046	145.46	5.26	145.38	145.26	94.82	1.045
NM70	191.59	1.06	191.60	188.30	94.24	1.061	147.70	3.32	147.68	147.07	95.08	1.058
NM80	193.67	0.49	193.69	190.42	94.02	1.073	148.09	3.88	148.06	148.63	94.72	1.069
NM90	194.42	-0.27	194.44	192.64	94.30	1.086	151.63	4.50	151.57	150.49	95.22	1.083

Table E23: Inference for the mean μ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	98.78	-0.57	98.78	99.22	94.86	1.000	69.77	-0.35	69.78	70.43	95.26	1.000
MI	101.12	-0.94	101.13	102.22	94.90	1.030	71.56	-0.26	71.57	72.20	95.22	1.025
NM10	98.87	-0.38	98.88	99.38	94.72	1.002	69.96	-0.33	69.97	70.55	95.26	1.002
NM20	99.15	-0.32	99.16	99.91	95.12	1.007	70.32	-0.30	70.33	70.90	95.42	1.007
NM30	100.33	-0.49	100.34	100.60	94.86	1.014	70.88	-0.36	70.89	71.42	95.40	1.014
NM40	101.08	-0.40	101.09	101.48	94.80	1.023	71.24	-0.20	71.25	71.98	95.26	1.022
NM50	101.94	-0.74	101.95	102.43	95.00	1.032	72.03	-0.32	72.04	72.68	95.30	1.032
NM60	103.17	-0.62	103.18	103.68	94.74	1.045	72.55	-0.13	72.56	73.48	95.42	1.043
NM70	104.45	-1.64	104.44	104.90	95.02	1.057	74.09	-0.21	74.10	74.40	94.98	1.056
NM80	105.86	-0.70	105.87	106.10	95.00	1.069	74.11	0.05	74.12	75.20	95.32	1.068
NM90	106.67	-1.21	106.67	107.35	95.26	1.082	75.17	-0.48	75.18	76.21	95.00	1.082

Table E24: Inference for the mean μ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.44	0.55	31.44	31.59	94.78	1.000	21.67	-0.20	21.67	22.35	95.58	1.000
MI	32.14	0.65	32.13	32.25	94.92	1.021	22.11	-0.14	22.11	22.81	95.38	1.020
NM10	31.49	0.52	31.49	31.65	94.80	1.002	21.68	-0.20	21.68	22.39	95.56	1.002
NM20	31.62	0.51	31.62	31.79	94.94	1.006	21.80	-0.19	21.80	22.49	95.46	1.006
NM30	31.83	0.58	31.83	32.01	94.90	1.013	21.96	-0.21	21.96	22.64	95.70	1.013
NM40	32.25	0.51	32.25	32.29	94.62	1.022	22.10	-0.29	22.10	22.84	95.90	1.022
NM50	32.47	0.58	32.47	32.60	94.62	1.032	22.27	-0.19	22.27	23.05	95.40	1.031
NM60	32.83	0.36	32.83	32.93	94.64	1.042	22.66	-0.23	22.66	23.31	95.58	1.043
NM70	33.27	0.41	33.27	33.31	94.62	1.054	22.84	-0.11	22.84	23.57	95.66	1.054
NM80	33.57	0.46	33.57	33.72	95.02	1.067	23.22	-0.14	23.22	23.85	95.36	1.067
NM90	34.02	0.32	34.02	34.18	94.76	1.082	23.64	-0.24	23.64	24.16	95.16	1.081

Table E25: Inference for the variance σ^2 based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	258.83	-38.93	255.91	248.15	88.88	1.000	194.38	-24.12	192.90	195.18	91.84	1.000
MI	284.65	33.87	282.66	291.93	93.72	1.176	207.06	17.15	206.37	216.01	94.90	1.107
NM10	261.82	-37.77	259.11	250.75	89.02	1.011	195.49	-23.53	194.08	197.14	91.94	1.010
NM20	269.30	-34.30	267.14	258.11	89.40	1.040	201.86	-21.91	200.69	202.54	91.72	1.038
NM30	278.33	-33.53	276.33	267.77	89.02	1.079	208.21	-20.45	207.23	210.23	91.76	1.077
NM40	287.98	-30.60	286.38	280.02	89.68	1.128	215.45	-17.92	214.72	219.61	91.96	1.125
NM50	302.08	-27.72	300.84	293.58	89.42	1.183	229.13	-17.83	228.46	229.41	92.02	1.175
NM60	315.28	-31.64	313.72	305.79	89.00	1.232	234.86	-18.84	234.13	239.54	92.14	1.227
NM70	336.49	-23.37	335.71	322.51	88.92	1.300	248.53	-16.26	248.02	251.01	90.96	1.286
NM80	342.53	-25.39	341.63	336.18	88.10	1.355	262.63	-18.33	262.02	261.57	90.64	1.340
NM90	351.31	-28.46	350.19	349.56	88.48	1.409	270.38	-19.92	269.67	272.56	91.40	1.396

Table E26: Inference for the variance σ^2 based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	142.58	-10.45	142.21	139.94	92.82	1.000	101.19	-5.35	101.06	99.47	94.14	1.000
MI	149.20	9.79	148.89	148.61	94.38	1.062	104.86	5.12	104.74	103.52	94.56	1.041
NM10	144.19	-10.52	143.82	141.27	93.08	1.009	102.25	-5.30	102.13	100.42	93.86	1.010
NM20	148.19	-8.79	147.95	145.16	93.12	1.037	105.29	-4.32	105.22	103.11	93.90	1.037
NM30	153.13	-8.51	152.91	150.43	93.00	1.075	108.97	-3.32	108.94	106.92	94.00	1.075
NM40	159.45	-8.03	159.26	156.75	93.26	1.120	111.80	-4.63	111.72	111.21	93.98	1.118
NM50	166.69	-8.58	166.48	163.59	93.02	1.169	117.31	-4.40	117.24	116.12	94.30	1.167
NM60	175.68	-5.11	175.62	171.54	92.72	1.226	123.15	-3.59	123.11	121.41	93.52	1.221
NM70	181.25	-4.39	181.22	179.37	93.30	1.282	128.33	-1.19	128.34	127.11	94.32	1.278
NM80	189.05	-5.47	188.99	187.09	92.62	1.337	131.68	-3.81	131.64	132.37	94.36	1.331
NM90	196.32	-7.99	196.18	194.73	92.60	1.391	139.57	-3.26	139.54	138.25	93.60	1.390

Table E27: Inference for the variance σ^2 based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	44.56	-1.36	44.54	44.66	94.98	1.000	30.78	-0.72	30.78	31.60	95.72	1.000
MI	45.57	0.67	45.57	45.73	95.04	1.024	31.45	0.31	31.45	32.29	95.92	1.022
NM10	44.95	-1.25	44.94	45.10	94.72	1.010	31.07	-0.80	31.06	31.90	95.66	1.010
NM20	46.24	-1.33	46.22	46.25	95.08	1.036	32.06	-0.59	32.06	32.73	95.64	1.036
NM30	47.67	-1.32	47.66	47.91	95.14	1.073	33.28	-0.83	33.28	33.89	95.58	1.073
NM40	49.71	-0.99	49.71	49.90	95.12	1.117	34.92	-0.55	34.92	35.30	95.72	1.117
NM50	51.74	-0.98	51.74	52.09	94.96	1.166	36.30	-0.99	36.29	36.83	94.72	1.165
NM60	54.06	-1.67	54.04	54.37	94.58	1.217	37.42	-0.22	37.42	38.50	95.74	1.218
NM70	56.55	-1.53	56.53	56.79	94.90	1.272	39.32	-0.64	39.31	40.19	95.28	1.272
NM80	59.01	-0.98	59.00	59.33	95.08	1.328	41.45	-0.77	41.45	41.95	94.86	1.328
NM90	61.42	0.23	61.43	61.99	95.00	1.388	42.73	-0.63	42.73	43.79	95.38	1.386

Table E28: Inference for the 84th percentile $\mu + \sigma$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	223.68	-27.81	221.96	217.29	92.56	1.000	171.38	-12.26	170.96	170.27	94.30	1.000
MI	227.99	-10.12	227.79	229.90	93.84	1.058	175.17	-1.86	175.18	177.30	94.96	1.041
NM10	224.98	-27.14	223.36	218.30	92.72	1.005	172.05	-11.92	171.65	171.04	94.32	1.005
NM20	227.38	-26.34	225.88	221.19	92.82	1.018	174.35	-11.59	173.98	173.17	94.26	1.017
NM30	231.59	-26.14	230.14	225.02	92.42	1.036	177.59	-11.09	177.26	176.26	94.26	1.035
NM40	236.06	-24.73	234.79	230.01	92.70	1.059	179.07	-10.08	178.81	180.12	94.52	1.058
NM50	238.18	-25.28	236.86	235.50	92.96	1.084	187.43	-11.08	187.12	184.17	93.96	1.082
NM60	245.40	-26.95	243.94	240.70	92.86	1.108	187.93	-11.19	187.62	188.55	94.44	1.107
NM70	254.27	-24.78	253.09	247.65	92.36	1.140	195.12	-12.70	194.73	193.38	93.72	1.136
NM80	260.95	-27.02	259.57	253.66	92.66	1.167	199.83	-14.02	199.36	198.06	93.40	1.163
NM90	260.37	-30.12	258.65	259.79	92.40	1.196	204.79	-14.76	204.28	203.15	93.42	1.193

Table E29: Inference for the 84th percentile $\mu + \sigma$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	123.33	-8.36	123.06	121.52	94.00	1.000	86.72	-4.31	86.62	86.26	94.56	1.000
MI	126.29	-3.80	126.25	125.20	94.22	1.030	88.95	-1.53	88.94	88.43	94.44	1.025
NM10	124.01	-8.27	123.74	122.03	93.86	1.004	87.17	-4.30	87.08	86.63	94.58	1.004
NM20	124.96	-7.48	124.75	123.56	94.26	1.017	88.02	-3.85	87.94	87.69	94.50	1.017
NM30	127.64	-7.69	127.42	125.67	94.12	1.034	89.83	-3.51	89.77	89.22	94.58	1.034
NM40	129.98	-7.60	129.77	128.25	94.02	1.055	90.79	-4.08	90.71	90.98	94.66	1.055
NM50	133.86	-8.50	133.60	131.10	93.94	1.079	93.21	-4.24	93.12	93.04	94.44	1.079
NM60	137.73	-7.02	137.57	134.46	93.98	1.107	95.19	-3.83	95.12	95.30	94.24	1.105
NM70	139.68	-7.93	139.47	137.88	94.04	1.135	98.10	-2.85	98.07	97.78	94.56	1.134
NM80	143.81	-7.87	143.61	141.33	93.94	1.163	98.94	-4.02	98.87	100.16	94.90	1.161
NM90	146.10	-10.03	145.78	144.82	94.06	1.192	102.43	-4.55	102.34	102.84	94.60	1.192

Table E30: Inference for the 84th percentile $\mu + \sigma$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	38.35	-0.38	38.36	38.69	95.02	1.000	26.65	-0.68	26.65	27.37	95.60	1.000
MI	39.24	0.23	39.24	39.50	95.08	1.021	27.13	-0.35	27.13	27.93	95.82	1.020
NM10	38.53	-0.35	38.53	38.86	95.20	1.004	26.73	-0.72	26.72	27.49	95.54	1.004
NM20	38.95	-0.42	38.95	39.32	95.06	1.016	27.10	-0.62	27.10	27.82	95.68	1.016
NM30	39.80	-0.37	39.81	39.99	94.82	1.033	27.69	-0.77	27.68	28.28	95.44	1.033
NM40	40.55	-0.30	40.56	40.81	94.70	1.055	28.12	-0.72	28.11	28.86	95.84	1.054
NM50	41.26	-0.24	41.26	41.73	95.06	1.078	28.71	-0.85	28.70	29.51	95.78	1.078
NM60	42.33	-0.84	42.32	42.71	95.06	1.104	29.41	-0.51	29.41	30.23	95.46	1.104
NM70	43.28	-0.76	43.28	43.78	95.36	1.131	30.21	-0.63	30.20	30.97	95.66	1.131
NM80	44.72	-0.47	44.73	44.91	95.02	1.161	31.13	-0.74	31.13	31.76	95.14	1.160
NM90	45.25	-0.04	45.26	46.13	95.52	1.192	31.77	-0.78	31.77	32.61	95.18	1.191

Table E31: Inference for the 95th percentile $\mu + 1.645\sigma$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	282.20	-46.04	278.45	272.14	92.20	1.000	214.79	-23.21	213.55	213.25	94.24	1.000
MI	287.75	-16.64	287.30	287.96	93.60	1.058	219.67	-6.24	219.60	222.07	95.00	1.041
NM10	284.35	-45.10	280.78	273.92	91.98	1.007	215.83	-22.71	214.66	214.62	93.96	1.006
NM20	288.75	-43.47	285.48	278.92	92.18	1.025	220.02	-22.05	218.94	218.38	93.94	1.024
NM30	295.93	-43.40	292.76	285.74	91.98	1.050	225.38	-21.29	224.40	223.83	93.84	1.050
NM40	303.53	-41.48	300.71	294.40	92.02	1.082	228.73	-19.70	227.90	230.54	94.20	1.081
NM50	308.60	-41.71	305.80	303.94	92.58	1.117	241.86	-21.13	240.96	237.66	93.56	1.114
NM60	321.40	-45.32	318.22	313.18	92.16	1.151	243.67	-21.81	242.71	245.29	94.02	1.150
NM70	335.73	-41.44	333.19	324.76	91.70	1.193	255.54	-23.03	254.53	253.55	93.12	1.189
NM80	345.34	-44.77	342.46	335.11	91.78	1.231	264.74	-25.56	263.53	261.63	93.02	1.227
NM90	346.83	-49.38	343.33	345.56	92.18	1.270	271.85	-27.19	270.52	270.22	93.08	1.267

Table E32: Inference for the 95th percentile $\mu + 1.645\sigma$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	155.50	-13.39	154.93	152.20	93.80	1.000	109.38	-6.87	109.17	108.04	94.56	1.000
MI	159.29	-5.64	159.20	156.82	94.22	1.030	112.18	-2.36	112.17	110.76	94.64	1.025
NM10	156.73	-13.35	156.18	153.13	93.92	1.006	110.16	-6.86	109.96	108.71	94.42	1.006
NM20	158.76	-12.09	158.31	155.83	94.24	1.024	111.82	-6.14	111.66	110.59	94.62	1.024
NM30	163.00	-12.33	162.54	159.59	94.02	1.049	114.85	-5.54	114.72	113.30	94.28	1.049
NM40	167.20	-12.24	166.77	164.14	93.76	1.078	116.70	-6.59	116.53	116.45	94.62	1.078
NM50	173.61	-13.51	173.10	169.17	93.48	1.111	120.87	-6.78	120.69	120.05	94.60	1.111
NM60	180.30	-11.15	179.97	174.91	93.56	1.149	124.69	-6.21	124.55	123.96	94.56	1.147
NM70	183.74	-12.00	183.37	180.77	93.34	1.188	128.97	-4.56	128.90	128.18	94.52	1.186
NM80	190.27	-12.50	189.88	186.68	93.66	1.227	130.98	-6.65	130.83	132.29	94.74	1.225
NM90	194.98	-15.71	194.36	192.64	93.96	1.266	137.10	-7.17	136.93	136.79	94.62	1.266

Table E33: Inference for the 95th percentile $\mu + 1.645\sigma$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	48.05	-0.98	48.05	48.46	95.00	1.000	33.43	-0.98	33.42	34.28	95.54	1.000
MI	49.15	-0.03	49.16	49.47	95.02	1.021	34.03	-0.49	34.03	34.98	95.72	1.020
NM10	48.36	-0.92	48.35	48.77	95.24	1.006	33.59	-1.05	33.58	34.50	95.48	1.006
NM20	49.17	-1.02	49.16	49.59	94.86	1.023	34.28	-0.89	34.27	35.08	95.56	1.023
NM30	50.56	-0.97	50.55	50.78	95.20	1.048	35.28	-1.12	35.26	35.92	95.26	1.048
NM40	51.87	-0.82	51.86	52.23	95.14	1.078	36.19	-1.00	36.18	36.94	95.70	1.078
NM50	53.21	-0.77	53.21	53.84	95.08	1.111	37.23	-1.28	37.21	38.07	95.70	1.111
NM60	55.03	-1.61	55.01	55.56	94.74	1.146	38.25	-0.70	38.25	39.32	95.38	1.147
NM70	56.72	-1.51	56.70	57.39	95.38	1.184	39.69	-0.96	39.68	40.60	95.74	1.184
NM80	59.06	-1.06	59.06	59.32	94.98	1.224	41.26	-1.13	41.25	41.95	95.00	1.224
NM90	60.19	-0.27	60.19	61.36	95.42	1.266	42.25	-1.13	42.24	43.37	95.76	1.265

Table E34: Inference for $\mu + \sigma^2/2$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 30$						Results for $n = 50$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	221.49	-19.01	220.70	216.90	92.66	1.000	170.31	-7.35	170.17	170.03	94.38	1.000
MI	231.71	16.91	231.12	238.29	94.62	1.099	176.83	13.51	176.33	180.82	95.50	1.063
NM10	222.92	-18.18	222.20	217.98	92.58	1.005	171.02	-6.95	170.89	170.83	94.44	1.005
NM20	225.68	-16.93	225.06	221.04	92.78	1.019	173.38	-6.32	173.28	173.02	94.40	1.018
NM30	229.87	-16.13	229.33	224.97	92.36	1.037	176.69	-5.50	176.63	176.18	94.44	1.036
NM40	234.40	-14.06	234.00	230.16	92.62	1.061	178.28	-4.12	178.25	180.16	94.58	1.060
NM50	236.99	-13.68	236.62	235.92	93.02	1.088	186.90	-4.40	186.86	184.30	93.88	1.084
NM60	244.35	-14.30	243.96	241.12	92.78	1.112	187.40	-4.16	187.38	188.69	94.60	1.110
NM70	253.99	-10.63	253.79	248.76	92.44	1.147	194.35	-4.81	194.31	193.74	93.78	1.139
NM80	260.01	-12.21	259.75	254.84	92.60	1.175	199.37	-5.28	199.32	198.46	93.56	1.167
NM90	259.20	-14.50	258.82	260.97	92.32	1.203	204.13	-5.46	204.08	203.56	93.38	1.197

Table E35: Inference for $\mu + \sigma^2/2$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	123.16	-5.79	123.03	121.48	94.00	1.000	86.60	-3.02	86.55	86.24	94.54	1.000
MI	126.95	3.96	126.90	126.45	94.50	1.041	89.15	2.30	89.13	88.86	94.80	1.030
NM10	123.85	-5.64	123.74	121.99	93.84	1.004	87.07	-2.98	87.03	86.62	94.60	1.004
NM20	124.88	-4.71	124.81	123.57	94.28	1.017	87.95	-2.46	87.93	87.69	94.56	1.017
NM30	127.53	-4.74	127.45	125.69	94.12	1.035	89.77	-2.02	89.76	89.24	94.52	1.035
NM40	129.97	-4.42	129.91	128.30	94.14	1.056	90.65	-2.51	90.62	90.99	94.60	1.055
NM50	133.90	-5.03	133.82	131.17	93.92	1.080	93.09	-2.52	93.07	93.06	94.44	1.079
NM60	137.79	-3.17	137.77	134.66	93.80	1.109	95.11	-1.93	95.10	95.35	94.30	1.106
NM70	139.71	-3.83	139.67	138.13	93.84	1.137	98.23	-0.81	98.24	97.89	94.50	1.135
NM80	144.12	-3.44	144.10	141.59	93.84	1.166	98.93	-1.85	98.92	100.23	94.92	1.162
NM90	146.08	-5.20	146.01	145.06	94.04	1.194	102.28	-2.11	102.27	102.94	94.72	1.194

Table E36: Inference for $\mu + \sigma^2/2$ based on fully masked $N(\mu = 0, \sigma^2 = 1)$ data

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	38.36	-0.13	38.36	38.69	95.06	1.000	26.64	-0.56	26.64	27.37	95.62	1.000
MI	39.28	0.98	39.27	39.53	95.14	1.022	27.13	0.01	27.13	27.94	95.82	1.021
NM10	38.53	-0.10	38.54	38.86	95.26	1.004	26.72	-0.60	26.71	27.49	95.50	1.004
NM20	38.96	-0.15	38.96	39.32	95.08	1.016	27.09	-0.49	27.09	27.82	95.64	1.016
NM30	39.80	-0.08	39.80	39.98	94.78	1.033	27.68	-0.63	27.67	28.28	95.42	1.033
NM40	40.56	0.01	40.56	40.81	94.74	1.055	28.11	-0.57	28.11	28.86	95.84	1.055
NM50	41.26	0.09	41.26	41.73	95.10	1.079	28.70	-0.69	28.70	29.51	95.78	1.078
NM60	42.32	-0.47	42.32	42.71	95.12	1.104	29.41	-0.34	29.41	30.23	95.46	1.104
NM70	43.26	-0.36	43.27	43.78	95.32	1.131	30.20	-0.43	30.20	30.97	95.64	1.132
NM80	44.73	-0.03	44.73	44.92	94.94	1.161	31.12	-0.52	31.12	31.76	95.16	1.160
NM90	45.26	0.44	45.26	46.15	95.44	1.193	31.76	-0.56	31.76	32.61	95.18	1.191

Table E37: Inference for the mean based on Exponential(mean = 1) data with $C = 2.996 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	99.45	-0.65	99.46	99.93	94.64	1.000	70.71	-1.16	70.71	70.63	94.86	1.000
MI.C90	104.36	-4.31	104.29	99.75	93.32	0.998	73.98	-3.12	73.92	70.59	93.48	0.999
MI.C80	108.75	0.95	108.76	100.50	92.78	1.006	76.28	-0.64	76.28	70.89	93.08	1.004
MI.D90	769.42	137.33	757.14	161.17	95.20	1.613	101.35	37.12	94.32	76.72	95.24	1.086
MI.D80	117.62	60.71	100.75	111.41	96.14	1.115	76.83	26.35	72.18	75.19	95.70	1.065
CENS	102.47	0.67	102.47	102.55	95.36	1.026	72.57	-0.22	72.57	72.52	95.04	1.027
NM10.1	99.54	-0.50	99.54	100.07	94.68	1.001	70.88	-1.06	70.87	70.72	94.62	1.001
NM10.2	99.56	-0.50	99.57	100.08	94.68	1.001	70.88	-1.08	70.87	70.73	94.64	1.001
NM20.1	99.94	-0.59	99.95	100.36	94.62	1.004	71.07	-1.00	71.07	70.94	94.72	1.004
NM20.2	99.99	-0.54	100.00	100.42	94.70	1.005	71.09	-1.06	71.09	70.98	94.66	1.005
NM30.1	100.12	-0.29	100.13	100.75	94.64	1.008	71.27	-0.85	71.27	71.20	94.80	1.008
NM30.2	100.31	-0.10	100.32	100.96	94.80	1.010	71.40	-0.92	71.40	71.33	94.76	1.010
NM40.1	100.41	-0.66	100.42	101.02	94.66	1.011	71.45	-0.72	71.45	71.44	94.82	1.011
NM40.2	100.67	-0.77	100.68	101.48	94.76	1.015	71.62	-0.78	71.62	71.77	94.88	1.016
NM50.1	100.90	0.12	100.91	101.36	94.82	1.014	71.61	-0.68	71.61	71.61	94.90	1.014
NM50.2	101.74	0.54	101.74	102.35	94.76	1.024	72.22	-0.80	72.22	72.27	94.58	1.023
NM60.1	100.92	-0.14	100.93	101.50	94.58	1.016	71.73	-0.74	71.73	71.72	95.10	1.015
NM60.2	102.24	0.22	102.25	103.19	94.66	1.033	72.60	-0.92	72.60	72.87	94.86	1.032
NM70.1	101.08	0.03	101.09	101.65	94.68	1.017	71.89	-0.46	71.89	71.83	94.68	1.017
NM70.2	103.82	0.72	103.83	104.32	94.68	1.044	73.55	-0.28	73.56	73.67	94.94	1.043
NM80.1	101.47	0.21	101.48	101.76	94.74	1.018	71.88	-0.47	71.89	71.90	95.10	1.018
NM80.2	105.49	1.02	105.50	105.64	94.80	1.057	74.64	-0.20	74.65	74.58	94.80	1.056
NM90.1	101.40	0.14	101.41	101.83	94.62	1.019	71.83	-0.55	71.84	71.94	94.88	1.019
NM90.2	107.09	1.21	107.09	107.17	94.68	1.072	75.43	-0.65	75.43	75.59	94.70	1.070

