

An Alternative Model-Based Seasonal Adjustment That Reduces Residual Seasonal Autocorrelation

Tucker McElroy*

Center for Statistical Research and Methodology
U.S. Census Bureau

Keywords: ARIMA, Seasonality, Signal extraction, Wiener-kolmogorov

JEL Classification: C01

* Correspondence: Tucker McElroy, Center for Statistical Research and Methodology, U.S. Census Bureau, 4600 Silver Hill Road, Washington, DC 20233. Tel: 301-763-3227; Fax: 301-763-8399; E-mail: tucker.s.mcelroy@census.gov. This paper has benefitted from the comments of two referees, as well as the associate editor. This paper is released to inform interested parties of ongoing research and to encourage discussion of work in progress. The views expressed on statistical, methodological, technical, and operational issues are those of the author and not necessarily those of the U.S. Census Bureau.

ABSTRACT

The Wiener-Kolmogorov (WK) signal extraction filter, extended to handle nonstationary signal and noise, has minimum Mean Squared Error (MSE) for Gaussian processes. However, the stochastic dynamics of the signal estimate typically differ from that of the target. The use of such filters, although widespread, has been observed to produce dips in the spectrum of the seasonal adjustments of seasonal time series. These spectral troughs correspond in practice to negative autocorrelations at lag 12 (or negative seasonal autocorrelation), a phenomenon corresponding to an annual stochastic cycle. So-called "square root" WK filters were introduced by Wecker (1979) in the case of stationary signal and noise, to ensure that the signal estimate shared the same stochastic dynamics as the original signal, and thereby remove spectral dips. This represents a different statistical philosophy: not only do we want to closely estimate a target quantity, but we desire that the dynamics of our estimate closely resemble those of the target. The MSE criterion ignores this aspect of the signal extraction problem, whereas the "dynamic matching" filters account for this issue at the cost of accruing additional MSE. This paper provides empirical documentation of the occurrence of negative seasonal autocorrelation in seasonally adjusted data, and provides matrix formulas for filters that match the dynamics of the desired signal, and are appropriate for finite samples of nonstationary time series. We apply these filters to 88 time series to produce seasonal adjustments that have greatly reduced incidences of negative seasonal autocorrelation.

1. INTRODUCTION

The principal task of seasonal adjustment methodology is to remove seasonality. First, seasonality must be defined; given a precise mathematical definition, one can then devise statistical methods to remove the seasonality. See Bell and Hillmer (1984) for a related discussion, and the antecedent Granger (1978a, 1978b). Although there are many alternative definitions available, there is generally agreement among seasonal adjusters that the presence of significant positive seasonal autocorrelation does *imply* the presence of seasonality, and is therefore a primitive concept – see Fase et al. (1973) and den Butter and Fase (1991) for a discussion of empirical criteria used to judge seasonal adjustments. Many seasonal adjusters require the presence of “nonstationary seasonality” – i.e., sample seasonal autocorrelations (i.e., autocorrelations at lags that are a multiple of the seasonal period) that decay extremely slowly over high lags – to classify a time series seasonal. Also see the discussion in Findley et al. (1998). For some practitioners, the presence of negative seasonal autocorrelation in the seasonal adjustment might also be cause for concern (Granger, 1978a). The paper at hand seeks to make several points: firstly, conventional signal extraction techniques typically remove nonstationary seasonality, leaving behind residual “stationary seasonal” effects; secondly, these residual seasonal effects are often associated with negative seasonal correlations; thirdly, this effect can be greatly attenuated by using modified “dynamic matching” filters, a model-based variant of classical seasonal adjustment filters.