Table E38: Inference for the mean based on Exponential(mean = 1) data with $C = 2.996 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.95	-0.02	31.96	31.62	94.86	1.000	22.38	0.45	22.37	22.37	94.82	1.000
MI.C90	33.51	-0.34	33.51	31.65	93.70	1.001	23.44	0.35	23.44	22.39	93.84	1.001
MI.C80	34.59	0.41	34.59	31.72	92.84	1.003	24.18	0.78	24.17	22.43	93.00	1.003
MI.D90	33.30	6.72	32.62	32.58	95.04	1.030	23.25	3.79	22.94	22.95	94.70	1.026
MI.D80	33.23	5.29	32.81	32.70	94.88	1.034	23.14	3.07	22.94	23.05	94.88	1.030
CENS	32.87	0.05	32.87	32.44	94.54	1.026	22.96	0.63	22.96	22.95	95.04	1.026
NM10.1	32.04	-0.02	32.04	31.66	94.76	1.001	22.41	0.48	22.41	22.40	95.02	1.001
NM10.2	32.06	-0.01	32.06	31.66	94.70	1.001	22.41	0.48	22.41	22.40	94.96	1.001
NM20.1	32.07	-0.03	32.07	31.76	94.88	1.004	22.45	0.46	22.45	22.47	94.90	1.004
NM20.2	32.07	-0.02	32.07	31.77	94.80	1.005	22.46	0.44	22.45	22.48	94.82	1.005
NM30.1	32.25	-0.08	32.25	31.87	94.80	1.008	22.55	0.51	22.55	22.55	95.08	1.008
NM30.2	32.30	-0.05	32.30	31.93	94.96	1.010	22.59	0.54	22.59	22.59	94.98	1.010
NM40.1	32.30	0.04	32.30	31.97	94.92	1.011	22.64	0.55	22.64	22.62	94.90	1.011
NM40.2	32.38	0.08	32.39	32.12	94.90	1.016	22.77	0.55	22.76	22.72	94.84	1.016
NM50.1	32.38	-0.04	32.39	32.04	94.68	1.013	22.73	0.54	22.73	22.67	94.94	1.013
NM50.2	32.72	0.08	32.72	32.35	94.50	1.023	23.02	0.58	23.02	22.88	94.54	1.023
NM60.1	32.52	0.01	32.52	32.09	95.02	1.015	22.74	0.51	22.74	22.71	95.04	1.015
NM60.2	33.11	0.04	33.11	32.62	94.68	1.031	23.09	0.50	23.08	23.08	95.18	1.031
NM70.1	32.46	0.02	32.46	32.13	94.70	1.016	22.78	0.51	22.78	22.73	94.96	1.016
NM70.2	33.26	0.18	33.26	32.96	94.92	1.042	23.52	0.51	23.52	23.31	94.74	1.042
NM80.1	32.56	-0.05	32.57	32.16	94.48	1.017	22.74	0.61	22.74	22.75	94.96	1.017
NM80.2	33.69	0.20	33.69	33.36	94.62	1.055	23.58	0.73	23.57	23.60	95.12	1.055
NM90.1	32.63	-0.01	32.64	32.18	94.64	1.018	22.73	0.56	22.73	22.77	95.04	1.018
NM90.2	34.29	0.19	34.29	33.82	94.40	1.070	23.85	0.53	23.84	23.92	94.84	1.069

Table E39: Inference for the mean based on Exponential(mean = 1) data with $C = 2.303 =$ 90th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	98.35	0.62	98.35	100.06	94.66	1.000	70.32	-0.15	70.32	70.70	94.68	1.000
MI.C90	107.68	1.95	107.67	100.59	93.02	1.005	77.00	0.86	77.00	70.99	92.36	1.004
MI.C80	114.37	13.20	113.62	102.25	92.82	1.022	80.03	5.28	79.86	71.60	91.84	1.013
MI.D90	121.11	59.58	105.46	111.53	96.26	1.115	77.59	27.31	72.63	75.28	95.62	1.065
MI.D80	109.22	41.89	100.88	108.86	97.02	1.088	74.73	19.96	72.03	74.78	95.80	1.058
CENS	104.65	3.99	104.59	105.61	95.44	1.055	74.66	1.57	74.66	74.59	95.02	1.055
NM10.1	98.59	0.71	98.60	100.25	94.72	1.002	70.43	-0.05	70.44	70.84	94.76	1.002
NM10.2	98.59	0.69	98.60	100.26	94.70	1.002	70.44	-0.06	70.45	70.84	94.74	1.002
NM20.1	98.95	0.85	98.96	100.73	94.80	1.007	70.63	-0.10	70.63	71.16	94.70	1.007
NM20.2	98.93	0.78	98.93	100.78	94.86	1.007	70.70	-0.09	70.71	71.20	94.80	1.007
NM30.1	99.98	1.65	99.98	101.42	94.86	1.014	71.22	-0.02	71.22	71.60	94.64	1.013
NM30.2	100.15	1.63	100.14	101.61	94.70	1.015	71.37	0.01	71.38	71.74	94.78	1.015
NM40.1	100.51	1.43	100.51	102.01	94.94	1.019	71.63	0.29	71.64	72.05	94.72	1.019
NM40.2	101.00	1.41	101.00	102.50	95.00	1.024	71.94	0.32	71.95	72.40	94.90	1.024
NM50.1	100.68	1.75	100.68	102.56	95.16	1.025	72.04	0.45	72.04	72.42	94.72	1.024
NM50.2	101.57	1.63	101.57	103.60	95.02	1.035	72.86	0.44	72.87	73.17	94.76	1.035
NM60.1	101.52	3.00	101.48	103.10	95.02	1.030	72.85	0.67	72.85	72.72	94.34	1.029
NM60.2	103.56	3.12	103.53	105.04	95.24	1.050	74.58	0.72	74.58	74.08	94.38	1.048
NM70.1	101.88	2.48	101.86	103.35	94.88	1.033	72.90	0.78	72.90	72.93	94.74	1.032
NM70.2	105.65	3.26	105.61	106.63	95.00	1.066	75.26	0.92	75.26	75.20	94.78	1.064
NM80.1	102.00	2.56	101.98	103.58	95.06	1.035	73.10	0.96	73.10	73.10	94.58	1.034
NM80.2	106.88	2.67	106.85	108.46	94.96	1.084	76.48	1.30	76.48	76.56	94.78	1.083
NM90.1	102.19	2.73	102.16	103.79	94.96	1.037	73.27	0.99	73.27	73.23	94.40	1.036
NM90.2	109.69	4.74	109.60	111.02	95.10	1.109	78.28	0.06	78.29	78.06	94.50	1.104

Table E40: Inference for the mean based on Exponential(mean = 1) data with $C = 2.303 =$ 90th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.15	0.18	31.16	31.63	95.34	1.000	22.26	0.11	22.26	22.36	95.04	1.000
MI.C90	33.86	0.29	33.87	31.71	93.34	1.003	24.23	0.07	24.24	22.42	92.84	1.002
MI.C80	35.19	1.27	35.17	31.85	92.12	1.007	25.02	0.67	25.01	22.50	92.22	1.006
MI.D90	32.40	5.29	31.97	32.70	95.56	1.034	22.98	2.61	22.84	23.04	95.32	1.030
MI.D80	32.55	4.14	32.29	32.72	95.16	1.034	23.04	2.16	22.94	23.07	95.26	1.032
CENS	32.96	0.38	32.97	33.34	95.12	1.054	23.64	0.20	23.64	23.57	95.04	1.054
NM10.1	31.19	0.21	31.19	31.69	95.24	1.002	22.32	0.12	22.32	22.40	95.04	1.002
NM10.2	31.20	0.20	31.20	31.69	95.26	1.002	22.32	0.13	22.32	22.41	95.04	1.002
NM20.1	31.42	0.31	31.42	31.84	95.30	1.007	22.44	0.08	22.44	22.51	95.06	1.007
NM20.2	31.45	0.34	31.45	31.86	95.36	1.007	22.44	0.08	22.44	22.52	95.04	1.007
NM30.1	31.51	0.34	31.51	32.03	95.06	1.013	22.62	0.06	22.62	22.64	95.04	1.013
NM30.2	31.59	0.39	31.59	32.10	95.12	1.015	22.61	0.07	22.61	22.69	95.30	1.015
NM40.1	31.82	0.26	31.83	32.22	95.10	1.019	22.67	0.13	22.67	22.78	95.10	1.019
NM40.2	32.05	0.29	32.05	32.38	95.00	1.024	22.80	0.10	22.80	22.89	95.08	1.024
NM50.1	32.01	0.29	32.01	32.38	95.02	1.024	22.85	0.26	22.85	22.90	95.20	1.024
NM50.2	32.49	0.37	32.49	32.72	94.88	1.034	23.03	0.31	23.03	23.13	94.90	1.034
NM60.1	32.18	0.21	32.18	32.50	95.28	1.028	22.90	0.12	22.90	22.98	94.86	1.028
NM60.2	32.91	0.30	32.92	33.11	95.54	1.047	23.31	0.08	23.31	23.41	95.22	1.047
NM70.1	32.06	0.27	32.06	32.59	95.24	1.031	23.04	0.20	23.04	23.04	94.64	1.030
NM70.2	32.99	0.51	32.99	33.61	95.26	1.063	23.78	0.16	23.78	23.76	95.30	1.062
NM80.1	32.18	0.18	32.18	32.66	95.30	1.033	23.06	0.09	23.06	23.09	95.02	1.033
NM80.2	33.81	0.18	33.82	34.19	95.18	1.081	24.15	0.14	24.15	24.18	95.02	1.081
NM90.1	32.22	0.16	32.22	32.71	95.28	1.034	23.05	0.11	23.05	23.13	95.16	1.034
NM90.2	34.33	0.14	34.33	34.91	95.22	1.104	24.50	0.11	24.50	24.68	95.22	1.104

Table E41: Inference for the log scale mean μ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	97.91	-0.60	97.92	99.25	95.00	1.000	70.53	-0.68	70.53	70.41	95.18	1.000
MI.C90	98.99	-2.34	98.98	99.03	94.82	0.998	71.27	-1.85	71.25	70.30	94.96	0.998
MI.C80	100.33	-0.42	100.34	99.42	94.48	1.002	71.71	-0.62	71.71	70.49	94.78	1.001
MI.D90	98.80	-0.78	98.80	102.24	95.34	1.030	70.55	-0.64	70.55	70.60	95.24	1.003
MI.D80	98.10	-0.67	98.10	99.85	95.08	1.006	70.61	-0.68	70.61	70.63	95.18	1.003
CENS	98.59	0.11	98.60	99.80	95.04	1.006	70.89	-0.37	70.89	70.73	95.18	1.005
NM10.1	97.93	-0.58	97.94	99.26	94.90	1.000	70.54	-0.70	70.54	70.41	95.16	1.000
NM10.2	97.93	-0.58	97.93	99.26	94.92	1.000	70.54	-0.70	70.54	70.41	95.16	1.000
NM20.1	97.97	-0.57	97.98	99.28	94.96	1.000	70.54	-0.66	70.54	70.43	95.26	1.000
NM20.2	97.97	-0.58	97.98	99.29	94.94	1.000	70.55	-0.67	70.55	70.44	95.24	1.000
NM30.1	97.99	-0.49	98.00	99.32	95.06	1.001	70.56	-0.60	70.56	70.46	95.16	1.001
NM30.2	98.02	-0.50	98.03	99.34	95.04	1.001	70.58	-0.61	70.59	70.47	95.12	1.001
NM40.1	98.01	-0.44	98.02	99.36	94.88	1.001	70.59	-0.66	70.59	70.47	95.14	1.001
NM40.2	98.05	-0.46	98.06	99.40	95.04	1.002	70.60	-0.73	70.60	70.49	95.20	1.001
NM50.1	98.10	-0.39	98.11	99.40	94.96	1.002	70.64	-0.61	70.65	70.50	95.22	1.001
NM50.2	98.16	-0.38	98.17	99.49	95.08	1.002	70.72	-0.62	70.72	70.56	95.04	1.002
NM60.1	98.23	-0.41	98.23	99.43	94.94	1.002	70.65	-0.50	70.65	70.54	95.14	1.002
NM60.2	98.50	-0.45	98.51	99.60	94.96	1.004	70.76	-0.57	70.77	70.66	94.96	1.004
NM70.1	98.23	-0.30	98.24	99.48	95.00	1.002	70.69	-0.63	70.69	70.53	95.06	1.002
NM70.2	98.59	-0.46	98.60	99.78	94.86	1.005	70.91	-0.84	70.91	70.75	95.06	1.005
NM80.1	98.17	-0.37	98.18	99.48	94.94	1.002	70.78	-0.49	70.79	70.57	95.16	1.002
NM80.2	98.68	-0.52	98.69	100.05	94.92	1.008	71.17	-0.42	71.17	70.99	94.96	1.008
NM90.1	98.18	-0.25	98.19	99.53	95.04	1.003	70.69	-0.55	70.69	70.57	95.24	1.002
NM90.2	99.11	-0.34	99.12	100.59	94.98	1.013	71.14	-0.69	71.15	71.32	95.42	1.013

Table E42: Inference for the log scale mean μ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.59	-0.43	31.59	31.59	95.20	1.000	22.46	-0.07	22.46	22.35	94.64	1.000
MI.C90	31.81	-0.69	31.81	31.59	95.00	1.000	22.63	-0.15	22.63	22.36	94.48	1.000
MI.C80	32.22	-0.28	32.22	31.62	94.76	1.001	22.78	-0.07	22.78	22.37	94.50	1.001
MI.D90	31.60	-0.42	31.60	31.61	95.18	1.001	22.46	-0.06	22.46	22.37	94.64	1.001
MI.D80	31.61	-0.44	31.61	31.63	95.12	1.001	22.46	-0.05	22.46	22.38	94.64	1.001
CENS	31.69	-0.38	31.69	31.71	95.10	1.004	22.52	0.02	22.53	22.44	94.64	1.004
NM10.1	31.59	-0.42	31.59	31.59	95.22	1.000	22.46	-0.06	22.46	22.36	94.66	1.000
NM10.2	31.59	-0.42	31.59	31.59	95.20	1.000	22.46	-0.06	22.46	22.36	94.66	1.000
NM20.1	31.60	-0.42	31.60	31.60	95.34	1.000	22.47	-0.06	22.47	22.36	94.54	1.000
NM20.2	31.60	-0.42	31.60	31.60	95.26	1.000	22.47	-0.06	22.48	22.36	94.54	1.000
NM30.1	31.59	-0.42	31.59	31.61	95.22	1.001	22.47	-0.06	22.47	22.37	94.70	1.001
NM30.2	31.61	-0.42	31.61	31.61	95.22	1.001	22.48	-0.06	22.49	22.37	94.64	1.001
NM40.1	31.62	-0.40	31.62	31.62	95.30	1.001	22.47	-0.06	22.47	22.37	94.68	1.001
NM40.2	31.66	-0.40	31.66	31.63	95.48	1.001	22.49	-0.06	22.49	22.38	94.60	1.001
NM50.1	31.60	-0.40	31.60	31.63	95.20	1.001	22.48	-0.04	22.48	22.38	94.72	1.001
NM50.2	31.64	-0.41	31.64	31.66	95.34	1.002	22.51	-0.05	22.51	22.40	94.62	1.002
NM60.1	31.65	-0.42	31.65	31.64	95.14	1.001	22.47	-0.03	22.47	22.39	94.66	1.001
NM60.2	31.72	-0.44	31.72	31.69	95.20	1.003	22.52	-0.01	22.52	22.43	94.58	1.003
NM70.1	31.64	-0.44	31.64	31.64	95.20	1.002	22.50	-0.01	22.50	22.39	94.78	1.002
NM70.2	31.74	-0.44	31.74	31.75	95.14	1.005	22.58	-0.03	22.58	22.47	94.62	1.005
NM80.1	31.63	-0.41	31.63	31.65	95.38	1.002	22.50	-0.04	22.50	22.40	94.78	1.002
NM80.2	31.78	-0.44	31.78	31.84	95.22	1.008	22.68	-0.03	22.69	22.53	94.76	1.008
NM90.1	31.61	-0.41	31.61	31.65	95.26	1.002	22.50	-0.02	22.50	22.40	94.76	1.002
NM90.2	31.96	-0.46	31.96	32.00	95.30	1.013	22.70	-0.01	22.70	22.65	94.84	1.013

Table E43: Inference for the log scale variance σ^2 based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	139.89	-10.04	139.54	140.00	93.62	1.000	99.13	-5.98	98.96	99.40	93.90	1.000
MI.C90	149.41	-14.14	148.76	139.95	91.98	1.000	105.08	-9.13	104.69	99.35	92.56	1.000
MI.C80	153.14	-7.06	152.99	141.48	91.74	1.011	107.75	-4.53	107.66	100.09	92.08	1.007
MI.D90	18835.31	346.71	18834.00	620.67	95.06	4.433	101.57	-1.55	101.57	100.57	94.22	1.012
MI.D80	141.77	-0.38	141.78	142.67	94.02	1.019	100.13	-1.67	100.13	100.43	94.18	1.010
CENS	146.24	-6.57	146.11	146.52	93.54	1.047	103.28	-4.51	103.19	103.72	93.86	1.043
NM10.1	140.00	-9.98	139.66	140.10	93.48	1.001	99.24	-6.05	99.06	99.46	93.82	1.001
NM10.2	140.00	-9.99	139.66	140.11	93.50	1.001	99.25	-6.05	99.07	99.46	93.86	1.001
NM20.1	140.18	-9.88	139.84	140.37	93.52	1.003	99.35	-5.87	99.19	99.66	94.00	1.003
NM20.2	140.18	-9.93	139.85	140.40	93.58	1.003	99.40	-5.91	99.24	99.68	93.96	1.003
NM30.1	140.83	-9.56	140.52	140.77	93.44	1.006	99.71	-5.61	99.56	99.94	93.88	1.005
NM30.2	140.97	-9.59	140.66	140.89	93.40	1.006	99.86	-5.63	99.71	100.03	93.96	1.006
NM40.1	141.21	-9.30	140.92	141.23	93.58	1.009	99.45	-5.89	99.29	100.21	93.98	1.008
NM40.2	141.62	-9.34	141.32	141.53	93.50	1.011	99.66	-6.11	99.48	100.41	94.02	1.010
NM50.1	141.82	-9.10	141.54	141.69	93.50	1.012	100.19	-5.68	100.04	100.54	93.82	1.011
NM50.2	142.79	-9.09	142.51	142.34	93.60	1.017	100.78	-5.71	100.63	101.00	94.08	1.016
NM60.1	141.85	-9.14	141.57	142.08	93.36	1.015	100.45	-5.11	100.33	100.88	93.96	1.015
NM60.2	143.29	-9.29	143.00	143.27	93.14	1.023	101.38	-5.36	101.24	101.72	94.18	1.023
NM70.1	142.52	-8.60	142.28	142.52	93.62	1.018	100.17	-5.71	100.02	101.07	93.92	1.017
NM70.2	144.72	-9.12	144.45	144.50	93.60	1.032	101.71	-6.37	101.52	102.47	94.06	1.031
NM80.1	141.84	-8.90	141.57	142.79	93.80	1.020	100.92	-5.13	100.80	101.35	93.84	1.020
NM80.2	144.52	-9.35	144.24	145.96	93.88	1.043	103.38	-5.04	103.27	103.67	93.82	1.043
NM90.1	142.45	-8.36	142.22	143.12	93.70	1.022	100.83	-5.38	100.70	101.49	93.96	1.021
NM90.2	147.46	-8.60	147.23	147.79	93.84	1.056	104.36	-5.85	104.21	104.81	94.24	1.054

Table E44: Inference for the log scale variance σ^2 based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	45.28	-1.43	45.26	44.66	94.18	1.000	31.68	-0.28	31.69	31.61	95.14	1.000
MI.C90	48.19	-2.18	48.14	44.71	92.68	1.001	33.46	-0.42	33.46	31.67	93.50	1.002
MI.C80	49.09	-0.83	49.09	44.85	92.30	1.004	34.37	-0.05	34.38	31.74	92.68	1.004
MI.D90	45.38	-0.76	45.38	44.80	94.42	1.003	31.74	0.07	31.75	31.70	95.20	1.003
MI.D80	45.41	-0.66	45.41	44.87	94.40	1.005	31.89	0.14	31.90	31.75	95.14	1.004
CENS	47.11	-1.22	47.10	46.50	94.60	1.041	32.75	0.11	32.75	32.92	95.24	1.041
NM10.1	45.34	-1.42	45.32	44.69	94.22	1.001	31.72	-0.26	31.72	31.64	95.14	1.001
NM10.2	45.35	-1.42	45.33	44.69	94.24	1.001	31.72	-0.26	31.72	31.64	95.16	1.001
NM20.1	45.41	-1.42	45.39	44.77	94.18	1.003	31.77	-0.24	31.77	31.69	95.20	1.003
NM20.2	45.43	-1.42	45.42	44.78	94.08	1.003	31.77	-0.25	31.77	31.70	95.18	1.003
NM30.1	45.48	-1.43	45.47	44.89	94.36	1.005	31.82	-0.25	31.82	31.78	95.12	1.005
NM30.2	45.55	-1.42	45.54	44.92	94.34	1.006	31.84	-0.24	31.85	31.80	95.12	1.006
NM40.1	45.68	-1.32	45.67	45.02	94.08	1.008	31.91	-0.23	31.91	31.87	95.22	1.008
NM40.2	45.75	-1.32	45.74	45.12	94.06	1.010	31.99	-0.24	31.99	31.94	95.06	1.010
NM50.1	45.84	-1.32	45.83	45.16	94.30	1.011	31.95	-0.14	31.96	31.97	95.00	1.011
NM50.2	46.07	-1.36	46.06	45.37	94.34	1.016	32.19	-0.17	32.19	32.12	94.82	1.016
NM60.1	45.97	-1.41	45.95	45.28	94.12	1.014	31.94	-0.13	31.95	32.06	95.20	1.014
NM60.2	46.36	-1.49	46.34	45.67	94.22	1.023	32.17	-0.05	32.18	32.34	95.08	1.023
NM70.1	45.92	-1.56	45.89	45.39	94.22	1.016	32.13	-0.02	32.14	32.14	95.10	1.017
NM70.2	46.59	-1.54	46.57	46.05	94.32	1.031	32.66	-0.06	32.66	32.61	95.20	1.032
NM80.1	46.15	-1.35	46.13	45.49	94.42	1.019	32.36	-0.13	32.36	32.21	94.86	1.019
NM80.2	47.20	-1.42	47.19	46.54	94.12	1.042	33.25	-0.13	33.25	32.95	94.72	1.042
NM90.1	46.02	-1.37	46.01	45.57	94.50	1.020	32.22	-0.06	32.23	32.26	95.12	1.020
NM90.2	47.59	-1.55	47.57	47.09	94.14	1.054	33.38	-0.04	33.38	33.35	94.98	1.055

Table E45: Inference for the mean $e^{\mu+\sigma^2/2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	199.62	2.64	199.62	201.99	93.52	1.000	141.26	-0.04	141.28	142.59	94.68	1.000
MI.C90	212.68	-0.33	212.70	202.88	92.60	1.004	148.84	-3.08	148.82	142.68	93.68	1.001
MI.C80	223.29	12.10	222.98	207.41	92.50	1.027	153.52	4.16	153.48	144.50	93.32	1.013
MI.D90							2×10^{15}	3×10^{13}	2×10^{15}	9×10^{14}	95.00	6×10^{12}
MI.D80	230.21	18.01	229.53	219.26	94.36	1.086	143.77	6.05	143.66	145.09	95.06	1.018
CENS	209.04	7.64	208.92	209.54	93.98	1.037	146.38	2.09	146.38	147.15	94.66	1.032
NM10.1	199.80	2.74	199.80	202.12	93.60	1.001	141.38	-0.11	141.40	142.65	94.62	1.000
NM10.2	199.77	2.72	199.78	202.12	93.60	1.001	141.39	-0.11	141.41	142.65	94.68	1.000
NM20.1	200.13	2.88	200.13	202.41	93.58	1.002	141.49	0.11	141.50	142.87	94.72	1.002
NM20.2	200.14	2.82	200.14	202.44	93.72	1.002	141.56	0.08	141.57	142.89	94.90	1.002
NM30.1	200.74	3.33	200.74	202.90	93.64	1.005	141.89	0.45	141.91	143.19	94.70	1.004
NM30.2	200.95	3.32	200.95	203.05	93.66	1.005	142.10	0.45	142.12	143.30	94.62	1.005
NM40.1	201.39	3.68	201.38	203.43	93.78	1.007	141.80	0.11	141.81	143.43	94.94	1.006
NM40.2	201.81	3.68	201.79	203.82	93.72	1.009	141.96	-0.17	141.97	143.66	94.74	1.008
NM50.1	202.30	4.03	202.28	203.96	93.58	1.010	142.63	0.44	142.64	143.80	94.64	1.008
NM50.2	203.36	4.17	203.34	204.84	93.86	1.014	143.38	0.46	143.39	144.40	94.52	1.013
NM60.1	202.98	4.04	202.96	204.38	93.70	1.012	142.90	1.11	142.91	144.20	94.88	1.011
NM60.2	204.99	4.10	204.97	206.02	93.54	1.020	144.02	0.88	144.03	145.29	94.72	1.019
NM70.1	203.55	4.72	203.52	204.93	93.88	1.015	142.89	0.40	142.91	144.33	94.78	1.012
NM70.2	206.80	4.42	206.78	207.69	93.98	1.028	144.93	-0.32	144.95	146.19	94.56	1.025
NM80.1	202.86	4.28	202.84	205.14	93.92	1.016	143.96	1.19	143.97	144.69	94.68	1.015
NM80.2	206.98	4.14	206.96	209.77	93.84	1.039	147.32	1.67	147.32	148.04	94.62	1.038
NM90.1	203.19	4.98	203.14	205.58	93.84	1.018	143.37	0.85	143.38	144.80	94.78	1.015
NM90.2	210.88	5.52	210.83	213.14	94.10	1.055	148.01	0.63	148.03	149.96	94.76	1.052

Table E46: Inference for the mean $e^{\mu+\sigma^2/2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	63.79	-0.65	63.79	63.80	94.88	1.000	45.14	0.28	45.14	45.16	95.22	1.000
MI.C90	66.82	-1.43	66.81	63.86	93.82	1.001	47.08	0.15	47.08	45.23	93.86	1.001
MI.C80	68.96	0.64	68.96	64.14	93.56	1.005	48.35	0.73	48.35	45.34	93.46	1.004
MI.D90	63.98	0.13	63.98	64.02	95.00	1.003	45.22	0.68	45.22	45.28	95.30	1.003
MI.D80	64.05	0.40	64.06	64.18	95.24	1.006	45.34	0.85	45.34	45.37	95.36	1.005
CENS	65.56	-0.33	65.57	65.62	95.02	1.028	46.19	0.76	46.19	46.45	95.48	1.029
NM10.1	63.83	-0.63	63.84	63.84	94.78	1.001	45.16	0.30	45.16	45.19	95.22	1.001
NM10.2	63.84	-0.63	63.85	63.84	94.82	1.001	45.16	0.30	45.17	45.19	95.18	1.001
NM20.1	63.92	-0.63	63.93	63.92	94.96	1.002	45.27	0.33	45.27	45.25	95.12	1.002
NM20.2	63.96	-0.63	63.96	63.93	94.96	1.002	45.28	0.32	45.28	45.26	95.20	1.002
NM30.1	63.96	-0.64	63.96	64.03	94.94	1.004	45.27	0.32	45.28	45.33	95.12	1.004
NM30.2	64.08	-0.62	64.08	64.08	95.02	1.004	45.36	0.32	45.36	45.36	95.16	1.004
NM40.1	64.23	-0.50	64.23	64.18	94.82	1.006	45.34	0.34	45.35	45.42	95.28	1.006
NM40.2	64.41	-0.48	64.41	64.30	94.78	1.008	45.46	0.32	45.47	45.51	95.14	1.008
NM50.1	64.25	-0.50	64.25	64.31	94.80	1.008	45.42	0.44	45.42	45.52	95.28	1.008
NM50.2	64.58	-0.54	64.58	64.57	95.10	1.012	45.73	0.41	45.73	45.71	95.22	1.012
NM60.1	64.56	-0.59	64.56	64.43	94.78	1.010	45.38	0.47	45.39	45.61	95.26	1.010
NM60.2	65.13	-0.67	65.14	64.93	94.98	1.018	45.75	0.57	45.75	45.98	95.24	1.018
NM70.1	64.46	-0.76	64.46	64.53	94.92	1.011	45.65	0.60	45.65	45.70	95.44	1.012
NM70.2	65.34	-0.70	65.34	65.42	94.94	1.025	46.35	0.55	46.35	46.32	95.00	1.026
NM80.1	64.61	-0.52	64.62	64.63	94.88	1.013	45.82	0.47	45.82	45.75	95.10	1.013
NM80.2	66.02	-0.57	66.02	66.10	95.20	1.036	47.12	0.51	47.12	46.79	95.02	1.036
NM90.1	64.45	-0.55	64.46	64.71	94.86	1.014	45.71	0.55	45.71	45.81	95.28	1.014
NM90.2	66.79	-0.69	66.80	67.02	95.10	1.050	47.30	0.63	47.30	47.46	95.36	1.051

Table E47: Inference for the variance $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	2169.14	235.84	2156.49	2095.21	89.68	1.000	1432.47	98.98	1429.19	1420.84	92.00	1.000
MI.C90	2540.54	336.20	2518.44	2255.95	87.78	1.077	1578.45	117.07	1574.26	1456.29	90.86	1.025
MI.C80	2898.65	611.67	2833.66	2541.08	89.02	1.213	1693.70	248.05	1675.61	1536.73	91.24	1.082
MI.D90							5×10^{63}	7×10^{61}	5×10^{63}	8×10^{63}	93.24	6×10^{60}
MI.D80	5×10^9	7×10^7	5×10^9	2×10^9	92.20	8×10^5	1555.37	261.43	1533.40	1547.27	93.50	1.089
CENS	2422.44	335.13	2399.39	2261.60	89.66	1.079	1543.33	139.50	1537.17	1503.86	92.08	1.058
NM10.1	2173.35	237.68	2160.53	2097.81	89.64	1.001	1435.22	98.49	1431.98	1421.76	92.14	1.001
NM10.2	2172.65	237.46	2159.85	2097.75	89.64	1.001	1435.38	98.55	1432.13	1421.82	92.16	1.001
NM20.1	2179.61	240.46	2166.52	2103.56	89.88	1.004	1437.48	101.47	1434.03	1425.66	92.16	1.003
NM20.2	2178.28	239.79	2165.25	2103.76	89.70	1.004	1438.16	101.20	1434.74	1425.96	92.22	1.004
NM30.1	2196.71	248.98	2182.78	2114.49	89.64	1.009	1446.89	106.79	1443.09	1431.72	92.22	1.008
NM30.2	2197.24	249.41	2183.26	2116.83	89.62	1.010	1450.00	107.31	1446.17	1433.38	92.10	1.009
NM40.1	2221.42	256.10	2206.83	2125.67	89.70	1.015	1443.16	102.18	1439.68	1434.61	92.36	1.010
NM40.2	2223.05	257.53	2208.30	2131.94	89.50	1.018	1444.62	99.47	1441.34	1437.38	92.32	1.012
NM50.1	2234.04	263.15	2218.71	2137.06	89.50	1.020	1462.05	108.51	1458.16	1441.98	92.28	1.015
NM50.2	2248.29	268.95	2232.36	2152.16	89.44	1.027	1473.97	110.70	1469.95	1450.81	92.14	1.021
NM60.1	2247.40	264.79	2231.97	2145.14	89.60	1.024	1465.97	117.42	1461.40	1449.79	92.42	1.020
NM60.2	2273.70	272.38	2257.56	2171.73	89.30	1.037	1479.14	117.60	1474.61	1464.91	92.16	1.031
NM70.1	2264.32	276.41	2247.61	2158.25	89.58	1.030	1464.57	108.44	1460.69	1450.42	92.22	1.021
NM70.2	2319.48	284.75	2302.16	2202.15	89.64	1.051	1495.50	105.65	1491.91	1475.53	92.24	1.038
NM80.1	2249.01	267.85	2233.23	2158.77	89.70	1.030	1486.13	120.67	1481.37	1458.72	92.06	1.027
NM80.2	2321.53	280.43	2304.76	2228.80	89.60	1.064	1530.57	133.84	1524.86	1505.95	92.04	1.060
NM90.1	2263.38	279.01	2246.34	2169.82	89.38	1.036	1475.12	115.52	1470.74	1459.47	92.26	1.027
NM90.2	2408.49	313.34	2388.26	2286.69	89.48	1.091	1543.02	124.90	1538.11	1527.50	91.94	1.075

Table E48: Inference for the variance $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile)

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	624.10	15.25	623.97	618.38	94.00	1.000	437.12	14.11	436.94	436.75	94.88	1.000
MI.C90	671.08	15.85	670.96	621.48	92.68	1.005	464.59	17.24	464.31	438.54	93.20	1.004
MI.C80	697.96	46.86	696.46	629.72	92.06	1.018	482.56	27.83	481.80	441.34	92.50	1.011
MI.D90	630.13	32.92	629.33	625.06	94.44	1.011	439.55	22.85	439.00	439.59	94.80	1.007
MI.D80	633.04	41.66	631.73	629.43	94.38	1.018	442.51	27.65	441.68	441.59	94.88	1.011
CENS	652.93	21.53	652.64	646.31	94.00	1.045	453.60	20.86	453.17	456.33	94.96	1.045
NM10.1	625.18	15.54	625.05	618.89	94.00	1.001	437.54	14.30	437.35	437.09	94.80	1.001
NM10.2	625.34	15.65	625.21	618.92	94.00	1.001	437.56	14.31	437.36	437.11	94.82	1.001
NM20.1	626.26	15.66	626.13	620.11	94.06	1.003	438.75	14.79	438.54	437.99	94.72	1.003
NM20.2	626.64	15.73	626.51	620.29	93.98	1.003	438.80	14.66	438.60	438.10	94.74	1.003
NM30.1	627.03	15.68	626.90	621.83	94.04	1.006	439.25	14.70	439.05	439.19	94.78	1.006
NM30.2	628.44	15.96	628.30	622.50	94.06	1.007	440.03	14.78	439.82	439.64	94.76	1.007
NM40.1	630.79	17.57	630.61	624.10	94.08	1.009	440.39	15.01	440.18	440.61	94.88	1.009
NM40.2	632.66	17.93	632.47	625.78	93.88	1.012	441.81	14.92	441.60	441.74	94.68	1.011
NM50.1	632.28	17.75	632.10	626.13	94.24	1.013	441.45	16.31	441.19	442.16	94.86	1.012
NM50.2	636.28	17.63	636.10	629.56	94.02	1.018	445.42	16.22	445.17	444.59	94.60	1.018
NM60.1	636.10	16.94	635.94	627.87	94.14	1.015	441.08	16.54	440.81	443.50	94.94	1.015
NM60.2	643.58	16.66	643.43	634.33	93.96	1.026	445.45	18.02	445.13	448.25	95.00	1.026
NM70.1	634.03	14.74	633.92	629.15	94.26	1.017	444.78	18.31	444.44	444.85	94.80	1.019
NM70.2	645.71	16.39	645.56	640.64	94.16	1.036	453.63	18.39	453.30	452.77	94.70	1.037
NM80.1	638.29	17.95	638.10	631.04	94.02	1.020	447.69	16.98	447.41	445.66	94.84	1.020
NM80.2	657.02	19.08	656.81	649.39	93.90	1.050	463.34	18.44	463.02	458.63	94.48	1.050
NM90.1	635.55	17.41	635.38	632.05	93.94	1.022	445.44	17.80	445.13	446.52	94.76	1.022
NM90.2	663.99	18.27	663.80	659.78	94.18	1.067	465.53	19.91	465.15	466.23	94.64	1.068

Table E49: Inference for the 84th percentile $e^{\mu+\sigma}$ based on $LN(\mu=0,\sigma^2=1)$ data with $C=5.18=$ 95th percentile

	Results for $n=100$						Results for $n=200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	327.79	-2.39	327.81	331.76	93.48	1.000	232.57	-3.43	232.57	234.68	94.68	1.000
MI.C90	347.90	-9.67	347.80	332.28	92.40	1.002	244.73	-9.48	244.57	234.62	93.60	1.000
MI.C80	363.25	8.92	363.17	338.19	92.26	1.019	251.81	1.59	251.83	237.10	93.26	1.010
MI.D90	2×10^{95}	3×10^{93}	2×10^{95}	7×10^{95}	95.22	2×10^{93}	867.05	15.49	867.00	300.80	95.00	1.282
MI.D80	335.53	16.89	335.14	341.46	94.22	1.029	235.79	5.13	235.76	238.01	94.88	1.014
CENS	342.05	5.19	342.04	343.49	94.04	1.035	240.56	-0.21	240.58	241.99	94.62	1.031
NM10.1	328.07	-2.24	328.09	331.96	93.48	1.001	232.75	-3.55	232.75	234.78	94.60	1.000
NM10.2	328.03	-2.26	328.05	331.96	93.48	1.001	232.77	-3.55	232.77	234.79	94.62	1.000
NM20.1	328.59	-2.03	328.61	332.43	93.64	1.002	232.92	-3.19	232.92	235.14	94.74	1.002
NM20.2	328.60	-2.12	328.63	332.48	93.68	1.002	233.04	-3.25	233.04	235.18	94.84	1.002
NM30.1	329.49	-1.35	329.52	333.19	93.64	1.004	233.54	-2.66	233.55	235.65	94.54	1.004
NM30.2	329.86	-1.38	329.89	333.44	93.60	1.005	233.89	-2.67	233.90	235.82	94.58	1.005
NM40.1	330.44	-0.81	330.47	334.01	93.86	1.007	233.40	-3.19	233.40	236.06	94.92	1.006
NM40.2	331.18	-0.85	331.21	334.66	93.64	1.009	233.68	-3.68	233.67	236.45	94.78	1.008
NM50.1	331.91	-0.29	331.95	334.85	93.66	1.009	234.70	-2.71	234.71	236.63	94.54	1.008
NM50.2	333.66	-0.16	333.69	336.28	93.88	1.014	235.92	-2.72	235.93	237.62	94.44	1.012
NM60.1	332.99	-0.29	333.03	335.53	93.58	1.011	235.13	-1.62	235.15	237.27	94.86	1.011
NM60.2	336.32	-0.32	336.35	338.18	93.50	1.019	237.02	-2.05	237.03	239.07	94.68	1.019
NM70.1	333.81	0.78	333.84	336.37	93.74	1.014	235.13	-2.77	235.14	237.50	94.64	1.012
NM70.2	339.04	0.05	339.08	340.87	93.90	1.027	238.47	-4.07	238.46	240.58	94.50	1.025
NM80.1	332.75	0.11	332.78	336.73	93.78	1.015	236.79	-1.52	236.81	238.05	94.64	1.014
NM80.2	339.43	-0.39	339.46	344.29	93.90	1.038	242.35	-0.91	242.37	243.53	94.50	1.038
NM90.1	333.16	1.20	333.19	337.40	93.86	1.017	235.89	-2.07	235.90	238.24	94.78	1.015
NM90.2	345.50	1.58	345.53	349.63	94.12	1.054	243.52	-2.70	243.53	246.70	94.68	1.051

Table E50: Inference for the 84th percentile $e^{\mu+\sigma}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	105.15	-1.77	105.14	105.16	94.88	1.000	74.40	0.12	74.41	74.45	95.20	1.000
MI.C90	110.11	-3.26	110.07	105.24	93.86	1.001	77.61	-0.18	77.61	74.54	93.88	1.001
MI.C80	113.57	-0.01	113.58	105.65	93.52	1.005	79.67	0.68	79.68	74.71	93.46	1.004
MI.D90	105.41	-0.64	105.42	105.48	94.92	1.003	74.52	0.70	74.53	74.62	95.30	1.002
MI.D80	105.52	-0.29	105.53	105.72	95.12	1.005	74.72	0.94	74.72	74.77	95.36	1.004
CENS	108.04	-1.30	108.04	108.14	95.00	1.028	76.13	0.89	76.13	76.56	95.46	1.028
NM10.1	105.22	-1.74	105.21	105.21	94.78	1.001	74.44	0.14	74.45	74.48	95.16	1.001
NM10.2	105.23	-1.73	105.23	105.22	94.80	1.001	74.45	0.15	74.45	74.48	95.22	1.001
NM20.1	105.37	-1.75	105.37	105.35	94.98	1.002	74.61	0.20	74.62	74.58	95.12	1.002
NM20.2	105.42	-1.74	105.42	105.37	95.06	1.002	74.63	0.18	74.64	74.60	95.18	1.002
NM30.1	105.43	-1.75	105.42	105.54	94.96	1.004	74.62	0.18	74.63	74.72	95.16	1.004
NM30.2	105.62	-1.73	105.62	105.62	94.98	1.004	74.76	0.19	74.77	74.77	95.14	1.004
NM40.1	105.86	-1.54	105.86	105.77	94.78	1.006	74.74	0.21	74.75	74.87	95.32	1.006
NM40.2	106.16	-1.50	106.16	105.98	94.82	1.008	74.94	0.18	74.95	75.01	95.14	1.008
NM50.1	105.90	-1.54	105.90	105.99	94.80	1.008	74.87	0.39	74.87	75.03	95.30	1.008
NM50.2	106.44	-1.62	106.44	106.43	95.08	1.012	75.37	0.33	75.38	75.34	95.14	1.012
NM60.1	106.40	-1.69	106.40	106.19	94.80	1.010	74.81	0.42	74.81	75.18	95.26	1.010
NM60.2	107.34	-1.83	107.33	107.01	94.96	1.018	75.41	0.59	75.42	75.78	95.24	1.018
NM70.1	106.25	-1.97	106.24	106.35	94.98	1.011	75.24	0.63	75.25	75.32	95.44	1.012
NM70.2	107.70	-1.90	107.70	107.81	94.92	1.025	76.39	0.55	76.40	76.35	95.04	1.026
NM80.1	106.48	-1.59	106.48	106.52	94.84	1.013	75.52	0.41	75.52	75.41	95.08	1.013
NM80.2	108.79	-1.70	108.79	108.93	95.10	1.036	77.67	0.47	77.68	77.13	95.06	1.036
NM90.1	106.23	-1.62	106.23	106.64	94.84	1.014	75.34	0.56	75.34	75.51	95.28	1.014
NM90.2	110.09	-1.90	110.08	110.45	95.10	1.050	77.97	0.66	77.97	78.22	95.36	1.051

Table E51: Inference for the 95th percentile $e^{\mu+1.645\sigma}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 5.18 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	787.63	-7.93	787.67	793.99	93.14	1.000	555.13	-9.97	555.10	560.75	94.34	1.000
MI.C90	851.74	-20.91	851.57	798.26	91.68	1.005	592.52	-23.34	592.12	561.40	92.80	1.001
MI.C80	896.62	33.42	896.08	818.22	91.52	1.031	613.95	8.56	613.95	569.51	92.84	1.016
MI.D90							220524.60	3132.47	220524.41	21467.98	94.78	38.284
MI.D80	820.28	58.71	818.26	831.96	93.92	1.048	566.91	18.94	566.65	572.10	94.64	1.020
CENS	833.38	14.55	833.34	832.03	93.24	1.048	580.85	-0.42	580.91	584.29	94.36	1.042
NM10.1	788.48	-7.49	788.52	794.61	93.20	1.001	555.76	-10.31	555.72	561.07	94.22	1.001
NM10.2	788.39	-7.56	788.43	794.62	93.24	1.001	555.82	-10.30	555.78	561.08	94.20	1.001
NM20.1	790.00	-6.87	790.05	796.13	93.10	1.003	556.31	-9.29	556.29	562.19	94.44	1.003
NM20.2	790.01	-7.11	790.05	796.28	93.08	1.003	556.64	-9.46	556.61	562.31	94.46	1.003
NM30.1	793.35	-4.86	793.41	798.57	93.20	1.006	558.43	-7.77	558.44	563.82	94.34	1.005
NM30.2	794.31	-4.91	794.37	799.31	93.04	1.007	559.46	-7.77	559.46	564.34	94.38	1.006
NM40.1	796.58	-3.25	796.65	801.20	93.20	1.009	557.66	-9.28	557.63	565.12	94.28	1.008
NM40.2	798.71	-3.27	798.78	803.15	93.24	1.012	558.53	-10.54	558.49	566.29	94.24	1.010
NM50.1	800.78	-1.73	800.86	803.90	93.24	1.012	561.94	-7.85	561.95	566.99	94.24	1.011
NM50.2	806.21	-1.16	806.29	808.18	93.08	1.018	565.62	-7.77	565.62	569.90	94.24	1.016
NM60.1	803.41	-1.67	803.49	806.10	93.04	1.015	563.32	-4.77	563.36	569.00	94.32	1.015
NM60.2	812.86	-1.44	812.94	813.96	92.72	1.025	568.81	-5.80	568.83	574.30	94.34	1.024
NM70.1	806.54	1.44	806.62	808.81	93.06	1.019	562.89	-7.99	562.89	569.77	94.30	1.016
NM70.2	821.89	0.01	821.97	822.01	93.22	1.035	572.78	-11.18	572.73	578.75	94.04	1.032
NM80.1	803.00	-0.54	803.08	809.98	93.26	1.020	568.05	-4.44	568.09	571.53	94.24	1.019
NM80.2	821.97	-1.28	822.05	831.81	93.28	1.048	584.05	-2.50	584.10	587.34	94.04	1.047
NM90.1	805.14	2.60	805.21	812.11	93.16	1.023	565.73	-5.99	565.76	572.16	94.32	1.020
NM90.2	841.10	4.79	841.17	846.75	93.32	1.066	588.35	-7.08	588.36	595.94	94.52	1.063