Regarding the first point, there are many experts who view the expression “stationary seasonality” as an oxymoron, since in their opinion quasi-periodic effects that are mean-reverting will have a pattern that changes too quickly to be classified as seasonal (Bell and Hillmer, 1984). Seasonality has been conceptually linked with nonstationarity and high positive seasonal autocorrelations. Note that, in contrast, business cycles are typically stationary; if a business cycle’s period were equal to one year, so that its modal frequency coincided with the first seasonal frequency, we would observe the cycle as a “stationary seasonal”. Therefore, to avoid the offense caused to some readers by the term

“stationary seasonality”, we will use the term “annual cyclicity” instead. In this paper, we consider seasonality in a broad mathematical sense – and not the sense more narrowly prescribed by economic considerations – as constituting quasi-periodic behavior, stationary or nonstationary as the case may be, associated with the seasonal frequencies. This is indicated by large autocorrelation values (either positive or negative) at lags that are a multiple of the seasonal periodicity. When these correlations, as a function of the lag, exhibit geometric decay, we refer to the phenomenon more specifically as annual cyclicity; this is just a convenient choice of statistical nomenclature, and should not dismay any readers on economic grounds. Likewise, we shall not refer to the phenomenon observed by Nerlove (1964) and others – namely that seasonal adjustments tend to exhibit negative seasonal autocorrelations – as residual seasonality, but rather as residual annual cyclicity.¹

Now if the reader may grant that the presence of nonzero seasonal autocorrelations in a seasonally adjusted series may – when the correlations are sufficiently large – be a possible cause for concern, it is natural to ask: how does this phenomenon arise? It is well-known that MSE optimal filters produce exactly this type of behavior in seasonally adjusted series, as described in Nerlove (1964), Sims (1978), Tukey (1978), and Bell and Hillmer (1984). We review this material in Section 2, also demonstrating that the typical effect of this optimal filtering is to yield spectral troughs at seasonal frequencies, which is manifested in the time domain as negative seasonal autocorrelations. The general concept here has been established by older literature, which is reviewed, but we provide some specific illustrations pertinent to our case.

Now for many authors, the recognition of the phenomena is an end to the matter (Sims (1978) and Tukey (1978) express this opinion, as well as Bell and Hillmer (1984)), although an alternative filter that avoids this behavior – at the cost of some additional MSE – was proposed by Wecker (1979) and reiterated in Ansley and Wecker (1984).² The chief objections to their proposal are: (i) one

¹ Properly speaking, a cycle corresponds to a spectral peak, and hence to positive autocorrelations of appropriate lag, whereas a spectral trough with negative autocorrelations is sometimes referred to as an anti-cycle. Here we use the terms cycle and cyclicity to embrace both peak and trough at once.

² The authors demonstrate that their proposed filters minimize MSE subject to the constraint of matching the target signal’s dynamics. A similar result can be derived for the case of nonstationary signal and noise processes.

should not deviate from MSE optimal filtering, since this gives the minimal overall error; (ii) their suggestion is developed for bi-infinite stationary time series, so its practical implementation is unclear. A combination of these two issues has prevented the Ansley and Wecker (1984) method – hereafter AW – from becoming widely used. We provide some discussion of (i) in Section 2, arguing that the objective of seasonal adjustment is not simply the most “accurate” signal extraction possible, but one that produces a filter output whose dynamics are most compatible with a non-seasonal time series. Ultimately, this is a matter of personal opinion, as is much in the field of seasonal adjustment; there are pros and cons to using MSE optimal filtering like any other statistical tool.

To address (ii), this paper extends the AW approach to signal extraction problems with nonstationary signal and nonstationary noise, from a finite sample of data. The theory is developed in Section 3. Since we no longer treat bi-infinite time series, it is more appropriate to match the autocovariance matrices of random vectors, rather than actual spectral densities, and therefore we introduce the concept of “dynamic matching”. This means that the second-order dynamics, i.e., the autocovariance function (acf) at a range of lags, is replicated. The formulas that are derived are demonstrated to have the desired properties when the noise is stationary. Given the usual model-based approach – the fitting of component models through a structural (Gersch and Kitagawa, 1983) or decomposition (Hillmer and Tiao, 1982) method – one can directly plug the acf quantities into the formula and obtain the alternative seasonal adjustments.

A fairly large empirical study is conducted in Section 4. There we assess the presence of residual annual cyclicity in economic time series via the tool of the sample acf viewed at seasonal lags. Comparing the results of MSE optimal filtering against the AW filters of this paper, we observe an overall reduction in the presence of annual cyclicity in seasonally adjustments. A secondary assessment of the AW filters is completed through revision variances and spectral analysis – a comparison of gain and phase delay plots for concurrent filters from both methods. Proofs are contained in the Appendix, and Section 5 concludes. Although the proposal of alternative filters is tendentious and controversial, and not suitable to everyone’s taste, we yet feel that the AW method is an intriguing technique that will be appealing to some practitioners.