Table E52: Inference for the 95th percentile $e^{\mu+1.645\sigma}$ based on $LN(\mu=0,\sigma^2=1)$ data with $C=5.18=$ 95th percentile

	Results for $n=1000$						Results for $n=2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	251.76	-4.39	251.75	251.07	94.62	1.000	177.50	0.45	177.52	177.75	94.94	1.000
MI.C90	267.44	-7.82	267.35	251.39	93.12	1.001	187.37	-0.06	187.39	178.07	93.42	1.002
MI.C80	276.44	1.23	276.46	252.67	92.76	1.006	193.40	2.51	193.41	178.60	92.82	1.005
MI.D90	252.64	-0.60	252.67	252.11	94.64	1.004	177.92	2.37	177.92	178.31	95.04	1.003
MI.D80	252.99	0.64	253.01	252.82	94.72	1.007	178.63	3.17	178.62	178.74	94.98	1.006
CENS	261.16	-2.95	261.17	260.61	94.66	1.038	183.05	2.64	183.05	184.52	95.26	1.038
NM10.1	252.03	-4.31	252.01	251.24	94.64	1.001	177.64	0.51	177.66	177.87	95.08	1.001
NM10.2	252.07	-4.27	252.06	251.25	94.60	1.001	177.66	0.52	177.67	177.87	95.08	1.001
NM20.1	252.46	-4.31	252.45	251.67	94.74	1.002	178.11	0.67	178.13	178.18	95.00	1.002
NM20.2	252.62	-4.29	252.60	251.73	94.74	1.003	178.16	0.62	178.17	178.22	94.98	1.003
NM30.1	252.71	-4.32	252.70	252.28	94.62	1.005	178.22	0.63	178.24	178.61	95.02	1.005
NM30.2	253.25	-4.26	253.24	252.51	94.62	1.006	178.56	0.64	178.58	178.77	95.02	1.006
NM40.1	254.00	-3.72	254.00	253.02	94.58	1.008	178.62	0.71	178.64	179.10	94.90	1.008
NM40.2	254.75	-3.63	254.75	253.63	94.62	1.010	179.18	0.64	179.20	179.53	94.78	1.010
NM50.1	254.35	-3.71	254.35	253.72	94.54	1.011	178.98	1.19	178.99	179.63	95.12	1.011
NM50.2	255.91	-3.90	255.90	255.00	94.66	1.016	180.47	1.06	180.48	180.54	95.22	1.016
NM60.1	255.68	-4.12	255.68	254.35	94.64	1.013	178.83	1.29	178.84	180.10	95.10	1.013
NM60.2	258.37	-4.48	258.36	256.77	94.78	1.023	180.51	1.76	180.52	181.85	95.22	1.023
NM70.1	255.22	-4.91	255.20	254.86	94.84	1.015	180.11	1.88	180.11	180.53	95.26	1.016
NM70.2	259.48	-4.66	259.46	259.13	94.70	1.032	183.46	1.69	183.47	183.53	94.84	1.033
NM80.1	256.18	-3.82	256.18	255.42	94.78	1.017	181.08	1.30	181.09	180.84	94.92	1.017
NM80.2	262.95	-4.03	262.95	262.34	94.80	1.045	187.15	1.48	187.16	185.76	94.82	1.045
NM90.1	255.39	-3.94	255.39	255.80	94.88	1.019	180.44	1.67	180.45	181.13	95.16	1.019
NM90.2	266.24	-4.58	266.23	266.47	94.96	1.061	188.03	2.00	188.04	188.75	95.06	1.062

Table E53: Inference for the log scale mean μ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	99.68	-1.73	99.68	99.19	94.70	1.000	71.87	0.41	71.88	70.37	94.36	1.000
MI.C90	101.42	-2.30	101.40	99.29	94.30	1.001	73.00	0.19	73.00	70.42	94.12	1.001
MI.C80	105.18	3.28	105.14	100.00	93.78	1.008	74.87	2.68	74.83	70.72	93.60	1.005
MI.D90	99.73	-1.73	99.73	99.79	94.74	1.006	71.99	0.40	72.00	70.59	94.32	1.003
MI.D80	100.04	-1.56	100.04	100.03	94.76	1.008	72.03	0.40	72.03	70.74	94.52	1.005
CENS	100.80	-0.82	100.81	100.40	94.80	1.012	72.70	0.69	72.70	71.12	94.66	1.011
NM10.1	99.67	-1.69	99.66	99.21	94.74	1.000	71.87	0.42	71.87	70.38	94.48	1.000
NM10.2	99.66	-1.69	99.66	99.21	94.74	1.000	71.86	0.42	71.87	70.38	94.46	1.000
NM20.1	99.76	-1.69	99.75	99.25	94.62	1.001	71.91	0.41	71.92	70.41	94.42	1.001
NM20.2	99.77	-1.67	99.77	99.26	94.64	1.001	71.91	0.39	71.92	70.41	94.42	1.001
NM30.1	99.78	-1.79	99.78	99.29	94.76	1.001	71.95	0.37	71.96	70.44	94.42	1.001
NM30.2	99.79	-1.81	99.78	99.32	94.72	1.001	71.93	0.35	71.94	70.46	94.38	1.001
NM40.1	99.79	-1.78	99.78	99.36	94.60	1.002	72.00	0.53	72.00	70.51	94.42	1.002
NM40.2	99.77	-1.83	99.77	99.43	94.70	1.002	72.05	0.51	72.05	70.56	94.40	1.003
NM50.1	100.01	-1.73	100.01	99.44	94.68	1.003	72.11	0.52	72.12	70.56	94.32	1.003
NM50.2	100.12	-1.76	100.12	99.60	94.70	1.004	72.27	0.55	72.27	70.68	94.38	1.004
NM60.1	99.99	-1.43	99.99	99.56	94.70	1.004	72.05	0.47	72.06	70.60	94.32	1.003
NM60.2	100.31	-1.43	100.31	99.90	94.82	1.007	72.26	0.41	72.26	70.84	94.52	1.007
NM70.1	99.97	-1.53	99.97	99.60	94.70	1.004	72.16	0.41	72.16	70.63	94.50	1.004
NM70.2	100.45	-1.67	100.44	100.23	94.74	1.010	72.57	0.50	72.58	71.10	94.46	1.010
NM80.1	100.01	-1.47	100.01	99.67	94.56	1.005	72.27	0.47	72.27	70.67	94.38	1.004
NM80.2	101.11	-1.46	101.11	100.84	94.82	1.017	72.94	0.60	72.95	71.53	94.44	1.016
NM90.1	100.07	-1.54	100.07	99.70	94.56	1.005	72.24	0.45	72.25	70.70	94.50	1.005
NM90.2	101.74	-1.61	101.74	101.85	94.82	1.027	73.81	0.77	73.81	72.28	94.84	1.027

Table E54: Inference for the log scale mean μ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.23	-0.02	31.23	31.61	94.88	1.000	22.54	0.20	22.54	22.35	94.78	1.000
MI.C90	31.70	-0.05	31.71	31.63	94.64	1.001	22.92	0.23	22.92	22.36	94.54	1.001
MI.C80	32.20	0.46	32.20	31.69	94.20	1.003	23.34	0.51	23.33	22.40	94.26	1.002
MI.D90	31.27	-0.07	31.27	31.64	95.00	1.001	22.57	0.20	22.57	22.37	94.78	1.001
MI.D80	31.35	-0.01	31.35	31.70	94.82	1.003	22.56	0.20	22.56	22.41	94.94	1.003
CENS	31.58	0.16	31.58	31.93	94.80	1.010	22.77	0.32	22.77	22.58	94.86	1.010
NM10.1	31.23	-0.03	31.23	31.61	94.92	1.000	22.54	0.19	22.55	22.35	94.84	1.000
NM10.2	31.23	-0.03	31.24	31.61	94.94	1.000	22.55	0.19	22.55	22.35	94.78	1.000
NM20.1	31.24	-0.03	31.24	31.62	95.02	1.001	22.56	0.21	22.56	22.36	94.72	1.001
NM20.2	31.24	-0.04	31.24	31.62	94.94	1.001	22.56	0.20	22.56	22.36	94.76	1.001
NM30.1	31.27	0.00	31.27	31.64	95.02	1.001	22.57	0.18	22.58	22.37	94.76	1.001
NM30.2	31.26	-0.00	31.27	31.65	95.02	1.001	22.59	0.19	22.59	22.38	94.84	1.001
NM40.1	31.29	-0.01	31.30	31.66	95.04	1.002	22.60	0.21	22.60	22.39	94.74	1.002
NM40.2	31.31	-0.03	31.31	31.69	95.10	1.003	22.63	0.21	22.63	22.40	94.82	1.003
NM50.1	31.30	0.00	31.30	31.69	95.04	1.003	22.58	0.18	22.58	22.40	95.06	1.002
NM50.2	31.34	-0.04	31.34	31.74	95.10	1.004	22.62	0.19	22.62	22.44	94.84	1.004
NM60.1	31.35	0.02	31.36	31.71	94.90	1.003	22.60	0.22	22.60	22.42	94.94	1.003
NM60.2	31.43	-0.04	31.44	31.82	95.10	1.007	22.70	0.19	22.71	22.50	94.74	1.007
NM70.1	31.33	0.09	31.33	31.73	94.94	1.004	22.61	0.24	22.61	22.43	94.88	1.004
NM70.2	31.50	0.12	31.50	31.94	94.98	1.011	22.76	0.25	22.76	22.58	95.02	1.010
NM80.1	31.36	-0.00	31.36	31.74	94.92	1.004	22.69	0.25	22.69	22.44	94.80	1.004
NM80.2	31.77	-0.12	31.77	32.12	95.18	1.016	22.92	0.25	22.92	22.71	95.08	1.016
NM90.1	31.41	0.07	31.42	31.76	94.92	1.005	22.62	0.24	22.62	22.45	94.92	1.005
NM90.2	32.17	-0.00	32.17	32.46	95.04	1.027	23.16	0.21	23.16	22.95	94.92	1.027

Table E55: Inference for the log scale variance σ^2 based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	140.77	-11.19	140.34	139.84	93.12	1.000	98.48	-7.15	98.23	99.28	94.48	1.000
MI.C90	154.77	-9.50	154.49	141.11	91.04	1.009	108.43	-6.43	108.25	99.89	92.24	1.006
MI.C80	156.23	1.73	156.24	143.65	91.50	1.027	109.33	-0.88	109.34	100.95	92.12	1.017
MI.D90	144.43	-1.11	144.44	143.03	93.78	1.023	99.13	-2.85	99.10	100.31	94.66	1.010
MI.D80	143.42	-0.15	143.44	143.06	93.64	1.023	99.42	-1.93	99.41	100.64	94.84	1.014
CENS	151.96	-7.03	151.81	152.37	93.46	1.090	107.77	-5.99	107.61	107.71	94.24	1.085
NM10.1	140.95	-10.97	140.53	140.01	93.08	1.001	98.57	-7.12	98.32	99.39	94.46	1.001
NM10.2	140.95	-10.98	140.53	140.02	93.04	1.001	98.58	-7.13	98.33	99.39	94.48	1.001
NM20.1	141.55	-11.03	141.13	140.40	93.02	1.004	98.82	-7.19	98.57	99.66	94.46	1.004
NM20.2	141.59	-10.98	141.18	140.44	93.00	1.004	98.84	-7.22	98.58	99.69	94.52	1.004
NM30.1	142.08	-11.28	141.65	140.95	93.16	1.008	99.06	-7.33	98.80	100.07	94.58	1.008
NM30.2	142.32	-11.33	141.88	141.07	93.14	1.009	99.21	-7.37	98.94	100.15	94.60	1.009
NM40.1	142.25	-11.15	141.83	141.67	93.00	1.013	99.96	-6.70	99.74	100.64	94.28	1.014
NM40.2	142.61	-11.28	142.17	141.99	93.04	1.015	100.25	-6.75	100.03	100.87	94.26	1.016
NM50.1	142.19	-11.08	141.77	142.44	93.36	1.019	100.36	-6.79	100.14	101.17	94.54	1.019
NM50.2	143.14	-11.20	142.71	143.16	93.32	1.024	100.79	-6.72	100.58	101.70	94.26	1.024
NM60.1	144.51	-9.78	144.19	143.39	93.26	1.025	101.22	-6.93	100.99	101.69	94.20	1.024
NM60.2	145.89	-9.84	145.57	144.77	93.02	1.035	102.24	-7.09	102.00	102.66	93.88	1.034
NM70.1	144.39	-10.19	144.04	144.03	93.32	1.030	101.27	-7.18	101.03	102.16	94.34	1.029
NM70.2	147.47	-10.55	147.11	146.38	93.24	1.047	102.76	-7.02	102.53	103.90	94.54	1.046
NM80.1	144.95	-9.84	144.63	144.70	93.26	1.035	101.50	-7.00	101.26	102.61	94.40	1.033
NM80.2	149.14	-9.91	148.82	148.51	93.08	1.062	104.18	-6.75	103.97	105.38	94.54	1.061
NM90.1	144.52	-10.28	144.16	145.15	93.56	1.038	102.68	-7.03	102.45	102.97	94.20	1.037
NM90.2	149.56	-10.76	149.18	150.55	93.42	1.077	106.94	-6.63	106.74	106.95	94.02	1.077

Table E56: Inference for the log scale variance σ^2 based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90\text{th}$ percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	44.78	-0.60	44.78	44.69	94.66	1.000	31.45	-1.04	31.44	31.59	94.82	1.000
MI.C90	48.88	-0.24	48.89	44.88	92.60	1.004	34.28	-0.70	34.27	31.71	92.74	1.004
MI.C80	49.27	0.95	49.26	45.10	92.64	1.009	34.49	-0.10	34.49	31.85	93.08	1.008
MI.D90	44.89	0.08	44.89	44.91	94.46	1.005	31.60	-0.64	31.59	31.72	94.80	1.004
MI.D80	45.15	0.40	45.15	45.01	94.58	1.007	31.60	-0.55	31.60	31.79	95.00	1.006
CENS	48.41	0.18	48.42	48.44	94.78	1.084	34.09	-0.53	34.09	34.22	94.74	1.083
NM10.1	44.82	-0.61	44.82	44.74	94.58	1.001	31.47	-1.08	31.46	31.62	94.70	1.001
NM10.2	44.82	-0.62	44.82	44.74	94.56	1.001	31.47	-1.08	31.46	31.62	94.70	1.001
NM20.1	44.97	-0.60	44.97	44.87	94.52	1.004	31.61	-1.01	31.60	31.71	94.74	1.004
NM20.2	45.00	-0.63	45.00	44.88	94.56	1.004	31.62	-1.02	31.61	31.72	94.72	1.004
NM30.1	45.07	-0.50	45.07	45.06	94.88	1.008	31.78	-1.12	31.76	31.84	94.60	1.008
NM30.2	45.14	-0.52	45.14	45.10	94.82	1.009	31.80	-1.10	31.78	31.87	94.64	1.009
NM40.1	45.37	-0.53	45.37	45.29	94.70	1.013	31.90	-1.01	31.89	32.01	94.54	1.013
NM40.2	45.51	-0.57	45.51	45.39	94.70	1.016	31.96	-1.02	31.95	32.08	94.64	1.016
NM50.1	45.49	-0.47	45.49	45.54	94.70	1.019	32.02	-1.10	32.01	32.18	94.66	1.019
NM50.2	45.82	-0.57	45.82	45.77	94.66	1.024	32.17	-1.08	32.15	32.34	94.80	1.024
NM60.1	45.78	-0.44	45.79	45.78	94.84	1.024	32.30	-0.92	32.29	32.35	94.64	1.024
NM60.2	46.33	-0.58	46.33	46.22	94.66	1.034	32.50	-1.00	32.49	32.67	94.72	1.034
NM70.1	45.94	-0.13	45.94	46.02	94.84	1.030	32.39	-0.90	32.38	32.51	94.74	1.029
NM70.2	46.88	-0.06	46.89	46.80	94.82	1.047	33.00	-0.87	32.99	33.06	94.76	1.047
NM80.1	46.13	-0.49	46.13	46.19	94.94	1.034	32.60	-0.83	32.59	32.65	94.52	1.034
NM80.2	47.46	-0.78	47.46	47.42	94.98	1.061	33.41	-0.82	33.40	33.53	94.72	1.061
NM90.1	46.16	-0.21	46.17	46.37	94.72	1.037	32.69	-0.87	32.68	32.76	94.46	1.037
NM90.2	48.10	-0.42	48.11	48.14	95.02	1.077	33.86	-0.95	33.85	34.02	94.70	1.077

Table E57: Inference for the mean $e^{\mu+\sigma^2/2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	202.04	0.20	202.06	201.56	93.24	1.000	143.17	0.96	143.18	142.56	94.36	1.000
MI.C90	222.93	6.91	222.85	206.26	92.10	1.023	155.82	4.09	155.78	144.28	92.82	1.012
MI.C80	240.18	32.76	237.96	215.24	92.30	1.068	164.31	16.03	163.54	147.52	92.64	1.035
MI.D90	6×10^{21}	9×10^{19}	6×10^{21}	5×10^{21}	93.90	2×10^{19}	145.44	6.95	145.29	145.02	94.60	1.017
MI.D80	212.02	19.65	211.13	212.08	94.10	1.052	146.56	9.75	146.25	146.42	95.02	1.027
CENS	217.10	6.80	217.02	216.66	93.58	1.075	154.19	3.29	154.17	152.04	94.54	1.067
NM10.1	202.19	0.47	202.21	201.79	93.30	1.001	143.20	1.00	143.21	142.68	94.34	1.001
NM10.2	202.18	0.47	202.20	201.80	93.22	1.001	143.19	0.99	143.20	142.69	94.42	1.001
NM20.1	203.05	0.52	203.07	202.25	93.18	1.003	143.62	0.96	143.63	142.99	94.48	1.003
NM20.2	203.13	0.61	203.15	202.33	93.14	1.004	143.62	0.91	143.63	143.02	94.36	1.003
NM30.1	203.49	0.22	203.51	202.85	93.22	1.006	143.88	0.79	143.90	143.43	94.62	1.006
NM30.2	203.69	0.17	203.71	203.04	93.10	1.007	143.94	0.74	143.95	143.56	94.50	1.007
NM40.1	203.61	0.35	203.63	203.67	93.00	1.010	144.84	1.67	144.84	144.13	94.44	1.011
NM40.2	203.82	0.18	203.84	204.17	93.04	1.013	145.25	1.63	145.26	144.50	94.26	1.014
NM50.1	204.55	0.60	204.57	204.58	93.54	1.015	145.65	1.63	145.66	144.72	94.26	1.015
NM50.2	205.68	0.58	205.70	205.74	93.24	1.021	146.50	1.82	146.50	145.57	94.48	1.021
NM60.1	206.68	2.37	206.69	205.88	93.18	1.021	145.94	1.47	145.95	145.28	94.48	1.019
NM60.2	209.02	2.60	209.03	208.18	93.16	1.033	147.46	1.38	147.47	146.86	94.36	1.030
NM70.1	206.48	1.86	206.49	206.53	93.30	1.025	146.53	1.22	146.54	145.78	94.28	1.023
NM70.2	210.54	1.84	210.55	210.57	93.28	1.045	149.06	1.71	149.06	148.71	94.50	1.043
NM80.1	207.09	2.30	207.10	207.34	93.36	1.029	147.16	1.51	147.17	146.30	94.38	1.026
NM80.2	214.38	3.14	214.38	214.33	93.14	1.063	151.57	2.31	151.56	151.24	94.36	1.061
NM90.1	206.92	1.81	206.93	207.79	93.14	1.031	148.05	1.53	148.06	146.70	94.46	1.029
NM90.2	216.01	2.43	216.02	218.74	93.56	1.085	156.18	3.10	156.17	154.65	94.32	1.085

Table E58: Inference for the mean $e^{\mu+\sigma^2/2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	62.84	0.66	62.84	63.89	95.36	1.000	44.83	0.08	44.84	45.13	95.18	1.000
MI.C90	67.76	1.46	67.75	64.20	93.24	1.005	48.34	0.68	48.34	45.32	93.36	1.004
MI.C80	69.69	3.86	69.59	64.69	93.30	1.013	49.67	1.91	49.64	45.58	92.72	1.010
MI.D90	63.16	1.58	63.14	64.25	95.36	1.006	45.09	0.63	45.09	45.34	95.26	1.005
MI.D80	63.67	2.36	63.63	64.56	95.34	1.011	45.10	0.90	45.10	45.52	95.16	1.009
CENS	66.97	1.76	66.96	67.91	95.28	1.063	47.65	0.78	47.65	47.94	94.94	1.062
NM10.1	62.88	0.64	62.88	63.94	95.22	1.001	44.87	0.04	44.87	45.17	95.18	1.001
NM10.2	62.89	0.64	62.89	63.94	95.28	1.001	44.87	0.04	44.88	45.17	95.16	1.001
NM20.1	63.03	0.66	63.03	64.08	95.26	1.003	45.02	0.12	45.03	45.27	95.14	1.003
NM20.2	63.04	0.62	63.04	64.10	95.32	1.003	45.03	0.11	45.03	45.28	95.20	1.003
NM30.1	63.22	0.80	63.22	64.30	95.42	1.006	45.20	-0.00	45.21	45.41	95.18	1.006
NM30.2	63.27	0.78	63.27	64.36	95.40	1.007	45.26	0.02	45.27	45.46	95.02	1.007
NM40.1	63.54	0.77	63.54	64.55	95.24	1.010	45.40	0.14	45.40	45.59	95.06	1.010
NM40.2	63.70	0.72	63.70	64.71	95.34	1.013	45.55	0.13	45.55	45.71	95.20	1.013
NM50.1	63.65	0.84	63.65	64.81	95.30	1.014	45.40	0.02	45.41	45.77	94.98	1.014
NM50.2	64.05	0.71	64.05	65.17	95.26	1.020	45.64	0.06	45.64	46.03	94.98	1.020
NM60.1	64.09	0.90	64.09	65.08	95.16	1.019	45.69	0.24	45.70	45.97	95.18	1.018
NM60.2	64.81	0.73	64.81	65.78	95.26	1.030	46.17	0.14	46.17	46.47	95.44	1.030
NM70.1	64.16	1.28	64.15	65.34	95.54	1.023	45.80	0.28	45.80	46.14	94.98	1.022
NM70.2	65.50	1.45	65.49	66.63	95.40	1.043	46.79	0.36	46.80	47.04	94.88	1.042
NM80.1	64.37	0.85	64.37	65.52	95.18	1.025	46.24	0.38	46.24	46.29	94.88	1.026
NM80.2	66.68	0.51	66.69	67.65	95.52	1.059	47.61	0.42	47.62	47.81	95.22	1.059
NM90.1	64.61	1.20	64.60	65.71	95.38	1.029	46.02	0.33	46.02	46.40	95.22	1.028
NM90.2	68.31	1.06	68.31	69.15	95.14	1.082	48.48	0.27	48.48	48.84	95.20	1.082

Table E59: Inference for the variance $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	2148.49	216.46	2137.77	2085.79	88.92	1.000	1427.97	95.62	1424.91	1418.54	91.46	1.000
MI.C90	2772.22	545.54	2718.28	2486.24	87.92	1.192	1697.15	234.11	1681.10	1528.93	90.90	1.078
MI.C80	3299.21	1062.81	3123.65	3058.70	90.64	1.466	1903.76	453.04	1849.25	1681.33	92.50	1.185
MI.D90	6×10^{89}	9×10^{87}	6×10^{89}	2×10^{90}	91.78	9×19^{86}	1533.35	253.96	1512.33	1541.83	92.92	1.087
MI.D80	2738.92	791.59	2622.30	2787.07	92.76	1.336	1588.12	331.00	1553.40	1607.07	93.76	1.133
CENS	2493.03	357.94	2467.45	2388.78	88.72	1.145	1643.89	155.73	1636.67	1581.10	91.46	1.115
NM10.1	2152.97	220.47	2141.87	2090.35	88.70	1.002	1428.69	96.28	1425.58	1420.43	91.46	1.001
NM10.2	2153.18	220.46	2142.08	2090.42	88.72	1.002	1428.56	96.19	1425.46	1420.45	91.46	1.001
NM20.1	2172.06	224.36	2160.65	2099.49	88.88	1.007	1436.55	96.83	1433.43	1425.23	91.42	1.005
NM20.2	2172.98	225.50	2161.47	2100.73	88.86	1.007	1436.87	96.32	1433.78	1425.51	91.42	1.005
NM30.1	2172.68	222.74	2161.44	2108.77	88.78	1.011	1437.61	95.46	1434.58	1431.44	91.50	1.009
NM30.2	2176.23	223.12	2164.98	2111.50	88.64	1.012	1439.64	95.31	1436.63	1433.13	91.64	1.010
NM40.1	2177.67	225.21	2166.21	2122.20	88.86	1.017	1458.20	108.58	1454.30	1444.58	90.98	1.018
NM40.2	2179.85	224.57	2168.47	2128.38	88.82	1.020	1464.37	109.19	1460.44	1449.35	91.26	1.022
NM50.1	2187.18	229.22	2175.36	2137.45	88.88	1.025	1471.21	109.80	1467.25	1453.91	91.40	1.025
NM50.2	2203.98	233.29	2191.82	2153.94	88.78	1.033	1482.95	113.55	1478.75	1465.08	91.14	1.033
NM60.1	2258.14	261.95	2243.12	2169.86	88.78	1.040	1475.85	109.76	1471.91	1462.72	91.40	1.031
NM60.2	2296.67	272.12	2280.72	2202.90	88.78	1.056	1498.86	112.44	1494.79	1482.65	91.46	1.045
NM70.1	2245.81	254.71	2231.54	2178.26	88.94	1.044	1488.18	107.81	1484.42	1470.23	91.54	1.036
NM70.2	2314.21	270.33	2298.59	2236.07	88.64	1.072	1523.31	118.71	1518.83	1507.56	91.96	1.063
NM80.1	2263.00	263.22	2247.87	2194.38	88.88	1.052	1495.24	112.27	1491.17	1478.98	91.54	1.043
NM80.2	2384.56	296.88	2366.24	2293.99	88.36	1.100	1558.51	130.98	1553.15	1541.00	91.62	1.086
NM90.1	2241.95	254.38	2227.69	2198.47	88.96	1.054	1520.05	116.21	1515.75	1486.90	91.56	1.048
NM90.2	2372.40	288.49	2355.03	2339.67	88.80	1.122	1627.44	149.48	1620.72	1583.48	91.08	1.116

Table E60: Inference for the variance $e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	617.36	28.04	616.78	620.33	94.74	1.000	431.88	6.94	431.87	435.86	94.86	1.000
MI.C90	689.99	54.96	687.87	630.93	92.62	1.017	479.65	22.51	479.17	440.63	92.68	1.011
MI.C80	715.76	94.52	709.56	644.43	92.82	1.039	492.50	42.11	490.75	446.30	93.04	1.024
MI.D90	626.87	53.01	624.69	631.14	95.06	1.017	436.98	20.20	436.56	440.66	95.10	1.011
MI.D80	636.15	70.27	632.32	638.44	95.08	1.029	438.18	27.31	437.37	443.73	95.28	1.018
CENS	678.59	46.19	677.08	679.80	94.74	1.096	472.95	17.75	472.66	476.32	94.58	1.093
NM10.1	617.98	27.92	617.41	621.03	94.74	1.001	432.18	6.51	432.18	436.32	94.78	1.001
NM10.2	618.10	27.84	617.53	621.04	94.70	1.001	432.19	6.50	432.18	436.33	94.72	1.001
NM20.1	620.05	28.31	619.47	623.07	94.50	1.004	434.48	7.62	434.46	437.82	94.50	1.005
NM20.2	620.33	27.90	619.77	623.20	94.54	1.005	434.64	7.54	434.62	437.96	94.44	1.005
NM30.1	622.74	30.11	622.07	626.26	94.80	1.010	436.90	6.27	436.90	439.71	94.88	1.009
NM30.2	623.71	29.89	623.06	626.98	94.82	1.011	437.59	6.57	437.58	440.27	94.80	1.010
NM40.1	627.16	30.22	626.49	629.79	94.72	1.015	439.31	8.05	439.28	442.36	94.76	1.015
NM40.2	629.37	29.78	628.73	631.70	94.52	1.018	440.89	8.06	440.86	443.75	94.82	1.018
NM50.1	629.12	31.17	628.41	633.69	94.62	1.022	439.97	6.80	439.96	444.84	94.84	1.021
NM50.2	634.66	30.28	634.00	637.84	94.50	1.028	442.72	7.32	442.70	447.92	94.74	1.028
NM60.1	634.40	32.29	633.64	637.57	94.52	1.028	444.82	9.65	444.76	447.74	94.76	1.027
NM60.2	643.49	31.30	642.79	645.67	94.50	1.041	449.61	8.86	449.56	453.46	94.76	1.040
NM70.1	637.19	36.92	636.18	641.73	94.90	1.034	446.21	10.18	446.14	450.17	94.82	1.033
NM70.2	654.31	40.18	653.14	656.85	94.76	1.059	458.13	11.73	458.03	460.60	94.80	1.057
NM80.1	638.82	32.15	638.07	643.87	94.68	1.038	450.65	11.62	450.54	452.38	94.50	1.038
NM80.2	664.63	30.77	663.98	667.57	94.56	1.076	466.23	13.03	466.09	469.39	94.48	1.077
NM90.1	641.44	36.42	640.47	647.02	94.50	1.043	449.81	10.98	449.73	454.00	94.76	1.042
NM90.2	680.84	38.23	679.83	683.71	94.58	1.102	474.83	11.90	474.73	479.67	94.82	1.101

Table E61: Inference for the 84th percentile $e^{\mu+\sigma}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	332.21	-6.45	332.18	331.10	93.22	1.000	235.75	-1.74	235.77	234.69	94.34	1.000
MI.C90	363.84	0.38	363.87	336.54	91.72	1.016	255.65	1.46	255.67	236.82	92.68	1.009
MI.C80	389.05	40.02	387.03	348.58	92.00	1.053	268.26	19.80	267.56	241.31	92.48	1.028
MI.D90	2726.01	50.94	2725.81	683.43	93.82	2.064	238.70	6.69	238.63	237.97	94.56	1.014
MI.D80	344.87	20.84	344.27	344.68	93.80	1.041	240.22	10.70	240.00	239.94	94.94	1.022
CENS	355.59	3.25	355.62	354.88	93.50	1.072	253.22	1.43	253.24	249.98	94.50	1.065
NM10.1	332.44	-6.03	332.42	331.46	93.26	1.001	235.81	-1.68	235.82	234.89	94.40	1.001
NM10.2	332.43	-6.03	332.41	331.47	93.26	1.001	235.79	-1.69	235.81	234.90	94.38	1.001
NM20.1	333.78	-6.01	333.76	332.20	93.06	1.003	236.45	-1.75	236.47	235.40	94.44	1.003
NM20.2	333.89	-5.87	333.87	332.32	93.12	1.004	236.46	-1.84	236.48	235.45	94.40	1.003
NM30.1	334.60	-6.56	334.56	333.18	93.16	1.006	236.93	-2.05	236.95	236.12	94.54	1.006
NM30.2	334.93	-6.66	334.89	333.50	93.12	1.007	237.02	-2.14	237.03	236.34	94.52	1.007
NM40.1	334.77	-6.36	334.74	334.52	93.08	1.010	238.43	-0.67	238.45	237.24	94.44	1.011
NM40.2	335.14	-6.67	335.11	335.34	93.06	1.013	239.11	-0.76	239.13	237.85	94.36	1.013
NM50.1	336.28	-5.94	336.26	335.99	93.40	1.015	239.73	-0.75	239.76	238.20	94.22	1.015
NM50.2	338.15	-6.06	338.13	337.88	93.12	1.021	241.11	-0.48	241.13	239.59	94.34	1.021
NM60.1	339.40	-3.26	339.42	337.96	93.22	1.021	240.27	-1.08	240.29	239.11	94.40	1.019
NM60.2	343.20	-3.02	343.22	341.72	93.12	1.032	242.75	-1.30	242.78	241.71	94.28	1.030
NM70.1	339.15	-4.09	339.16	339.04	93.30	1.024	241.15	-1.51	241.17	239.93	94.30	1.022
NM70.2	345.77	-4.43	345.77	345.64	93.32	1.044	245.26	-0.80	245.29	244.74	94.42	1.043
NM80.1	340.10	-3.42	340.12	340.32	93.32	1.028	242.18	-1.04	242.21	240.77	94.38	1.026
NM80.2	351.84	-2.47	351.87	351.69	93.10	1.062	249.36	0.10	249.39	248.87	94.34	1.060
NM90.1	339.99	-4.19	340.00	341.08	93.12	1.030	243.56	-1.09	243.58	241.41	94.50	1.029
NM90.2	354.92	-3.67	354.93	358.96	93.50	1.084	256.84	1.19	256.86	254.42	94.28	1.084

Table E62: Inference for the 84th percentile $e^{\mu+\sigma}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	103.55	0.40	103.56	105.29	95.38	1.000	73.91	-0.20	73.91	74.41	95.20	1.000
MI.C90	111.59	1.36	111.59	105.73	93.20	1.004	79.68	0.61	79.68	74.68	93.34	1.004
MI.C80	114.66	5.06	114.56	106.47	93.20	1.011	81.84	2.52	81.81	75.09	92.70	1.009
MI.D90	104.01	1.67	104.00	105.83	95.36	1.005	74.31	0.58	74.31	74.73	95.24	1.004
MI.D80	104.83	2.83	104.80	106.31	95.32	1.010	74.32	0.97	74.32	75.01	95.12	1.008
CENS	110.31	2.11	110.30	111.88	95.24	1.063	78.54	0.89	78.54	79.01	94.92	1.062
NM10.1	103.62	0.38	103.63	105.37	95.24	1.001	73.96	-0.27	73.97	74.46	95.20	1.001
NM10.2	103.63	0.36	103.64	105.37	95.30	1.001	73.97	-0.27	73.98	74.47	95.14	1.001
NM20.1	103.87	0.40	103.88	105.61	95.28	1.003	74.22	-0.14	74.23	74.63	95.16	1.003
NM20.2	103.88	0.33	103.89	105.63	95.22	1.003	74.23	-0.15	74.24	74.65	95.14	1.003
NM30.1	104.17	0.63	104.18	105.96	95.38	1.006	74.51	-0.35	74.52	74.87	95.14	1.006
NM30.2	104.25	0.59	104.26	106.06	95.40	1.007	74.61	-0.31	74.62	74.94	95.00	1.007
NM40.1	104.70	0.58	104.71	106.37	95.26	1.010	74.84	-0.12	74.85	75.16	95.00	1.010
NM40.2	104.97	0.47	104.98	106.63	95.32	1.013	75.09	-0.13	75.09	75.36	95.20	1.013
NM50.1	104.89	0.69	104.89	106.81	95.34	1.014	74.85	-0.32	74.86	75.46	95.04	1.014
NM50.2	105.54	0.46	105.55	107.40	95.24	1.020	75.24	-0.26	75.24	75.89	94.94	1.020
NM60.1	105.60	0.78	105.61	107.24	95.20	1.019	75.32	0.04	75.32	75.78	95.16	1.018
NM60.2	106.80	0.48	106.81	108.39	95.30	1.030	76.10	-0.13	76.11	76.60	95.44	1.029
NM70.1	105.70	1.39	105.70	107.67	95.54	1.023	75.50	0.11	75.51	76.06	94.94	1.022
NM70.2	107.91	1.64	107.90	109.79	95.40	1.043	77.13	0.23	77.14	77.55	94.90	1.042
NM80.1	106.07	0.68	106.08	107.96	95.22	1.025	76.23	0.27	76.23	76.30	94.94	1.025
NM80.2	109.89	0.07	109.90	111.47	95.50	1.059	78.49	0.31	78.49	78.82	95.24	1.059
NM90.1	106.45	1.26	106.45	108.28	95.38	1.028	75.86	0.18	75.87	76.49	95.24	1.028
NM90.2	112.57	0.97	112.58	113.95	95.16	1.082	79.91	0.06	79.92	80.51	95.18	1.082

Table E63: Inference for the 95th percentile $e^{\mu+1.645\sigma}$ based on $LN(\mu=0,\sigma^2=1)$ data with $C=3.602=90$th percentile

	Results for $n=100$						Results for $n=200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	795.10	-17.31	794.99	792.17	92.80	1.000	558.99	-8.76	558.97	560.54	93.86	1.000
MI.C90	894.71	12.72	894.71	813.36	91.12	1.027	621.42	5.18	621.46	568.48	92.14	1.014
MI.C80	958.37	120.03	950.92	852.39	91.82	1.076	653.21	55.31	650.93	582.69	92.32	1.040
MI.D90	2021812.42	28640.33	2021811.74	320046.09	93.82	404.011	569.14	19.78	568.85	571.75	94.40	1.020
MI.D80	839.58	71.98	836.57	838.69	93.86	1.059	574.02	32.10	573.18	577.74	94.74	1.031
CENS	868.68	11.97	868.68	867.27	92.82	1.095	614.64	1.39	614.70	608.48	94.18	1.086
NM10.1	795.95	-16.09	795.87	793.28	92.90	1.001	559.25	-8.57	559.24	561.16	93.80	1.001
NM10.2	795.95	-16.10	795.87	793.31	92.90	1.001	559.22	-8.61	559.21	561.17	93.78	1.001
NM20.1	800.16	-15.93	800.08	795.61	92.60	1.004	561.20	-8.76	561.19	562.73	93.78	1.004
NM20.2	800.47	-15.56	800.40	795.94	92.66	1.005	561.25	-8.98	561.24	562.87	93.74	1.004
NM30.1	802.52	-17.24	802.42	798.68	92.48	1.008	562.50	-9.51	562.47	564.96	93.82	1.008
NM30.2	803.60	-17.45	803.49	799.57	92.54	1.009	562.97	-9.73	562.94	565.59	93.92	1.009
NM40.1	803.18	-16.61	803.09	802.84	92.50	1.013	567.50	-5.63	567.53	568.44	93.74	1.014
NM40.2	804.52	-17.32	804.41	805.13	92.62	1.016	569.46	-5.81	569.48	570.15	93.48	1.017
NM50.1	806.52	-15.56	806.45	807.39	92.68	1.019	571.16	-5.83	571.19	571.44	94.28	1.019
NM50.2	812.13	-15.67	812.06	812.71	92.60	1.026	574.89	-5.06	574.93	575.33	94.06	1.026
NM60.1	818.33	-7.68	818.38	813.65	92.80	1.027	573.57	-6.55	573.59	574.31	93.98	1.025
NM60.2	829.02	-6.75	829.07	824.12	92.58	1.040	580.69	-6.96	580.71	581.52	93.74	1.037
NM70.1	817.38	-9.97	817.41	817.05	92.86	1.031	575.90	-7.72	575.90	576.88	93.96	1.029
NM70.2	837.04	-10.09	837.06	835.26	92.76	1.054	587.26	-5.68	587.29	590.08	94.12	1.053
NM80.1	820.59	-7.94	820.63	821.07	92.72	1.036	578.54	-6.42	578.56	579.48	93.98	1.034
NM80.2	853.38	-4.51	853.45	852.00	92.58	1.076	598.49	-3.09	598.54	601.41	93.94	1.073
NM90.1	818.98	-10.23	818.99	823.42	92.94	1.039	583.83	-6.36	583.85	581.54	94.22	1.037
NM90.2	858.74	-7.94	858.79	870.43	93.00	1.099	619.14	-0.02	619.21	615.71	93.84	1.098

Table E64: Inference for the 95th percentile $e^{\mu+1.645\sigma}$ based on $LN(\mu = 0, \sigma^2 = 1)$ data with $C = 3.602 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	248.10	1.10	248.12	251.44	95.10	1.000	175.79	-1.48	175.80	177.62	95.22	1.000
MI.C90	272.41	4.75	272.40	252.91	92.58	1.006	192.96	1.25	192.97	178.49	92.44	1.005
MI.C80	279.74	14.88	279.37	255.10	92.74	1.015	197.79	6.43	197.71	179.67	92.42	1.012
MI.D90	249.49	5.58	249.45	253.15	95.06	1.007	176.98	1.13	176.99	178.60	95.34	1.006
MI.D80	251.78	9.00	251.64	254.44	95.10	1.012	177.04	2.35	177.04	179.33	95.30	1.010
CENS	268.91	6.03	268.87	271.95	95.12	1.082	190.25	1.67	190.26	191.93	94.92	1.081
NM10.1	248.31	1.03	248.33	251.70	95.12	1.001	175.95	-1.65	175.96	177.79	95.30	1.001
NM10.2	248.35	1.00	248.37	251.70	95.16	1.001	175.96	-1.66	175.97	177.80	95.28	1.001
NM20.1	249.10	1.11	249.12	252.42	95.08	1.004	176.75	-1.28	176.76	178.31	95.18	1.004
NM20.2	249.18	0.93	249.20	252.48	95.06	1.004	176.78	-1.32	176.80	178.37	95.20	1.004
NM30.1	249.96	1.73	249.98	253.51	95.36	1.008	177.65	-1.86	177.66	179.03	95.06	1.008
NM30.2	250.24	1.62	250.26	253.80	95.42	1.009	177.91	-1.75	177.92	179.24	95.02	1.009
NM40.1	251.58	1.62	251.60	254.77	95.06	1.013	178.57	-1.22	178.58	179.95	95.22	1.013
NM40.2	252.40	1.36	252.43	255.52	94.90	1.016	179.19	-1.25	179.20	180.50	95.08	1.016
NM50.1	252.20	1.92	252.22	256.14	95.08	1.019	178.78	-1.74	178.79	180.88	95.02	1.018
NM50.2	254.19	1.37	254.21	257.78	95.06	1.025	179.85	-1.60	179.86	182.07	95.22	1.025
NM60.1	254.23	2.18	254.25	257.48	94.86	1.024	180.28	-0.76	180.29	181.86	95.24	1.024
NM60.2	257.72	1.45	257.74	260.68	95.20	1.037	182.27	-1.18	182.29	184.14	95.18	1.037
NM70.1	254.76	3.88	254.76	258.83	95.24	1.029	180.82	-0.58	180.84	182.72	95.20	1.029
NM70.2	261.08	4.59	261.06	264.65	95.12	1.053	185.35	-0.24	185.37	186.82	94.94	1.052
NM80.1	255.81	1.95	255.83	259.73	94.96	1.033	182.73	-0.13	182.75	183.48	94.72	1.033
NM80.2	266.10	0.48	266.13	269.21	95.12	1.071	188.88	0.04	188.90	190.27	95.06	1.071
NM90.1	256.72	3.54	256.72	260.72	95.00	1.037	182.10	-0.37	182.11	184.08	95.16	1.036
NM90.2	272.71	2.90	272.72	275.55	95.04	1.096	192.49	-0.62	192.51	194.59	95.30	1.096

Table E65: Inference for the mean μ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	101.30	2.27	101.28	99.19	94.28	1.000	70.26	0.58	70.26	70.50	94.70	1.000
MI.C90	102.12	0.47	102.13	98.96	94.08	0.998	70.76	-0.39	70.77	70.42	94.68	0.999
MI.C80	103.55	2.77	103.52	99.41	93.82	1.002	71.53	0.84	71.53	70.61	94.28	1.002
MI.D90	103.09	2.05	103.08	106.03	94.54	1.069	70.30	0.59	70.30	70.67	94.80	1.002
MI.D80	101.40	2.22	101.38	99.82	94.48	1.006	70.28	0.56	70.29	70.72	94.70	1.003
CENS	102.05	3.02	102.01	99.75	94.18	1.006	70.40	0.93	70.40	70.83	94.66	1.005
NM10.1	101.35	2.29	101.33	99.23	94.32	1.000	70.25	0.62	70.26	70.53	94.62	1.000
NM10.2	101.33	2.29	101.32	99.23	94.38	1.000	70.25	0.62	70.26	70.53	94.62	1.000
NM20.1	101.48	2.38	101.46	99.31	94.40	1.001	70.32	0.74	70.32	70.60	94.64	1.001
NM20.2	101.41	2.40	101.40	99.33	94.40	1.001	70.37	0.72	70.37	70.61	94.62	1.002
NM30.1	101.61	2.66	101.58	99.44	94.36	1.002	70.34	0.69	70.34	70.63	94.78	1.002
NM30.2	101.54	2.66	101.52	99.51	94.52	1.003	70.45	0.66	70.46	70.68	94.84	1.002
NM40.1	101.66	2.63	101.64	99.48	94.30	1.003	70.31	0.77	70.31	70.68	94.92	1.003
NM40.2	101.70	2.55	101.68	99.63	94.36	1.004	70.53	0.81	70.53	70.80	95.06	1.004
NM50.1	101.72	2.74	101.70	99.54	94.42	1.003	70.33	0.78	70.33	70.70	94.78	1.003
NM50.2	101.76	2.74	101.73	99.85	94.26	1.007	70.62	0.85	70.63	70.93	94.68	1.006
NM60.1	101.75	2.73	101.73	99.55	94.36	1.004	70.26	0.73	70.26	70.71	94.86	1.003
NM60.2	101.91	2.70	101.89	100.08	94.34	1.009	70.77	0.77	70.78	71.08	94.82	1.008
NM70.1	101.74	2.76	101.71	99.57	94.24	1.004	70.30	0.81	70.30	70.73	94.70	1.003
NM70.2	101.96	2.77	101.93	100.38	94.44	1.012	70.82	0.95	70.83	71.32	94.60	1.012
NM80.1	101.84	2.74	101.81	99.58	94.40	1.004	70.33	0.86	70.33	70.75	94.76	1.003
NM80.2	102.65	2.93	102.62	100.76	94.50	1.016	71.35	1.05	71.35	71.58	94.60	1.015
NM90.1	101.88	2.86	101.85	99.61	94.36	1.004	70.41	0.83	70.41	70.74	94.76	1.003
NM90.2	102.80	3.03	102.76	101.19	94.36	1.020	71.86	1.11	71.86	71.87	94.62	1.019