2. RESIDUAL ANNUAL CYCLICALITY

This section endeavors to first demonstrate how residual annual cyclicity arises from the use of bi-infinite WK filters. Although the general concept is handled in Ansley and Wecker (1984) and Maravall (1987), we provide specific illustrations that are pertinent to our context. Secondly, we set forth the lag 12 sample acf as a practical empirical measure of annual cyclicity in seasonal adjustments, and show how a notion of statistical significance can be associated with this quantity.

It has long been known that WK seasonal adjustments, i.e., seasonal adjustments that arise from using a model-based WK methodology for filtering, tend to have negative autocorrelations at lag 12. The accompanying phenomenon in frequency domain is a dip in the pseudo-spectral density (i.e., the spectral density of a stationary component process divided by the appropriate unit root factors) in the neighborhood of the seasonal frequencies. This was first noticed and documented in Nerlove (1964); subsequent literature includes Grether and Nerlove (1970), Wecker (1979), Bell and Hillmer (1984), Ansley and Wecker (1984), Maravall (1987), and Findley and Martin (2006). We refer to this phenomenon as annual cyclicity – we avoid denoting it as residual seasonality, because the periodic behavior is virtually always stationary, and hence is not seasonal in the economic sense of the term. Some practitioners have also referred to the phenomenon as “over-adjustment”, though we avoid using this term, since it is vague, also referring to problems in seasonal adjustment arising from using poorly fitted models (e.g., see McElroy (2008b)). The correspondence between the time domain and frequency domain manifestations of annual cyclicity is given mathematically by the formula

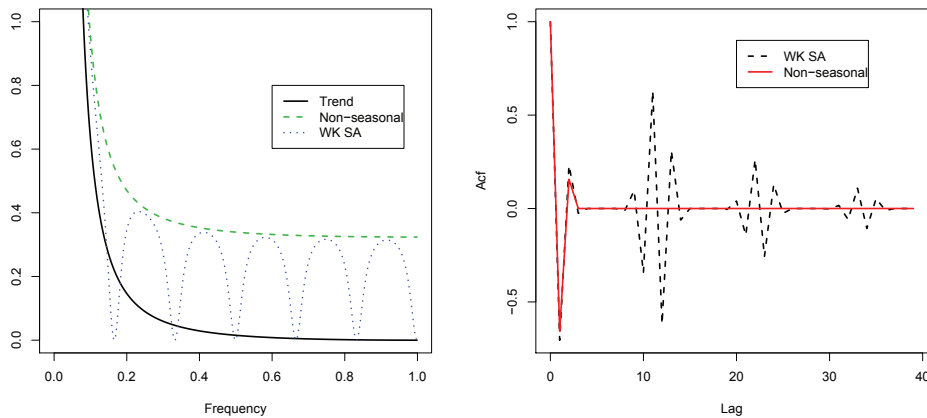
$$\gamma_{12} = \frac{2}{2\pi} \int_0^\pi f(\lambda) \cos(12\lambda) d\lambda,$$

which expresses the lag 12 autocovariance γ_{12} in terms of the spectral density f ; the well-known shape of the cosine function positively weights values of f in

the neighborhood of $\pi/6$, while negatively down-weighting values at least $\pi/12$ away. Although one cannot say that spectral seasonal troughs in f *must* imply $\gamma_{12} < 0$, nonetheless this tends to be true in practice. This is demonstrated for some explicit examples below.