Table E66: Inference for the mean μ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.60	0.08	31.61	31.61	94.86	1.000	21.92	-0.14	21.92	22.35	95.44	1.000
MI.C90	31.78	-0.09	31.79	31.61	94.72	1.000	22.04	-0.19	22.04	22.36	95.34	1.000
MI.C80	32.14	0.07	32.14	31.63	94.40	1.001	22.22	0.01	22.22	22.37	94.84	1.001
MI.D90	31.61	0.07	31.61	31.63	94.86	1.001	21.93	-0.13	21.93	22.36	95.42	1.001
MI.D80	31.60	0.06	31.60	31.65	94.86	1.001	21.93	-0.17	21.93	22.38	95.52	1.001
CENS	31.69	0.12	31.69	31.73	94.86	1.004	21.98	-0.07	21.99	22.44	95.50	1.004
NM10.1	31.61	0.07	31.61	31.62	95.00	1.000	21.92	-0.12	21.92	22.36	95.40	1.000
NM10.2	31.61	0.07	31.61	31.62	95.00	1.000	21.92	-0.12	21.92	22.36	95.40	1.000
NM20.1	31.63	0.08	31.63	31.64	94.86	1.001	21.93	-0.16	21.93	22.37	95.48	1.001
NM20.2	31.63	0.08	31.64	31.64	94.82	1.001	21.94	-0.16	21.94	22.38	95.50	1.001
NM30.1	31.64	0.12	31.64	31.66	94.76	1.002	21.95	-0.10	21.95	22.39	95.32	1.002
NM30.2	31.66	0.13	31.66	31.68	94.72	1.002	21.97	-0.10	21.97	22.40	95.14	1.002
NM40.1	31.65	0.11	31.65	31.67	94.98	1.002	21.96	-0.12	21.96	22.40	95.48	1.002
NM40.2	31.71	0.11	31.71	31.73	94.98	1.004	22.01	-0.13	22.01	22.44	95.58	1.004
NM50.1	31.70	0.10	31.70	31.68	94.86	1.002	21.96	-0.09	21.96	22.41	95.48	1.002
NM50.2	31.83	0.10	31.83	31.78	94.70	1.006	22.02	-0.06	22.02	22.48	95.36	1.006
NM60.1	31.68	0.12	31.68	31.69	94.82	1.003	21.98	-0.11	21.99	22.41	95.42	1.003
NM60.2	31.85	0.17	31.85	31.86	94.94	1.008	22.05	-0.08	22.05	22.53	95.14	1.008
NM70.1	31.65	0.09	31.65	31.69	94.90	1.003	21.94	-0.11	21.94	22.41	95.46	1.003
NM70.2	31.86	0.10	31.86	31.95	95.02	1.011	22.05	-0.09	22.05	22.60	95.42	1.011
NM80.1	31.70	0.14	31.70	31.70	94.78	1.003	21.97	-0.06	21.98	22.42	95.40	1.003
NM80.2	32.10	0.16	32.10	32.07	94.88	1.015	22.16	0.02	22.16	22.68	95.32	1.015
NM90.1	31.62	0.11	31.63	31.70	94.84	1.003	21.97	-0.09	21.97	22.42	95.56	1.003
NM90.2	32.06	0.11	32.07	32.19	95.18	1.018	22.24	-0.02	22.24	22.77	95.38	1.019

Table E67: Inference for the variance σ^2 based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	139.98	-11.13	139.55	139.85	93.48	1.000	98.85	-3.48	98.80	99.65	94.50	1.000
MI.C90	150.43	-15.42	149.65	139.76	91.50	0.999	105.98	-5.82	105.83	99.68	92.72	1.000
MI.C80	155.31	-7.17	155.16	141.46	91.18	1.012	109.23	-1.21	109.23	100.41	92.52	1.008
MI.D90	51050.17	1631.16	51029.21	2402.43	94.90	17.179	99.49	0.42	99.50	100.43	94.68	1.008
MI.D80	146.44	-0.72	146.45	143.31	94.28	1.025	99.86	0.75	99.87	100.67	94.76	1.010
CENS	146.87	-7.60	146.69	146.37	93.18	1.047	103.40	-1.71	103.40	104.01	94.46	1.044
NM10.1	140.26	-11.02	139.84	140.31	93.42	1.003	99.40	-3.25	99.35	100.00	94.56	1.003
NM10.2	140.27	-11.01	139.85	140.33	93.42	1.003	99.37	-3.26	99.33	100.01	94.52	1.004
NM20.1	141.68	-10.65	141.30	141.36	93.14	1.011	99.99	-2.73	99.97	100.76	94.52	1.011
NM20.2	141.92	-10.57	141.54	141.51	93.12	1.012	100.03	-2.78	100.00	100.86	94.46	1.012
NM30.1	142.80	-9.31	142.51	142.54	93.36	1.019	100.48	-2.94	100.44	101.43	94.52	1.018
NM30.2	143.24	-9.27	142.95	143.05	93.40	1.023	100.85	-3.03	100.82	101.78	94.62	1.021
NM40.1	143.95	-9.41	143.65	143.22	93.36	1.024	100.93	-2.56	100.91	101.96	94.32	1.023
NM40.2	145.31	-9.67	145.01	144.40	93.36	1.033	101.79	-2.46	101.77	102.84	94.24	1.032
NM50.1	144.32	-8.89	144.06	143.77	93.26	1.028	101.71	-2.49	101.69	102.29	94.48	1.026
NM50.2	146.69	-8.88	146.43	146.07	93.06	1.044	103.53	-2.23	103.51	103.95	94.06	1.043
NM60.1	144.86	-8.93	144.60	144.08	93.34	1.030	102.10	-2.70	102.08	102.48	94.32	1.028
NM60.2	148.81	-9.04	148.55	147.71	93.10	1.056	104.99	-2.68	104.97	105.09	94.22	1.055
NM70.1	144.86	-8.84	144.60	144.31	93.20	1.032	101.97	-2.33	101.95	102.67	94.56	1.030
NM70.2	150.73	-8.70	150.50	149.52	92.86	1.069	105.81	-1.94	105.80	106.42	94.24	1.068
NM80.1	144.74	-8.91	144.48	144.47	93.22	1.033	102.05	-2.08	102.04	102.81	94.24	1.032
NM80.2	151.27	-8.32	151.06	151.34	93.28	1.082	107.12	-1.54	107.12	107.70	94.08	1.081
NM90.1	145.07	-8.38	144.84	144.69	92.94	1.035	102.21	-2.25	102.19	102.88	94.68	1.032
NM90.2	153.47	-7.89	153.28	153.06	93.22	1.094	108.08	-1.67	108.08	108.86	94.14	1.092

Table E68: Inference for the variance σ^2 based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	44.83	-0.51	44.83	44.70	95.00	1.000	31.64	-0.53	31.64	31.61	94.88	1.000
MI.C90	47.61	-0.92	47.61	44.76	93.02	1.001	33.72	-0.56	33.72	31.66	93.84	1.002
MI.C80	48.80	-0.23	48.81	44.88	92.90	1.004	34.86	0.17	34.86	31.74	92.84	1.004
MI.D90	44.99	0.14	45.00	44.84	94.84	1.003	31.64	-0.18	31.64	31.69	94.98	1.003
MI.D80	44.97	0.25	44.98	44.91	95.06	1.005	31.82	-0.23	31.82	31.74	94.94	1.004
CENS	46.45	-0.32	46.45	46.54	94.96	1.041	33.06	-0.21	33.07	32.91	95.12	1.041
NM10.1	44.88	-0.55	44.88	44.84	94.78	1.003	31.72	-0.47	31.72	31.71	94.92	1.003
NM10.2	44.90	-0.56	44.90	44.85	94.78	1.003	31.73	-0.47	31.73	31.72	94.84	1.003
NM20.1	45.14	-0.52	45.14	45.16	94.90	1.010	31.92	-0.60	31.92	31.93	94.98	1.010
NM20.2	45.25	-0.51	45.25	45.21	94.90	1.011	31.97	-0.63	31.97	31.96	95.00	1.011
NM30.1	45.64	-0.33	45.64	45.48	94.58	1.017	32.15	-0.36	32.15	32.16	94.92	1.017
NM30.2	45.91	-0.26	45.91	45.64	94.62	1.021	32.27	-0.37	32.27	32.27	94.92	1.021
NM40.1	45.77	-0.36	45.78	45.69	94.74	1.022	32.30	-0.44	32.30	32.30	95.16	1.022
NM40.2	46.29	-0.36	46.29	46.08	94.82	1.031	32.57	-0.47	32.57	32.58	95.32	1.031
NM50.1	45.78	-0.41	45.78	45.83	95.06	1.025	32.47	-0.32	32.47	32.41	94.98	1.025
NM50.2	46.61	-0.40	46.61	46.56	94.82	1.042	32.98	-0.20	32.98	32.93	95.08	1.042
NM60.1	45.87	-0.29	45.87	45.93	94.82	1.027	32.55	-0.38	32.55	32.47	95.02	1.027
NM60.2	47.08	-0.14	47.08	47.11	94.72	1.054	33.57	-0.29	33.57	33.30	95.12	1.054
NM70.1	46.02	-0.45	46.02	45.98	94.82	1.029	32.53	-0.39	32.53	32.51	95.02	1.029
NM70.2	47.96	-0.41	47.96	47.65	94.68	1.066	33.72	-0.33	33.72	33.69	95.10	1.066
NM80.1	46.13	-0.23	46.13	46.04	94.78	1.030	32.72	-0.18	32.72	32.56	95.14	1.030
NM80.2	48.59	-0.10	48.60	48.22	94.96	1.079	34.46	0.05	34.46	34.10	94.86	1.079
NM90.1	46.00	-0.34	46.01	46.08	94.70	1.031	32.67	-0.31	32.67	32.58	95.06	1.031
NM90.2	48.97	-0.37	48.97	48.73	95.02	1.090	34.65	-0.10	34.65	34.47	94.86	1.090

Table E69: Inference for the 84th percentile $\mu + \sigma$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	121.99	-5.79	121.86	121.49	94.46	1.000	84.84	-2.39	84.82	86.35	95.82	1.000
MI.C90	128.88	-10.58	128.46	121.34	92.84	0.999	89.28	-4.93	89.16	86.34	94.32	1.000
MI.C80	133.19	-4.76	133.12	122.05	92.30	1.005	92.06	-1.69	92.05	86.68	93.80	1.004
MI.D90	223.91	10.47	223.69	134.19	95.26	1.105	85.08	-0.75	85.08	86.69	95.68	1.004
MI.D80	122.25	-1.98	122.25	122.65	94.96	1.010	85.20	-0.79	85.21	86.85	95.80	1.006
CENS	126.61	-3.51	126.57	125.20	94.44	1.031	87.07	-1.26	87.07	88.86	95.60	1.029
NM10.1	122.23	-5.73	122.11	121.76	94.60	1.002	85.06	-2.25	85.04	86.55	95.64	1.002
NM10.2	122.20	-5.71	122.08	121.77	94.60	1.002	85.05	-2.25	85.03	86.56	95.66	1.002
NM20.1	123.18	-5.50	123.07	122.36	94.36	1.007	85.46	-1.89	85.45	86.99	95.40	1.007
NM20.2	123.14	-5.45	123.03	122.48	94.48	1.008	85.59	-1.93	85.58	87.06	95.28	1.008
NM30.1	123.90	-4.59	123.83	123.03	94.36	1.013	85.76	-2.05	85.74	87.38	95.68	1.012
NM30.2	123.96	-4.57	123.89	123.41	94.40	1.016	86.18	-2.13	86.16	87.65	95.72	1.015
NM40.1	124.47	-4.70	124.39	123.42	94.38	1.016	85.84	-1.79	85.83	87.69	95.76	1.016
NM40.2	125.19	-4.96	125.11	124.32	94.38	1.023	86.70	-1.72	86.69	88.34	95.52	1.023
NM50.1	124.78	-4.34	124.71	123.73	94.42	1.018	86.21	-1.76	86.21	87.87	95.48	1.018
NM50.2	125.93	-4.43	125.87	125.48	94.46	1.033	87.72	-1.61	87.71	89.13	95.50	1.032
NM60.1	125.08	-4.40	125.02	123.91	94.40	1.020	86.18	-1.93	86.16	87.99	95.50	1.019
NM60.2	127.31	-4.62	127.24	126.73	94.32	1.043	88.63	-1.96	88.62	90.01	95.22	1.042
NM70.1	125.06	-4.32	125.00	124.04	94.52	1.021	86.24	-1.66	86.24	88.10	95.34	1.020
NM70.2	128.37	-4.46	128.30	128.16	94.46	1.055	89.16	-1.42	89.15	91.05	95.04	1.054
NM80.1	125.23	-4.37	125.17	124.14	94.40	1.022	86.35	-1.49	86.35	88.17	95.58	1.021
NM80.2	129.88	-4.12	129.82	129.71	94.52	1.068	90.78	-1.15	90.79	92.14	95.24	1.067
NM90.1	125.46	-4.00	125.41	124.25	94.44	1.023	86.62	-1.61	86.62	88.22	95.60	1.022
NM90.2	131.21	-3.89	131.16	131.30	94.46	1.081	92.26	-1.19	92.26	93.25	95.30	1.080

Table E70: Inference for the 84th percentile $\mu + \sigma$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	38.77	-0.43	38.77	38.71	94.58	1.000	27.08	-0.53	27.07	27.38	95.46	1.000
MI.C90	40.41	-0.87	40.40	38.75	93.76	1.001	28.28	-0.63	28.27	27.41	94.54	1.001
MI.C80	41.65	-0.42	41.65	38.82	92.96	1.003	29.17	-0.10	29.18	27.46	93.66	1.003
MI.D90	38.85	-0.17	38.85	38.79	94.52	1.002	27.09	-0.37	27.09	27.43	95.34	1.002
MI.D80	38.81	-0.16	38.81	38.86	94.66	1.004	27.18	-0.45	27.18	27.47	95.44	1.004
CENS	39.65	-0.31	39.66	39.79	94.78	1.028	27.84	-0.31	27.84	28.14	95.50	1.028
NM10.1	38.81	-0.46	38.81	38.80	94.68	1.002	27.12	-0.48	27.12	27.44	95.54	1.002
NM10.2	38.82	-0.47	38.82	38.80	94.58	1.002	27.12	-0.48	27.12	27.44	95.46	1.002
NM20.1	38.97	-0.44	38.98	38.98	94.90	1.007	27.22	-0.59	27.22	27.57	95.42	1.007
NM20.2	39.03	-0.43	39.04	39.02	94.90	1.008	27.27	-0.60	27.26	27.59	95.58	1.008
NM30.1	39.19	-0.31	39.19	39.17	94.44	1.012	27.37	-0.41	27.37	27.70	95.54	1.012
NM30.2	39.36	-0.26	39.36	39.29	94.32	1.015	27.46	-0.42	27.46	27.78	95.66	1.015
NM40.1	39.28	-0.34	39.28	39.29	94.78	1.015	27.46	-0.47	27.46	27.79	95.52	1.015
NM40.2	39.63	-0.34	39.63	39.59	94.82	1.023	27.68	-0.49	27.68	27.99	95.44	1.023
NM50.1	39.41	-0.37	39.41	39.37	94.62	1.017	27.53	-0.39	27.53	27.85	95.54	1.017
NM50.2	40.07	-0.37	40.07	39.93	94.52	1.032	27.89	-0.29	27.89	28.24	95.64	1.032
NM60.1	39.40	-0.28	39.40	39.43	94.72	1.019	27.63	-0.43	27.63	27.88	95.60	1.019
NM60.2	40.31	-0.18	40.31	40.34	94.84	1.042	28.26	-0.37	28.26	28.53	95.06	1.042
NM70.1	39.39	-0.40	39.39	39.47	94.78	1.020	27.49	-0.43	27.49	27.91	95.80	1.020
NM70.2	40.74	-0.40	40.74	40.78	94.86	1.054	28.29	-0.40	28.29	28.84	95.80	1.054
NM80.1	39.56	-0.25	39.57	39.50	94.74	1.020	27.67	-0.29	27.67	27.93	95.66	1.020
NM80.2	41.53	-0.18	41.54	41.27	94.84	1.066	28.89	-0.10	28.89	29.19	95.58	1.066
NM90.1	39.30	-0.33	39.30	39.52	94.98	1.021	27.63	-0.38	27.63	27.95	95.50	1.021
NM90.2	41.60	-0.37	41.60	41.77	94.98	1.079	29.14	-0.21	29.14	29.54	95.46	1.079

Table E71: Inference for the 95th percentile $\mu + 1.645\sigma$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	152.16	-10.99	151.78	152.16	94.50	1.000	106.15	-4.30	106.07	108.14	95.60	1.000
MI.C90	163.50	-17.70	162.55	152.04	92.50	0.999	113.53	-7.86	113.27	108.18	93.60	1.000
MI.C80	169.45	-9.62	169.20	152.98	91.50	1.005	117.50	-3.33	117.46	108.65	93.00	1.005
MI.D90	347.35	15.89	347.02	171.14	95.40	1.125	106.50	-1.62	106.50	108.64	95.60	1.005
MI.D80	152.50	-4.68	152.44	153.76	94.80	1.011	106.75	-1.66	106.75	108.87	95.70	1.007
CENS	159.49	-7.71	159.32	158.28	94.40	1.040	110.08	-2.68	110.06	112.33	95.50	1.039
NM10.1	152.53	-10.90	152.16	152.61	94.50	1.003	106.56	-4.10	106.49	108.48	95.60	1.003
NM10.2	152.49	-10.88	152.12	152.64	94.50	1.003	106.54	-4.10	106.48	108.49	95.50	1.003
NM20.1	154.08	-10.58	153.73	153.62	94.40	1.010	107.20	-3.58	107.15	109.21	95.50	1.010
NM20.2	154.11	-10.52	153.77	153.79	94.30	1.011	107.35	-3.63	107.30	109.33	95.50	1.011
NM30.1	155.20	-9.26	154.94	154.69	94.30	1.017	107.70	-3.82	107.65	109.87	95.60	1.016
NM30.2	155.41	-9.24	155.15	155.29	94.60	1.021	108.30	-3.94	108.24	110.29	95.60	1.020
NM40.1	156.18	-9.43	155.91	155.36	94.30	1.021	107.92	-3.44	107.88	110.38	95.80	1.021
NM40.2	157.45	-9.81	157.16	156.76	94.20	1.030	109.16	-3.35	109.12	111.40	95.50	1.030
NM50.1	156.63	-8.91	156.40	155.85	94.40	1.024	108.60	-3.40	108.56	110.69	95.50	1.024
NM50.2	158.70	-9.05	158.46	158.57	94.40	1.042	110.88	-3.20	110.85	112.64	95.50	1.042
NM60.1	157.15	-8.99	156.91	156.16	94.40	1.026	108.68	-3.64	108.63	110.89	95.30	1.025
NM60.2	160.99	-9.35	160.74	160.50	94.00	1.055	112.38	-3.71	112.33	114.00	95.20	1.054
NM70.1	157.15	-8.89	156.92	156.38	94.30	1.028	108.72	-3.26	108.68	111.06	95.50	1.027
NM70.2	162.82	-9.12	162.58	162.67	94.30	1.069	113.23	-2.95	113.20	115.56	95.30	1.069
NM80.1	157.30	-8.96	157.06	156.54	94.40	1.029	108.85	-3.00	108.82	111.19	95.40	1.028
NM80.2	164.59	-8.67	164.38	164.96	94.50	1.084	115.44	-2.58	115.42	117.18	95.00	1.084
NM90.1	157.65	-8.42	157.44	156.72	94.40	1.030	109.22	-3.18	109.18	111.27	95.50	1.029
NM90.2	166.79	-8.35	166.59	167.24	94.40	1.099	117.37	-2.67	117.35	118.78	95.00	1.098

Table E72: Inference for the 95th percentile $\mu + 1.645\sigma$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 = 95$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	48.60	-0.76	48.59	48.48	94.70	1.000	34.09	-0.78	34.08	34.29	95.40	1.000
MI.C90	51.33	-1.38	51.32	48.55	93.50	1.001	36.11	-0.92	36.10	34.34	94.00	1.002
MI.C80	53.08	-0.74	53.08	48.66	92.90	1.004	37.47	-0.17	37.47	34.42	92.90	1.004
MI.D90	48.73	-0.32	48.73	48.61	94.60	1.003	34.10	-0.52	34.10	34.37	95.40	1.002
MI.D80	48.67	-0.30	48.68	48.71	94.60	1.005	34.25	-0.64	34.25	34.44	95.20	1.004
CENS	50.10	-0.59	50.11	50.30	95.30	1.037	35.40	-0.46	35.40	35.57	95.60	1.037
NM10.1	48.65	-0.80	48.65	48.63	94.80	1.003	34.16	-0.72	34.16	34.39	95.40	1.003
NM10.2	48.67	-0.81	48.67	48.63	94.70	1.003	34.17	-0.71	34.16	34.40	95.40	1.003
NM20.1	48.93	-0.77	48.93	48.94	94.80	1.010	34.34	-0.86	34.34	34.61	95.40	1.009
NM20.2	49.03	-0.76	49.03	49.00	94.60	1.011	34.41	-0.89	34.40	34.65	95.40	1.011
NM30.1	49.32	-0.58	49.32	49.25	94.70	1.016	34.59	-0.61	34.58	34.83	95.80	1.016
NM30.2	49.60	-0.51	49.60	49.44	94.70	1.020	34.72	-0.62	34.72	34.96	95.60	1.020
NM40.1	49.46	-0.62	49.46	49.46	95.00	1.020	34.73	-0.69	34.73	34.98	95.50	1.020
NM40.2	50.03	-0.63	50.03	49.92	94.80	1.030	35.07	-0.73	35.06	35.30	95.60	1.030
NM50.1	49.62	-0.67	49.62	49.60	94.60	1.023	34.87	-0.57	34.87	35.07	95.60	1.023
NM50.2	50.63	-0.67	50.63	50.46	94.70	1.041	35.44	-0.44	35.44	35.69	95.80	1.041
NM60.1	49.64	-0.55	49.64	49.69	95.00	1.025	35.01	-0.64	35.00	35.14	95.20	1.025
NM60.2	51.04	-0.40	51.05	51.10	95.20	1.054	36.04	-0.55	36.04	36.13	95.20	1.054
NM70.1	49.67	-0.72	49.67	49.75	94.80	1.026	34.83	-0.65	34.83	35.18	95.70	1.026
NM70.2	51.82	-0.72	51.82	51.76	94.90	1.068	36.12	-0.60	36.12	36.61	95.90	1.068
NM80.1	49.90	-0.49	49.90	49.81	95.10	1.027	35.10	-0.43	35.10	35.22	95.40	1.027
NM80.2	52.90	-0.41	52.90	52.49	95.20	1.083	37.02	-0.18	37.03	37.12	95.30	1.083
NM90.1	49.56	-0.61	49.56	49.84	94.90	1.028	35.04	-0.57	35.04	35.25	95.60	1.028
NM90.2	53.09	-0.68	53.09	53.20	95.00	1.097	37.36	-0.34	37.37	37.63	95.20	1.098

Table E73: Inference for $\mu + \sigma^2/2$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.645 =$ 95th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	121.69	-3.30	121.65	121.43	94.40	1.000	84.77	-1.16	84.78	86.35	95.80	1.000
MI.C90	128.56	-7.24	128.37	121.30	92.90	0.999	89.19	-3.30	89.14	86.35	94.30	1.000
MI.C80	133.34	-0.82	133.35	122.27	92.40	1.007	92.13	0.23	92.14	86.79	93.80	1.005
MI.D90	25524.55	817.63	25514.01	1252.11	95.40	10.311	85.14	0.81	85.15	86.80	95.70	1.005
MI.D80	123.47	1.86	123.47	123.33	95.00	1.016	85.30	0.94	85.30	86.98	95.80	1.007
CENS	126.54	-0.78	126.55	125.29	94.40	1.032	87.06	0.08	87.07	88.93	95.60	1.030
NM10.1	121.94	-3.22	121.91	121.71	94.60	1.002	84.99	-1.01	84.99	86.56	95.70	1.002
NM10.2	121.90	-3.21	121.87	121.72	94.70	1.002	84.98	-1.01	84.99	86.57	95.70	1.002
NM20.1	122.89	-2.94	122.87	122.33	94.40	1.007	85.42	-0.63	85.42	87.01	95.40	1.008
NM20.2	122.86	-2.89	122.83	122.44	94.50	1.008	85.55	-0.67	85.56	87.09	95.40	1.008
NM30.1	123.67	-2.00	123.66	123.03	94.50	1.013	85.70	-0.78	85.71	87.41	95.60	1.012
NM30.2	123.73	-1.97	123.73	123.41	94.50	1.016	86.13	-0.85	86.14	87.67	95.60	1.015
NM40.1	124.28	-2.07	124.28	123.43	94.50	1.016	85.80	-0.51	85.80	87.72	95.80	1.016
NM40.2	125.01	-2.29	125.00	124.32	94.50	1.024	86.67	-0.42	86.68	88.38	95.60	1.023
NM50.1	124.62	-1.70	124.63	123.75	94.40	1.019	86.18	-0.47	86.19	87.91	95.50	1.018
NM50.2	125.79	-1.70	125.79	125.50	94.60	1.034	87.70	-0.26	87.71	89.17	95.50	1.033
NM60.1	124.93	-1.74	124.93	123.94	94.40	1.021	86.14	-0.62	86.15	88.03	95.50	1.019
NM60.2	127.13	-1.82	127.13	126.77	94.30	1.044	88.60	-0.57	88.61	90.05	95.20	1.043
NM70.1	124.89	-1.66	124.89	124.08	94.50	1.022	86.20	-0.36	86.20	88.14	95.40	1.021
NM70.2	128.21	-1.58	128.22	128.22	94.50	1.056	89.15	-0.02	89.16	91.11	95.10	1.055
NM80.1	125.08	-1.71	125.08	124.18	94.50	1.023	86.32	-0.18	86.33	88.22	95.60	1.022
NM80.2	129.80	-1.23	129.80	129.79	94.70	1.069	90.79	0.28	90.79	92.21	95.30	1.068
NM90.1	125.34	-1.33	125.35	124.30	94.50	1.024	86.60	-0.30	86.61	88.27	95.60	1.022
NM90.2	131.19	-0.91	131.20	131.41	94.70	1.082	92.29	0.27	92.30	93.33	95.40	1.081

Table E74: Inference for $\mu+\sigma^2/2$ based on $N(\mu=0,\sigma^2=1)$ data with $C=1.645=$ 95th percentile

	Results for $n=1000$						Results for $n=2000$					
	RMSE $\times10^3$	Bias $\times10^3$	SD $\times10^3$	$\widehat{\text{SD}}$ $\times10^3$	Cvg. %	Rel. Len.	RMSE $\times10^3$	Bias $\times10^3$	SD $\times10^3$	$\widehat{\text{SD}}$ $\times10^3$	Cvg. %	Rel. Len.
CD	38.77	-0.18	38.77	38.71	94.50	1.000	27.07	-0.40	27.07	27.37	95.40	1.000
MI.C90	40.41	-0.55	40.41	38.75	93.70	1.001	28.27	-0.47	28.27	27.41	94.60	1.001
MI.C80	41.67	-0.04	41.67	38.83	93.00	1.003	29.18	0.10	29.18	27.47	93.70	1.003
MI.D90	38.86	0.14	38.86	38.80	94.50	1.002	27.10	-0.22	27.10	27.43	95.40	1.002
MI.D80	38.82	0.18	38.83	38.87	94.60	1.004	27.18	-0.28	27.18	27.48	95.50	1.004
CENS	39.66	-0.04	39.66	39.80	94.70	1.028	27.84	-0.17	27.84	28.14	95.50	1.028
NM10.1	38.80	-0.21	38.81	38.80	94.70	1.002	27.12	-0.36	27.12	27.44	95.50	1.002
NM10.2	38.81	-0.21	38.82	38.80	94.60	1.002	27.12	-0.35	27.12	27.44	95.50	1.002
NM20.1	38.97	-0.18	38.97	38.99	94.90	1.007	27.22	-0.46	27.22	27.57	95.40	1.007
NM20.2	39.03	-0.17	39.03	39.02	94.90	1.008	27.26	-0.48	27.26	27.59	95.60	1.008
NM30.1	39.19	-0.05	39.20	39.17	94.40	1.012	27.37	-0.28	27.37	27.70	95.60	1.012
NM30.2	39.36	0.00	39.37	39.30	94.30	1.015	27.46	-0.29	27.46	27.79	95.60	1.015
NM40.1	39.28	-0.07	39.28	39.30	94.80	1.015	27.46	-0.34	27.46	27.79	95.60	1.015
NM40.2	39.64	-0.07	39.64	39.59	94.80	1.023	27.68	-0.36	27.68	27.99	95.50	1.023
NM50.1	39.41	-0.10	39.42	39.38	94.70	1.017	27.53	-0.25	27.53	27.85	95.50	1.017
NM50.2	40.08	-0.10	40.08	39.94	94.50	1.032	27.89	-0.16	27.89	28.24	95.60	1.032
NM60.1	39.40	-0.02	39.41	39.44	94.70	1.019	27.63	-0.30	27.63	27.88	95.60	1.019
NM60.2	40.31	0.10	40.32	40.35	94.80	1.042	28.26	-0.23	28.26	28.53	95.00	1.042
NM70.1	39.39	-0.14	39.39	39.47	94.80	1.020	27.49	-0.30	27.49	27.91	95.70	1.020
NM70.2	40.74	-0.11	40.75	40.79	94.80	1.054	28.29	-0.26	28.30	28.84	95.80	1.054
NM80.1	39.57	0.02	39.57	39.51	94.80	1.020	27.67	-0.15	27.67	27.94	95.70	1.020
NM80.2	41.54	0.11	41.54	41.28	94.90	1.066	28.90	0.04	28.90	29.19	95.60	1.066
NM90.1	39.30	-0.06	39.31	39.53	95.00	1.021	27.63	-0.25	27.63	27.95	95.40	1.021
NM90.2	41.60	-0.07	41.61	41.77	94.90	1.079	29.13	-0.06	29.14	29.54	95.50	1.079

Table E75: Inference for the mean μ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	98.01	-1.17	98.01	99.14	94.72	1.000	69.97	-2.11	69.95	70.45	94.88	1.000
MI.C90	99.75	-1.86	99.74	99.22	94.40	1.001	71.47	-2.37	71.44	70.50	94.44	1.001
MI.C80	102.31	3.72	102.25	99.95	93.78	1.008	72.34	0.02	72.35	70.79	94.42	1.005
MI.D90	98.21	-1.14	98.22	99.74	94.88	1.006	70.02	-2.13	70.00	70.67	94.98	1.003
MI.D80	98.18	-1.28	98.19	99.96	94.90	1.008	70.01	-2.11	69.99	70.83	94.88	1.005
CENS	98.97	-0.26	98.98	100.35	94.98	1.012	70.66	-1.76	70.64	71.21	94.96	1.011
NM10.1	98.13	-1.08	98.13	99.21	94.58	1.001	70.02	-2.07	70.00	70.49	94.84	1.001
NM10.2	98.12	-1.09	98.13	99.21	94.56	1.001	70.04	-2.07	70.01	70.49	94.86	1.001
NM20.1	98.12	-0.99	98.13	99.35	94.98	1.002	70.11	-2.09	70.08	70.57	94.82	1.002
NM20.2	98.15	-0.99	98.15	99.37	94.90	1.002	70.14	-2.11	70.12	70.58	94.74	1.002
NM30.1	98.19	-0.95	98.19	99.50	94.86	1.004	70.14	-2.03	70.12	70.68	94.98	1.003
NM30.2	98.15	-1.02	98.16	99.56	94.78	1.004	70.24	-1.95	70.22	70.73	95.00	1.004
NM40.1	98.40	-0.68	98.41	99.68	94.70	1.005	70.24	-2.05	70.22	70.77	94.92	1.005
NM40.2	98.57	-0.73	98.58	99.84	94.62	1.007	70.43	-2.05	70.41	70.88	94.88	1.006
NM50.1	98.68	-0.78	98.69	99.75	94.78	1.006	70.25	-2.19	70.22	70.80	94.84	1.005
NM50.2	99.11	-0.81	99.12	100.09	94.74	1.010	70.50	-2.30	70.47	71.04	95.04	1.008
NM60.1	98.52	-0.75	98.53	99.81	94.78	1.007	70.42	-1.80	70.41	70.91	94.94	1.007
NM60.2	99.19	-0.97	99.19	100.42	95.00	1.013	71.00	-1.65	70.99	71.38	94.76	1.013
NM70.1	98.82	-0.49	98.83	99.91	94.84	1.008	70.29	-1.96	70.27	70.92	94.88	1.007
NM70.2	99.42	-0.65	99.42	100.92	94.76	1.018	71.03	-1.94	71.01	71.66	94.90	1.017
NM80.1	98.57	-0.55	98.58	99.94	94.70	1.008	70.43	-1.93	70.41	70.95	94.96	1.007
NM80.2	99.93	-0.64	99.94	101.45	94.74	1.023	71.54	-2.03	71.52	72.03	94.82	1.022
NM90.1	98.49	-0.63	98.50	99.95	94.88	1.008	70.47	-1.84	70.45	70.98	94.86	1.008
NM90.2	99.92	-0.92	99.92	102.05	95.08	1.029	72.32	-1.86	72.31	72.50	94.68	1.029

Table E76: Inference for the mean μ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	31.40	-0.27	31.40	31.60	95.04	1.000	22.29	-0.02	22.29	22.34	95.20	1.000
MI.C90	31.87	-0.22	31.87	31.63	94.62	1.001	22.59	-0.11	22.59	22.36	94.62	1.001
MI.C80	32.52	0.25	32.52	31.69	94.16	1.003	23.01	0.20	23.01	22.39	94.32	1.002
MI.D90	31.42	-0.24	31.43	31.64	95.22	1.001	22.31	-0.04	22.31	22.37	94.98	1.001
MI.D80	31.51	-0.31	31.51	31.70	95.08	1.003	22.33	-0.02	22.34	22.41	95.00	1.003
CENS	31.69	-0.13	31.69	31.93	95.00	1.010	22.50	0.03	22.50	22.57	94.88	1.010
NM10.1	31.44	-0.29	31.44	31.61	95.02	1.000	22.30	-0.03	22.30	22.36	95.10	1.000
NM10.2	31.44	-0.29	31.44	31.62	95.02	1.000	22.30	-0.03	22.30	22.36	95.10	1.001
NM20.1	31.48	-0.27	31.48	31.65	95.16	1.002	22.30	-0.02	22.31	22.38	95.10	1.002
NM20.2	31.48	-0.27	31.48	31.66	95.12	1.002	22.31	-0.02	22.31	22.39	95.04	1.002
NM30.1	31.47	-0.23	31.47	31.70	95.04	1.003	22.37	-0.01	22.37	22.42	95.28	1.003
NM30.2	31.50	-0.24	31.50	31.72	94.92	1.004	22.39	-0.03	22.40	22.43	95.24	1.004
NM40.1	31.55	-0.25	31.55	31.74	95.16	1.004	22.40	0.03	22.41	22.44	95.06	1.004
NM40.2	31.59	-0.23	31.59	31.79	95.00	1.006	22.43	0.02	22.44	22.48	95.14	1.006
NM50.1	31.58	-0.18	31.58	31.77	94.98	1.005	22.44	0.01	22.45	22.46	95.02	1.005
NM50.2	31.71	-0.17	31.71	31.89	94.92	1.009	22.54	-0.01	22.55	22.54	95.00	1.009
NM60.1	31.56	-0.25	31.57	31.79	95.06	1.006	22.43	0.00	22.43	22.48	95.02	1.006
NM60.2	31.78	-0.25	31.78	31.99	95.06	1.012	22.54	-0.02	22.54	22.62	95.26	1.012
NM70.1	31.55	-0.17	31.55	31.81	95.24	1.007	22.44	0.01	22.44	22.49	95.20	1.006
NM70.2	31.87	-0.14	31.88	32.14	95.14	1.017	22.65	-0.06	22.65	22.72	95.22	1.017
NM80.1	31.55	-0.19	31.56	31.82	95.14	1.007	22.45	0.01	22.45	22.50	94.90	1.007
NM80.2	32.04	-0.06	32.04	32.31	94.90	1.023	22.75	-0.08	22.75	22.84	95.42	1.022
NM90.1	31.61	-0.16	31.61	31.83	95.06	1.007	22.41	0.03	22.41	22.50	95.10	1.007
NM90.2	32.24	-0.03	32.25	32.52	95.00	1.029	22.83	-0.05	22.83	22.99	95.20	1.029

Table E77: Inference for the variance σ^2 based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	142.17	-11.97	141.68	139.73	92.68	1.000	99.98	-4.90	99.87	99.51	94.20	1.000
MI.C90	155.16	-10.84	154.80	140.92	91.12	1.009	109.31	-4.12	109.24	100.12	92.26	1.006
MI.C80	156.75	0.73	156.76	143.51	91.44	1.027	109.86	1.25	109.86	101.16	92.52	1.017
MI.D90	145.12	-2.03	145.12	142.64	93.30	1.021	100.80	-0.66	100.81	100.53	94.10	1.010
MI.D80	144.17	-1.47	144.18	142.86	93.52	1.022	100.79	0.39	100.80	100.88	94.44	1.014
CENS	154.12	-7.99	153.93	152.24	92.98	1.090	107.73	-3.31	107.69	108.00	94.06	1.085
NM10.1	142.67	-11.54	142.21	140.36	92.76	1.005	100.56	-4.70	100.46	99.94	93.84	1.004
NM10.2	142.64	-11.56	142.18	140.37	92.76	1.005	100.55	-4.70	100.45	99.95	93.84	1.004
NM20.1	143.73	-11.16	143.31	141.82	92.94	1.015	101.28	-4.69	101.18	100.94	93.94	1.014
NM20.2	143.80	-11.15	143.38	141.90	92.88	1.016	101.33	-4.73	101.23	100.99	94.10	1.015
NM30.1	144.80	-10.88	144.40	143.49	93.12	1.027	102.66	-4.53	102.57	102.12	93.92	1.026
NM30.2	145.16	-11.05	144.75	143.77	93.10	1.029	102.92	-4.33	102.84	102.35	93.96	1.029
NM40.1	146.57	-9.81	146.26	145.09	93.04	1.038	102.94	-4.49	102.85	103.11	93.84	1.036
NM40.2	147.21	-9.92	146.89	145.87	93.06	1.044	103.38	-4.51	103.29	103.68	94.04	1.042
NM50.1	148.10	-10.19	147.76	146.08	92.80	1.045	104.19	-5.12	104.08	103.77	93.78	1.043
NM50.2	149.44	-10.30	149.10	147.76	93.22	1.057	105.36	-5.41	105.23	104.96	93.80	1.055
NM60.1	149.12	-10.06	148.79	146.82	93.16	1.051	104.17	-3.57	104.12	104.47	93.90	1.050
NM60.2	151.85	-10.64	151.49	149.78	93.08	1.072	106.21	-3.16	106.17	106.70	94.20	1.072
NM70.1	148.79	-9.06	148.53	147.51	93.04	1.056	104.54	-4.13	104.47	104.76	93.98	1.053
NM70.2	152.63	-9.54	152.34	152.19	92.94	1.089	108.24	-3.92	108.18	108.20	94.18	1.087
NM80.1	150.20	-9.27	149.92	147.89	93.00	1.058	104.24	-4.05	104.18	105.06	94.32	1.056
NM80.2	156.29	-9.55	156.02	154.48	93.48	1.106	108.49	-4.27	108.41	109.77	93.92	1.103
NM90.1	149.59	-9.61	149.30	148.15	92.90	1.060	105.40	-3.61	105.35	105.33	94.06	1.058
NM90.2	156.95	-10.29	156.63	156.58	92.66	1.121	111.71	-3.66	111.66	111.42	94.04	1.120

Table E78: Inference for the variance σ^2 based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	44.16	-0.92	44.15	44.68	95.12	1.000	31.21	-1.17	31.19	31.59	95.18	1.000
MI.C90	48.21	-0.41	48.21	44.87	93.40	1.004	34.18	-1.24	34.16	31.70	92.88	1.004
MI.C80	48.89	0.69	48.89	45.09	92.94	1.009	34.48	-0.60	34.48	31.83	92.84	1.008
MI.D90	44.42	-0.06	44.43	44.90	95.28	1.005	31.31	-0.84	31.30	31.72	95.32	1.004
MI.D80	44.44	0.04	44.45	44.99	95.48	1.007	31.37	-0.67	31.37	31.78	95.44	1.006
CENS	47.80	-0.32	47.80	48.41	95.52	1.083	34.01	-0.94	34.00	34.20	95.12	1.083
NM10.1	44.37	-0.99	44.36	44.86	95.24	1.004	31.36	-1.20	31.34	31.72	95.18	1.004
NM10.2	44.37	-1.00	44.36	44.86	95.30	1.004	31.36	-1.20	31.34	31.72	95.18	1.004
NM20.1	44.77	-0.92	44.76	45.31	95.36	1.014	31.63	-1.17	31.61	32.03	95.22	1.014
NM20.2	44.80	-0.92	44.80	45.34	95.36	1.015	31.65	-1.18	31.63	32.05	95.22	1.015
NM30.1	45.34	-0.75	45.34	45.84	95.10	1.026	32.13	-1.14	32.12	32.40	95.18	1.026
NM30.2	45.45	-0.78	45.45	45.94	95.08	1.028	32.20	-1.18	32.18	32.47	95.18	1.028
NM40.1	45.56	-0.82	45.56	46.29	95.32	1.036	32.36	-0.92	32.35	32.73	95.04	1.036
NM40.2	45.74	-0.78	45.73	46.54	95.36	1.042	32.55	-0.96	32.54	32.91	94.86	1.042
NM50.1	46.31	-0.50	46.32	46.63	95.04	1.044	32.65	-1.03	32.64	32.95	95.20	1.043
NM50.2	46.89	-0.49	46.90	47.17	95.08	1.056	33.01	-1.10	32.99	33.34	95.06	1.055
NM60.1	46.18	-0.82	46.18	46.84	95.32	1.048	32.85	-1.06	32.83	33.11	95.04	1.048
NM60.2	47.00	-0.84	47.00	47.82	95.20	1.070	33.65	-1.13	33.64	33.81	94.88	1.070
NM70.1	46.38	-0.52	46.39	47.02	95.46	1.052	33.15	-1.02	33.14	33.23	95.10	1.052
NM70.2	48.11	-0.42	48.11	48.56	95.26	1.087	34.16	-1.19	34.14	34.31	94.88	1.086
NM80.1	46.53	-0.56	46.53	47.14	95.54	1.055	33.14	-1.02	33.12	33.31	95.18	1.055
NM80.2	48.34	-0.26	48.34	49.29	95.48	1.103	34.62	-1.29	34.60	34.82	94.88	1.102
NM90.1	46.66	-0.49	46.67	47.24	95.34	1.057	33.07	-0.93	33.06	33.39	95.18	1.057
NM90.2	49.63	-0.17	49.64	50.01	95.12	1.119	35.02	-1.10	35.01	35.33	94.72	1.119

Table E79: Inference for the 84th percentile $\mu+\sigma$ based on $N(\mu=0,\sigma^2=1)$ data with $C=1.282=$ 90th percentile

	Results for $n=100$						Results for $n=200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	121.81	-9.72	121.43	121.43	94.54	1.000	85.55	-5.82	85.36	86.28	95.08	1.000
MI.C90	131.25	-11.21	130.78	121.81	92.96	1.003	92.89	-6.36	92.68	86.55	92.76	1.003
MI.C80	134.70	-0.67	134.71	123.03	92.52	1.013	94.20	-1.71	94.19	87.14	92.76	1.010
MI.D90	122.31	-5.93	122.18	122.54	94.76	1.009	85.80	-4.21	85.71	86.79	95.18	1.006
MI.D80	122.49	-6.03	122.35	122.95	94.86	1.013	85.78	-3.87	85.70	87.11	95.22	1.010
CENS	128.83	-7.23	128.64	129.22	94.76	1.064	90.47	-4.87	90.35	91.64	94.94	1.062
NM10.1	122.29	-9.44	121.93	121.81	94.48	1.003	85.89	-5.70	85.71	86.55	94.74	1.003
NM10.2	122.27	-9.46	121.92	121.82	94.50	1.003	85.92	-5.69	85.74	86.55	94.72	1.003
NM20.1	122.72	-9.20	122.39	122.73	94.62	1.011	86.43	-5.73	86.25	87.18	94.88	1.010
NM20.2	122.81	-9.18	122.48	122.80	94.76	1.011	86.52	-5.77	86.34	87.23	94.86	1.011
NM30.1	123.38	-9.05	123.06	123.79	94.74	1.019	87.07	-5.62	86.90	87.93	94.94	1.019
NM30.2	123.53	-9.22	123.20	124.05	94.84	1.022	87.38	-5.45	87.22	88.13	94.94	1.021
NM40.1	124.48	-8.30	124.22	124.77	94.76	1.028	87.44	-5.62	87.27	88.57	95.04	1.026
NM40.2	125.19	-8.43	124.92	125.48	94.74	1.033	88.04	-5.65	87.86	89.08	95.10	1.032
NM50.1	125.87	-8.65	125.59	125.41	94.62	1.033	88.05	-6.12	87.84	89.00	94.90	1.032
NM50.2	127.36	-8.79	127.07	126.92	94.30	1.045	89.14	-6.40	88.92	90.07	94.96	1.044
NM60.1	125.84	-8.59	125.56	125.87	94.50	1.037	88.36	-4.95	88.23	89.41	94.96	1.036
NM60.2	128.57	-9.20	128.25	128.57	94.76	1.059	90.44	-4.65	90.33	91.38	95.08	1.059
NM70.1	126.40	-7.81	126.17	126.29	94.84	1.040	88.27	-5.40	88.11	89.61	94.80	1.039
NM70.2	129.51	-8.37	129.25	130.56	94.94	1.075	91.56	-5.38	91.41	92.70	94.58	1.074
NM80.1	126.29	-8.02	126.05	126.53	94.74	1.042	88.46	-5.32	88.31	89.80	94.84	1.041
NM80.2	131.93	-8.49	131.67	132.66	94.48	1.093	92.68	-5.64	92.52	94.17	94.90	1.091
NM90.1	125.93	-8.26	125.67	126.71	94.76	1.044	89.01	-5.04	88.88	89.96	94.82	1.043
NM90.2	132.46	-9.19	132.16	134.83	94.74	1.110	95.52	-5.26	95.39	95.77	94.72	1.110