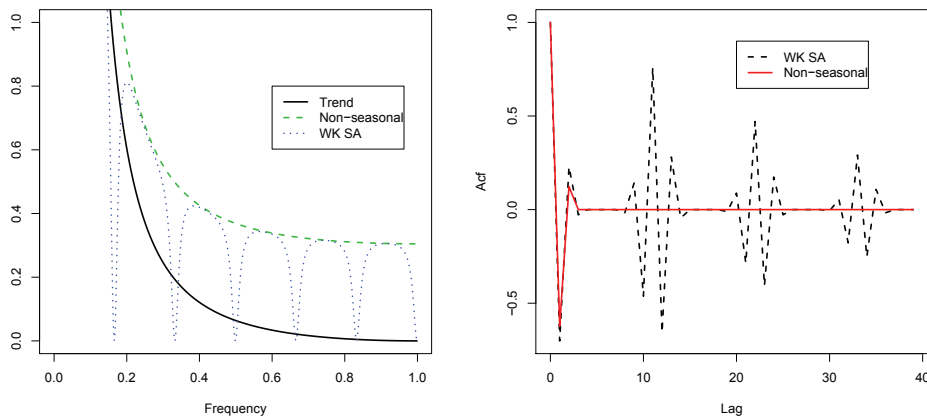
Consider the case of a Box-Jenkins Airline model (Box and Jenkins, 1976) fitted to a time series, from which a WK seasonal adjustment is generated according to the canonical decomposition of Hillmer and Tiao (1982). Letting f_Y, f_T, f_S, f_{SA} denote the pseudo-spectra of the data process, the trend, the seasonal, and the non-seasonal (equivalent to the sum of the trend and irregular spectra) respectively, the pseudo-spectrum of the WK estimate of the non-seasonal is f_{SA}^2/f_Y . This is an $I(2)$ process, so after two non-seasonal differences, we can meaningfully plot its acf. The formulas for these quantities follow at once from Hillmer and Tiao (1982). Rather than reproduce them here, we provide plots of pseudo-spectra and acfs for two choices of parameters – namely, (0.63, 0.42) and (0.36, 0.62) for the nonseasonal and seasonal moving average parameters respectively – since this palpably illustrates the phenomenon of residual annual cyclicity. These choices of parameters correspond to two of the time series, m00190 and x3, discussed below. Figures 1 and 2 show the spectral dips at seasonal frequencies in f_{SA}^2/f_Y (indicated as “WK SA”). We omit plotting f_Y and f_S for clarity. The corresponding behavior in the acf is immediately evident as well.

When does the WK estimate’s spectra correspond to its target? As noted in Findley and Martin (2006), when f_S and f_{SA} have disjoint compact support, then trivially $f_{SA}^2/f_Y = f_{SA}$. Incidentally, WK signal extraction is also errorless in this case, i.e., the MSE is zero. Unfortunately, this situation *never* arises in the ARIMA model-based approach to signal extraction, since the pseudo-spectra of ARIMA processes – including structural processes encountered in a structural approach (Gersch and Kitagawa (1983) and Harvey (1989)) to the problem, as in the time series software STAMP (Koopman et al., 2000) – have a finite number of zeroes, not a continuum of zeroes. Thus the typical situation encountered by both SEATS (the seasonal adjustment program of the Bank of Spain – see Maravall and Caporello (2004)) and STAMP (commercial software for signal extraction) seasonal adjustments is essentially encapsulated by Figures 1 and 2.



Note: The left panel displays the pseudo-spectra corresponding to the canonical decomposition of the (0.63, 0.42) Airline model. Spectra for trend, true non-seasonal, and seasonally adjusted (WK) estimate are provided. The right panel displays autocorrelations for twice-differenced non-seasonal and seasonally adjusted components.

Figure 1 Pseudo-Spectra and Autocorrelations for Components of a (0.63, 0.42) Airline Model



Note: The left panel displays the pseudo-spectra corresponding to the canonical decomposition of the (0.36, 0.62) Airline model. Spectra for trend, true non-seasonal, and seasonally adjusted (WK) estimate are provided. The right panel displays autocorrelations for twice-differenced non-seasonal and seasonally adjusted components.

Figure 2 Pseudo-Spectra and Autocorrelations for Components of a (0.36, 0.62) Airline Model

Another observation is that annual cyclicity can also be manifested as negative autocorrelations at lags 24 and 36 and higher multiples of 12. However, these correlations tend to be diminished. Although we have scrutinized these higher lag correlations, no results on them are presented in this paper, since the story at lag 12 is the most compelling and interesting, both in theory and in practice.