Table E80: Inference for the 84th percentile $\mu + \sigma$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	38.39	-0.98	38.38	38.70	95.32	1.000	27.28	-0.73	27.27	27.37	95.20	1.000
MI.C90	41.21	-0.80	41.21	38.82	93.32	1.003	29.27	-0.92	29.25	27.44	93.56	1.003
MI.C80	42.55	0.13	42.56	39.00	92.80	1.008	30.06	-0.33	30.06	27.56	92.90	1.007
MI.D90	38.56	-0.61	38.56	38.85	95.36	1.004	27.36	-0.62	27.36	27.47	95.16	1.004
MI.D80	38.71	-0.67	38.71	38.98	95.20	1.007	27.42	-0.54	27.42	27.56	95.20	1.007
CENS	40.65	-0.58	40.65	41.08	95.18	1.061	28.98	-0.58	28.98	29.04	95.14	1.061
NM10.1	38.59	-1.03	38.58	38.82	95.34	1.003	27.37	-0.75	27.36	27.45	95.22	1.003
NM10.2	38.59	-1.04	38.58	38.82	95.30	1.003	27.37	-0.75	27.36	27.45	95.18	1.003
NM20.1	38.85	-0.98	38.84	39.10	95.42	1.010	27.49	-0.73	27.49	27.65	95.26	1.010
NM20.2	38.87	-0.98	38.86	39.13	95.40	1.011	27.51	-0.74	27.51	27.67	95.24	1.011
NM30.1	39.08	-0.87	39.07	39.44	95.30	1.019	27.86	-0.71	27.85	27.89	95.18	1.019
NM30.2	39.18	-0.89	39.17	39.53	95.30	1.021	27.94	-0.75	27.94	27.95	95.14	1.021
NM40.1	39.36	-0.92	39.36	39.73	95.14	1.026	28.04	-0.56	28.04	28.09	94.78	1.027
NM40.2	39.52	-0.88	39.51	39.96	95.18	1.032	28.20	-0.59	28.19	28.25	94.92	1.032
NM50.1	39.74	-0.69	39.74	39.94	94.96	1.032	28.28	-0.64	28.28	28.24	94.60	1.032
NM50.2	40.28	-0.69	40.28	40.43	94.96	1.045	28.64	-0.70	28.64	28.58	94.72	1.044
NM60.1	39.67	-0.92	39.66	40.08	95.12	1.036	28.31	-0.66	28.31	28.34	95.00	1.036
NM60.2	40.50	-0.95	40.49	40.96	95.02	1.058	28.92	-0.73	28.91	28.96	95.04	1.058
NM70.1	39.69	-0.70	39.69	40.19	95.42	1.038	28.48	-0.64	28.47	28.42	95.02	1.038
NM70.2	41.16	-0.64	41.16	41.58	95.28	1.074	29.37	-0.80	29.36	29.39	94.80	1.074
NM80.1	39.78	-0.74	39.78	40.27	95.26	1.041	28.49	-0.63	28.48	28.47	95.06	1.040
NM80.2	41.57	-0.48	41.58	42.26	95.00	1.092	29.78	-0.88	29.77	29.87	94.90	1.091
NM90.1	39.97	-0.68	39.97	40.34	95.32	1.042	28.37	-0.57	28.37	28.52	95.06	1.042
NM90.2	42.61	-0.43	42.61	42.97	95.30	1.110	30.11	-0.75	30.10	30.38	95.18	1.110

Table E81: Inference for the 95th percentile $\mu + 1.645\sigma$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	153.85	-15.24	153.11	152.08	93.90	1.000	107.48	-8.21	107.18	108.06	94.70	1.000
MI.C90	168.45	-17.25	167.58	152.68	92.00	1.004	118.68	-8.94	118.35	108.48	92.30	1.004
MI.C80	171.95	-3.51	171.93	154.29	91.70	1.015	120.05	-2.82	120.03	109.28	92.20	1.011
MI.D90	154.48	-9.02	154.24	153.62	94.10	1.010	107.90	-5.56	107.77	108.78	94.70	1.007
MI.D80	154.80	-9.10	154.55	154.12	94.30	1.013	107.85	-5.01	107.75	109.20	95.00	1.010
CENS	165.22	-11.73	164.82	164.68	94.20	1.083	115.48	-6.88	115.28	116.77	94.60	1.081
NM10.1	154.53	-14.84	153.83	152.71	93.90	1.004	108.05	-8.04	107.76	108.50	94.40	1.004
NM10.2	154.51	-14.85	153.81	152.72	93.90	1.004	108.08	-8.03	107.79	108.51	94.40	1.004
NM20.1	155.33	-14.49	154.67	154.19	93.90	1.014	108.90	-8.07	108.61	109.54	94.70	1.014
NM20.2	155.45	-14.47	154.79	154.30	93.90	1.015	109.02	-8.13	108.72	109.61	94.60	1.014
NM30.1	156.44	-14.28	155.81	155.92	94.30	1.025	110.05	-7.94	109.77	110.76	94.50	1.025
NM30.2	156.76	-14.52	156.10	156.31	94.20	1.028	110.47	-7.70	110.21	111.05	94.60	1.028
NM40.1	158.15	-13.22	157.61	157.50	94.20	1.036	110.56	-7.93	110.29	111.80	94.90	1.035
NM40.2	159.16	-13.40	158.61	158.55	94.40	1.043	111.37	-7.97	111.10	112.55	94.90	1.042
NM50.1	160.22	-13.73	159.65	158.55	94.00	1.042	111.68	-8.65	111.36	112.51	94.50	1.041
NM50.2	162.33	-13.94	161.74	160.79	93.80	1.057	113.30	-9.05	112.95	114.10	94.80	1.056
NM60.1	160.45	-13.64	159.88	159.30	94.30	1.047	111.97	-6.98	111.77	113.15	94.50	1.047
NM60.2	164.43	-14.52	163.80	163.28	94.10	1.074	114.94	-6.58	114.76	116.07	94.90	1.074
NM70.1	160.96	-12.53	160.49	159.94	94.20	1.052	112.02	-7.62	111.77	113.49	94.70	1.050
NM70.2	165.74	-13.35	165.22	166.23	94.30	1.093	116.89	-7.59	116.66	118.03	94.40	1.092
NM80.1	161.22	-12.84	160.73	160.35	94.30	1.054	112.13	-7.51	111.89	113.79	94.80	1.053
NM80.2	169.49	-13.55	168.96	169.28	94.50	1.113	118.21	-7.97	117.95	120.16	94.80	1.112
NM90.1	160.68	-13.18	160.15	160.64	94.20	1.056	113.10	-7.11	112.89	114.05	94.70	1.055
NM90.2	170.41	-14.52	169.81	172.36	94.40	1.133	122.31	-7.45	122.10	122.44	94.50	1.133

Table E82: Inference for the 95th percentile $\mu + 1.645\sigma$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 =$ 90th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	48.03	-1.43	48.01	48.47	95.50	1.000	34.10	-1.19	34.08	34.28	95.30	1.000
MI.C90	52.50	-1.18	52.49	48.66	92.80	1.004	37.29	-1.44	37.26	34.40	93.00	1.004
MI.C80	54.13	0.05	54.13	48.91	92.60	1.009	38.21	-0.67	38.21	34.57	92.60	1.008
MI.D90	48.30	-0.85	48.29	48.70	95.50	1.005	34.22	-1.00	34.21	34.43	95.50	1.004
MI.D80	48.46	-0.90	48.46	48.86	95.40	1.008	34.30	-0.88	34.30	34.54	95.40	1.008
CENS	51.74	-0.86	51.74	52.34	95.00	1.080	36.91	-0.98	36.90	37.00	95.10	1.079
NM10.1	48.32	-1.51	48.30	48.66	95.40	1.004	34.25	-1.22	34.23	34.41	95.20	1.004
NM10.2	48.32	-1.52	48.30	48.67	95.30	1.004	34.25	-1.21	34.23	34.41	95.20	1.004
NM20.1	48.75	-1.44	48.73	49.13	95.40	1.014	34.47	-1.18	34.46	34.74	95.40	1.014
NM20.2	48.78	-1.44	48.77	49.17	95.40	1.014	34.50	-1.20	34.49	34.77	95.40	1.014
NM30.1	49.19	-1.28	49.18	49.68	95.30	1.025	35.05	-1.16	35.03	35.13	95.50	1.025
NM30.2	49.34	-1.31	49.32	49.81	95.20	1.028	35.17	-1.21	35.15	35.22	95.40	1.027
NM40.1	49.59	-1.36	49.57	50.15	95.00	1.035	35.33	-0.94	35.32	35.46	94.90	1.035
NM40.2	49.81	-1.30	49.80	50.49	95.10	1.042	35.56	-0.98	35.55	35.70	94.90	1.042
NM50.1	50.24	-1.03	50.24	50.50	95.20	1.042	35.69	-1.06	35.68	35.70	94.90	1.042
NM50.2	51.03	-1.03	51.03	51.22	95.20	1.057	36.21	-1.14	36.20	36.21	95.00	1.056
NM60.1	50.13	-1.36	50.12	50.72	95.20	1.046	35.79	-1.09	35.78	35.87	95.00	1.046
NM60.2	51.32	-1.40	51.30	52.02	95.10	1.073	36.72	-1.19	36.71	36.78	94.90	1.073
NM70.1	50.21	-1.04	50.20	50.90	95.30	1.050	36.07	-1.06	36.06	35.99	95.20	1.050
NM70.2	52.40	-0.96	52.40	52.95	95.20	1.092	37.39	-1.28	37.37	37.43	95.10	1.092
NM80.1	50.35	-1.09	50.35	51.03	95.20	1.053	36.08	-1.05	36.06	36.08	95.00	1.053
NM80.2	52.93	-0.75	52.93	53.93	95.00	1.113	37.99	-1.39	37.97	38.11	95.00	1.112
NM90.1	50.61	-1.02	50.61	51.14	95.30	1.055	35.93	-0.95	35.92	36.15	95.20	1.055
NM90.2	54.49	-0.68	54.49	54.95	95.20	1.134	38.48	-1.21	38.47	38.84	95.30	1.133

Table E83: Inference for $\mu + \sigma^2/2$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 100$						Results for $n = 200$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	121.43	-7.15	121.23	121.36	94.50	1.000	85.40	-4.56	85.28	86.27	95.00	1.000
MI.C90	131.10	-7.28	130.91	121.95	93.10	1.005	92.81	-4.43	92.71	86.61	92.80	1.004
MI.C80	135.17	4.09	135.13	123.55	92.60	1.018	94.29	0.65	94.30	87.33	92.80	1.012
MI.D90	122.75	-2.16	122.75	122.97	94.90	1.013	85.79	-2.46	85.76	86.90	95.20	1.007
MI.D80	122.62	-2.01	122.61	123.34	94.90	1.016	85.79	-1.91	85.78	87.26	95.30	1.011
CENS	128.91	-4.25	128.85	129.37	94.70	1.066	90.39	-3.42	90.34	91.70	95.00	1.063
NM10.1	121.92	-6.86	121.74	121.75	94.50	1.003	85.75	-4.43	85.64	86.54	94.70	1.003
NM10.2	121.90	-6.87	121.72	121.76	94.60	1.003	85.77	-4.42	85.67	86.55	94.70	1.003
NM20.1	122.41	-6.57	122.25	122.68	94.70	1.011	86.29	-4.44	86.19	87.18	94.90	1.010
NM20.2	122.50	-6.56	122.33	122.75	94.90	1.011	86.38	-4.47	86.28	87.22	95.00	1.011
NM30.1	123.05	-6.39	122.90	123.75	94.80	1.020	86.95	-4.30	86.85	87.93	94.90	1.019
NM30.2	123.19	-6.55	123.03	124.01	94.90	1.022	87.26	-4.11	87.18	88.13	95.00	1.022
NM40.1	124.27	-5.59	124.16	124.77	94.70	1.028	87.31	-4.29	87.21	88.57	95.10	1.027
NM40.2	124.95	-5.69	124.84	125.47	94.70	1.034	87.91	-4.31	87.81	89.08	95.10	1.033
NM50.1	125.62	-5.88	125.50	125.42	94.70	1.033	87.89	-4.75	87.77	89.00	94.90	1.032
NM50.2	127.07	-5.96	126.95	126.92	94.50	1.046	88.98	-5.00	88.85	90.07	94.90	1.044
NM60.1	125.68	-5.78	125.56	125.90	94.60	1.037	88.26	-3.59	88.19	89.44	94.90	1.037
NM60.2	128.35	-6.29	128.21	128.56	94.80	1.059	90.35	-3.23	90.30	91.42	95.20	1.060
NM70.1	126.25	-5.02	126.16	126.34	94.90	1.041	88.14	-4.03	88.05	89.64	94.80	1.039
NM70.2	129.29	-5.42	129.19	130.59	94.90	1.076	91.43	-3.90	91.36	92.73	94.70	1.075
NM80.1	126.23	-5.18	126.14	126.59	94.80	1.043	88.34	-3.96	88.26	89.83	94.90	1.041
NM80.2	131.81	-5.41	131.71	132.71	94.50	1.094	92.57	-4.16	92.49	94.19	94.90	1.092
NM90.1	125.77	-5.44	125.67	126.76	94.60	1.045	88.93	-3.65	88.86	90.00	94.80	1.043
NM90.2	132.20	-6.07	132.07	134.86	94.80	1.111	95.46	-3.69	95.39	95.82	94.70	1.111

Table E84: Inference for $\mu + \sigma^2/2$ based on $N(\mu = 0, \sigma^2 = 1)$ data with $C = 1.282 = 90$th percentile

	Results for $n = 1000$						Results for $n = 2000$					
	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.	RMSE $\times 10^3$	Bias $\times 10^3$	SD $\times 10^3$	$\widehat{\text{SD}}$ $\times 10^3$	Cvg. %	Rel. Len.
CD	38.38	-0.73	38.37	38.70	95.30	1.000	27.27	-0.61	27.26	27.36	95.20	1.000
MI.C90	41.21	-0.43	41.21	38.83	93.40	1.003	29.25	-0.73	29.25	27.44	93.60	1.003
MI.C80	42.57	0.60	42.57	39.02	92.80	1.008	30.05	-0.10	30.05	27.56	92.90	1.007
MI.D90	38.56	-0.27	38.56	38.86	95.40	1.004	27.35	-0.46	27.35	27.47	95.10	1.004
MI.D80	38.71	-0.29	38.71	38.99	95.20	1.008	27.42	-0.35	27.42	27.56	95.20	1.007
CENS	40.65	-0.29	40.65	41.08	95.20	1.062	28.97	-0.44	28.97	29.04	95.10	1.061
NM10.1	38.57	-0.78	38.57	38.82	95.40	1.003	27.35	-0.63	27.35	27.45	95.20	1.003
NM10.2	38.57	-0.79	38.57	38.82	95.30	1.003	27.35	-0.63	27.35	27.45	95.20	1.003
NM20.1	38.83	-0.73	38.83	39.10	95.40	1.010	27.48	-0.60	27.48	27.65	95.20	1.010
NM20.2	38.85	-0.73	38.85	39.13	95.50	1.011	27.50	-0.61	27.50	27.66	95.30	1.011
NM30.1	39.07	-0.61	39.07	39.44	95.30	1.019	27.85	-0.58	27.84	27.88	95.20	1.019
NM30.2	39.17	-0.63	39.17	39.53	95.30	1.021	27.93	-0.62	27.93	27.94	95.10	1.021
NM40.1	39.35	-0.66	39.35	39.73	95.20	1.027	28.04	-0.43	28.04	28.09	94.80	1.027
NM40.2	39.50	-0.62	39.50	39.96	95.20	1.032	28.19	-0.46	28.19	28.25	94.90	1.032
NM50.1	39.74	-0.43	39.74	39.95	94.90	1.032	28.27	-0.50	28.27	28.24	94.60	1.032
NM50.2	40.28	-0.42	40.28	40.43	95.00	1.045	28.63	-0.56	28.62	28.58	94.70	1.044
NM60.1	39.66	-0.66	39.66	40.08	95.10	1.036	28.30	-0.53	28.30	28.34	95.00	1.036
NM60.2	40.49	-0.67	40.49	40.96	95.00	1.058	28.90	-0.59	28.90	28.96	95.10	1.058
NM70.1	39.69	-0.43	39.70	40.20	95.40	1.039	28.47	-0.50	28.47	28.41	95.10	1.038
NM70.2	41.16	-0.35	41.17	41.59	95.30	1.075	29.36	-0.66	29.36	29.39	94.80	1.074
NM80.1	39.77	-0.46	39.77	40.28	95.20	1.041	28.48	-0.49	28.48	28.47	95.10	1.040
NM80.2	41.57	-0.19	41.57	42.26	95.00	1.092	29.77	-0.73	29.76	29.86	94.90	1.091
NM90.1	39.96	-0.41	39.97	40.34	95.40	1.042	28.36	-0.43	28.36	28.52	95.10	1.042
NM90.2	42.61	-0.12	42.62	42.98	95.20	1.111	30.09	-0.60	30.09	30.37	95.20	1.110

APPENDIX F: ANALYSIS OF FULLY MASKED DATA ON TOTAL HOUSEHOLD INCOME AND HOUSEHOLD ALIMONEY PAYEMENT BASED ON DATA FROM THE 2000 U.S. CURRENT POPULATION SURVEY

In this appendix we present tables summarizing analysis of two variables of the 2000 U.S. Current Population Survey data under full data masking. The data and notations appearing in the tables are described in Section 7 of the paper.

Table F1. Analysis of *household total income* under full masking

	μ		σ^2		$e^{\mu+\sigma^2/2}$		$e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$		$e^{\mu+\sigma}$		$e^{\mu+1.645\sigma}$	
	Est	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.	Est $\times 10^{-6}$	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.
CD	10.495	1.000	0.981	1.000	58996	1.000	5805.083	1.000	97263	1.000	184262	1.000
MI	10.495	1.015	0.980	1.013	58977	1.016	5793.485	1.014	97231	1.016	184145	1.015
NM10U	10.494	1.001	0.980	1.003	58958	1.001	5790.372	1.000	97201	1.001	184097	1.001
NM10C	10.494	1.002	0.981	1.003	58962	1.001	5795.846	1.001	97207	1.002	184139	1.002
NM20U	10.495	1.008	0.983	1.015	59041	1.011	5827.769	1.017	97338	1.011	184492	1.013
NM20C	10.496	1.006	0.980	1.012	59048	1.009	5804.365	1.011	97349	1.009	184354	1.011
NM30U	10.493	1.015	0.980	1.028	58879	1.017	5769.845	1.019	97071	1.018	183817	1.021
NM30C	10.493	1.016	0.984	1.033	59003	1.022	5833.608	1.033	97277	1.022	184462	1.026
NM40U	10.493	1.032	0.987	1.059	59074	1.042	5872.531	1.063	97394	1.041	184845	1.049
NM40C	10.493	1.027	0.982	1.054	58932	1.035	5803.355	1.047	97159	1.035	184134	1.041
NM50U	10.495	1.048	0.983	1.087	59057	1.061	5829.819	1.082	97364	1.061	184535	1.071
NM50C	10.494	1.038	0.978	1.078	58889	1.050	5755.189	1.061	97086	1.051	183738	1.060
NM60U	10.489	1.076	0.990	1.139	58936	1.093	5878.810	1.134	97168	1.092	184631	1.109
NM60C	10.493	1.056	0.980	1.115	58892	1.074	5775.893	1.096	97092	1.074	183881	1.088
NM70U	10.485	1.108	0.991	1.201	58741	1.127	5848.492	1.179	96847	1.126	184076	1.150
NM70C	10.497	1.075	0.983	1.156	59196	1.106	5861.327	1.148	97595	1.106	184997	1.126
NM80U	10.483	1.149	0.984	1.277	58404	1.160	5710.635	1.210	96288	1.160	182554	1.191
NM80C	10.492	1.095	0.984	1.200	58919	1.129	5812.647	1.177	97138	1.129	184171	1.155
NM90U	10.481	1.198	0.951	1.367	57307	1.165	5216.701	1.163	94454	1.173	177174	1.205
NM90C	10.490	1.113	0.977	1.239	58616	1.149	5691.716	1.189	96635	1.150	182819	1.180

Table F2. Analysis of *household alimony payments* under full masking

	μ		σ^2		$e^{\mu+\sigma^2/2}$		$e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$		$e^{\mu+\sigma}$		$e^{\mu+1.645\sigma}$	
	Est	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.	Est $\times 10^{-6}$	Rel. Len.	Est	Rel. Len.	Est	Rel. Len.
CD	8.715	1.000	1.168	1.000	10930	1.000	264.812	1.000	17962	1.000	36069	1.000
MI	8.724	1.004	1.147	1.012	10988	1.037	280.919	1.329	18013	1.027	36058	1.039
NM10U	8.709	1.000	1.165	1.000	10847	0.993	259.675	0.981	17828	0.993	35767	0.993
NM10C	8.715	1.003	1.172	1.006	10947	1.006	267.222	1.014	17987	1.005	36164	1.007
NM20U	8.724	1.013	1.185	1.026	11115	1.035	280.355	1.079	18255	1.031	36834	1.037
NM20C	8.707	1.006	1.169	1.012	10843	1.000	260.912	0.996	17818	1.000	35788	1.001
NM30U	8.720	1.035	1.220	1.071	11272	1.081	303.136	1.204	18484	1.069	37683	1.089
NM30C	8.702	1.005	1.151	1.011	10695	0.986	247.352	0.946	17587	0.989	35136	0.987
NM40U	8.709	1.011	1.135	1.021	10690	0.991	241.386	0.934	17587	0.997	34968	0.993
NM40C	8.712	1.028	1.182	1.057	10977	1.043	272.638	1.078	18030	1.039	36358	1.049
NM50U	8.705	1.027	1.137	1.053	10650	1.008	240.305	0.955	17520	1.013	34857	1.013
NM50C	8.738	1.025	1.147	1.050	11058	1.046	262.552	1.040	18185	1.049	36280	1.051
NM60U	8.737	1.099	1.266	1.207	11733	1.219	350.650	1.533	19194	1.192	39661	1.242
NM60C	8.704	1.020	1.102	1.040	10458	0.983	219.756	0.872	17220	0.994	33890	0.986
NM70U	8.688	1.100	1.197	1.203	10795	1.112	269.324	1.183	17719	1.102	35888	1.133
NM70C	8.712	1.063	1.168	1.129	10896	1.084	262.985	1.107	17906	1.082	35951	1.098
NM80U	8.667	1.168	1.273	1.345	10981	1.214	310.293	1.475	17956	1.183	37181	1.249
NM80C	8.735	1.058	1.116	1.120	10863	1.075	242.053	1.024	17881	1.084	35339	1.088
NM90U	8.694	1.230	1.294	1.491	11397	1.331	343.876	1.762	18614	1.287	38769	1.383
NM90C	8.738	1.141	1.282	1.302	11842	1.303	365.256	1.710	19354	1.267	40176	1.337