Empirically speaking, actual seasonal adjustments – once appropriately differenced – can be assessed for the presence of negative lag 12 autocorrelation; all that is needed is to generate the sample acf plot for the differenced seasonal adjustments. Of course there is the issue of what constitutes a statistically significant value. We are trying to assess whether there is really any practical discrepancy between the lag 12 autocorrelations arising from the WK seasonal adjustments, versus the actual non-seasonal component itself. For the Airline model, the actual non-seasonal component follows an IMA(2, 2) process. Its two moving average parameters are determined by the Airline model parameters' maximum likelihood estimates (mles) and the canonical decomposition algorithm. Since we difference twice before computing the autocorrelations, we can plug the MA(2) parameter values into Bartlett's formula for the variance of the sample acf (formula 7.2.5 of Brockwell and Davis, 1991). Critical values assume a Gaussian distribution. Hence significant values can be interpreted as a rejection of the hypothesis that the twice-differenced seasonally adjusted estimate follows the given MA(2) model for the trend-irregular.

For example, consider series m00190 of Foreign Trade Imports (U.S. Census Bureau). An Airline model in the original scale – with outliers and trading day effects detected, but no Easter effect (see Findley et al., 1998) – was the best fitting model according to X-12-ARIMA, and the model parameters are (0.63, 0.42). The resulting MA(2) model for the twice-differenced non-seasonal is $1 - 1.58B + 0.60B^2$. Given the series has 155 observations, the critical value (based on a two-sided 0.95 confidence interval) is 0.218; the actual lag 12 sample autocorrelation is -0.382 (cf. Table 1). Now for these parameter values, the lag 12 autocorrelation for the bi-infinite sample WK non-seasonal component is -0.29 .³ The reason for the discrepancy is that the actual non-seasonal

³ This refers to the component model identified by the canonical decomposition routine, with parameters determined by algorithm from the Airline mles. We compute the autocorrelations of the process with spectrum f_{SA}^2/f_Y , appropriately differenced.

Table 1 Results for Foreign Trade, Retail, and Housing Series

Series	WK acf	DM acf	Crit	DM/WK	Rev
m00190	-0.382	-0.154	0.218	0.40	0.64
m12060	-0.264	-0.042	0.219	0.16	0.77
m12135	-0.148	-0.069	0.204	0.46	0.62
m12150	-0.230	-0.172	0.216	0.75	0.62
m12540	-0.027	-0.021	0.218	0.79	0.95
m21110	-0.151	-0.111	0.218	0.73	0.69
m21180	-0.198	-0.135	0.211	0.68	0.58
m21610	-0.220	-0.148	0.216	0.67	0.58
m22020	-0.260	-0.200	0.215	0.77	0.60
m3000	-0.250	0.007	0.211	0.03	0.64
m3010	-0.238	-0.089	0.213	0.37	0.55
m40020	-0.323	-0.079	0.195	0.24	0.59
m40110	-0.242	-0.238	0.219	0.98	0.93
m41140	-0.051	0.023	0.215	0.44	0.60
m41310	-0.314	-0.149	0.211	0.48	0.65
m42110	-0.409	-0.187	0.218	0.46	0.56
x00300	0.090	0.090	0.210	1.00	1.00
x10140	-0.311	-0.171	0.219	0.55	0.56
x11020	-0.134	-0.052	0.217	0.39	0.58
x12550	-0.060	0.057	0.213	0.96	0.58
x12600	-0.353	-0.274	0.215	0.78	0.58
x13200	-0.089	-0.052	0.217	0.58	0.65
x21000	-0.368	-0.305	0.218	0.83	0.53
x21030	-0.229	-0.229	0.214	1.00	1.00
x21150	-0.191	-0.150	0.217	0.79	0.61
x21500	-0.190	-0.122	0.214	0.64	0.57
x3020	-0.170	-0.002	0.215	0.01	0.53
x3	-0.023	0.097	0.212	4.29	0.57
x40000	-0.143	-0.050	0.210	0.35	0.62
x40030	-0.298	-0.225	0.214	0.75	0.58
x41020	-0.160	-0.068	0.214	0.43	0.55
x41120	-0.270	-0.126	0.215	0.47	0.56
x41140	-0.154	-0.154	0.217	1.00	1.00
x42100	-0.065	0.082	0.218	1.26	0.56
s0b44000	-0.291	-0.092	0.193	0.32	0.60
s0b44130	-0.343	-0.243	0.190	0.71	0.57

Table 1 Results for Foreign Trade, Retail, and Housing Series (Continued)

Series	WK acf	DM acf	Crit	DM/WK	Rev
s0b44312	− 0.255	−0.078	0.175	0.31	0.59
s0b44500	−0.093	0.031	0.192	0.33	0.57
s1fam	−0.180	−0.092	0.190	0.51	0.59
stot	−0.053	0.025	0.198	0.47	0.60
us1fam	− 0.213	−0.104	0.179	0.49	0.56
wtot	−0.039	−0.023	0.197	0.60	0.81
mw1fam	−0.055	−0.055	0.189	1.00	1.00
mwtot	−0.195	−0.195	0.197	1.00	1.00

Note: For time series of Foreign Trade Imports (m prefix), Foreign Trade Exports (x prefix), Retail (s0b prefix), and Housing Starts, the lag 12 sample autocorrelation of twice-differenced seasonal adjustments from both the WK and DM methodologies is given, along with the critical value associated with a 5 percent rejection rate (see Section 2 for discussion). The absolute ratio of the correlations is given in the fifth column. The sixth column gives the ratio of DM revision variance to WK revision variance, for the respective seasonal adjustments (see Section 4).

estimate is not computed from the single bi-infinite WK filter – as the computations involving f_{SA} presume – but rather a suite of time-varying filters of finite length, as explicitly described in McElroy (2008a). It is known that the model-based concurrent filter, for example, will produce an estimate whose dynamics match those of the target (Bell and Martin, 2004). Since the actual seasonal adjustment is an amalgam of concurrent and symmetric filters – and all the asymmetric filters in-between these extremes – we cannot expect the lag 12 sample acf to match the theoretical quantity (there is also statistical error involved).

So according to our criterion based on the Bartlett formula, significant annual cyclicity exists in the WK seasonal adjustment of series m00190. As another example, consider x3 of Foreign Trade Exports (U.S. Census Bureau), with an Airline model identified as best for the untransformed data, with Trading Day and Easter effects identified and removed. X-12-ARIMA estimated the parameters to be (0.36, 0.62), and the corresponding MA(2) for the non-seasonal was $1 - 1.33B + 0.356B^2$. Having the same length as m00190, the

critical value was 0.212 with lag 12 autocorrelation -0.02 (Table 1) – clearly nonsignificant. The bi-infinite sample WK non-seasonal component has lag 12 autocorrelation -0.19 , a wide discrepancy from the estimated value.

By repeating this latter exercise over a grid of parameters uniformly distributed over the unit square $[0, 1] \times [0, 1]$, one finds a remarkable relationship: to a high degree of precision, the lag 12 autocorrelation for the bi-infinite sample WK non-seasonal is equal to $(\theta_{12} - 1)/2$, where θ_{12} is the seasonal Airline model parameter. The non-seasonal Airline model parameter has negligible impact on this relationship. In the case of a stable seasonal (θ_{12} close to unity) we expect to have less negative correlation, but the issue becomes more prevalent as the seasonal becomes more chaotic.

Generalizing from these two examples, we summarize our observations:

- For Airline processes, the stochastic properties of the WK estimate of the non-seasonal differ from the properties of the true non-seasonal; this is true not only of estimates based on bi-infinite samples (a known result), but also from estimates based on finite sample (which is emphasized here).
- Characteristic seasonal spectral troughs in the seasonal adjustment's spectrum correspond to negative lag 12 autocorrelation.
- Actual annual cyclicity can be assessed through the sample acf with significance determined by the Bartlett formula. The phenomenon may or may not be significant.

In view of the last point, it is important to repeat the analysis over many time series, drawn from different sectors of the economy, in order to gauge the overall prevalence of annual cyclicity in seasonal adjustments. This is carried out in Section 4. But first in Section 3 we turn to a modified signal extraction procedure, in the spirit of AW, that strives to reduce the incidence of residual annual cyclicity.

3. MATHEMATICAL TREATMENT OF DYNAMIC-MATCHING FILTERS

In this section we discuss the DM filter, which is a $n \times n$ matrix that left-multiplies the data vector $Y = \{Y_1, \dots, Y_n\}'$. Section 3.1 sets out some basic conventions, and Section 3.2 gives explicit formulas for DM filter.

3.1 Defining the Component Models

We suppose that the signal and noise processes are ARIMA, with differencing operators $\delta^S(z)$ and $\delta^N(z)$ respectively. These polynomials include only unit roots, and are assumed to be of order d_S and d_N respectively. A crucial assumption is that the two polynomials share no common factors; in practice this is easily accomplished as follows. For the canonical decomposition approach to component modeling, we have in mind an ARIMA model for the data of the form

$$\delta(B)Y_t = \Psi(B)\epsilon_t =: W_t,$$

where $\Psi(z)$ is a rational function (with no poles on the unit circle), and ϵ_t is white noise. Since $Y_t = S_t + N_t$, it follows that each factor of $\delta(B)$ must appear as a “left-hand operator” in the ARIMA equation for either S_t or N_t (or both). Making *a priori* allocations of the factors of $\delta(z)$ to either the signal or the noise constitutes part of the definition of the components; we can choose to do this in such a way that no factors are shared. This is sensible too, since the left-hand operators $\delta^S(z)$ and $\delta^N(z)$ serve to define some of the key dynamics of the signal and noise processes, so that making the operators distinct serves to separate the components and assist in making them well-defined. For example, suppose that a time series has the fitted ARIMA model $(2, 1, 3) (0, 1, 1)_{12}$ given by

$$(1 - 2\rho \cos \omega B + \rho^2 B^2)(1 - B)^2 U(B)Y_t = \Theta(B)\epsilon_t,$$

where ρ and ω control the strength and location respectively of a cycle, $U(B) = 1 + B + B^2 + \dots + B^{11}$ is the annual summation operator associated with

nonstationary seasonality, $\Theta(z)$ is an order 15 polynomial with zero coefficients at certain particular lags, and ϵ_t is white noise. If we are interested in suppressing seasonality, then we naturally let $\delta^N(B) = U(B)$ – since this is associated with the seasonal frequencies – and $\delta^S(B) = (1 - B)^2$, which corresponds to the trend frequencies. If instead we want cycle estimation, then $\delta^N(B) = (1 - B)^2 U(B)$ and $\delta^S(B) = 1$; the $(1 - 2\rho \cos \omega B + \rho^2 B^2)$ operator will be an autoregressive operator defining the stationary signal component. If we wish to detrend the series, then $\delta^N(B) = (1 - B)^2$ and $\delta^S(B) = U(B)$.

Once $\delta(z)$ has been partitioned among the signal and noise appropriately, one typically assumes a balanced ARIMA process for each component, so that the “right-hand operator” in each ARIMA equation has order equal to the left-hand, i.e., we have an MA polynomial of order $d_S + p$ or $d_N + p$ for the signal or the noise, respectively, where p is the order of any autoregressive polynomials in the model. Note that because the factors are made distinct by construction, the order of $\delta(z)$ is $d = d_S + d_N$. The MA polynomials for the signal and noise component processes are then determined via partial fractions, as discussed in Hillmer and Tiao (1982).

Hence, we will proceed from the standpoint that ARIMA models have been found for each of the components, with the following notation:

$$\begin{aligned}\delta(B)Y_t &= W_t = \Psi(B)\epsilon_t, \\ \delta^S(B)S_t &= U_t = \Psi^S(B)\epsilon_t^S, \\ \delta^N(B)N_t &= V_t = \Psi^N(B)\epsilon_t^N.\end{aligned}$$

This corresponds to the classical signal extraction scenario (Hillmer and Tiao, 1982). For ARIMA models, one typically assumes that the leading coefficients of AR and MA polynomials are unity.

Next, for any polynomial g of order h we define $\Delta(g)$ to be the $(n - h) \times n$ matrix with entries given by $\Delta_{ij} = g_{i-j+h}$ (with the convention that $g_k = 0$ if $k < 0$ or $k > h$). This means that each row of this matrix consists of the coefficients of the polynomial g , horizontally shifted in an appropriate fashion. Alternatively, these can be defined via $\Delta(g) = [0 \ 1_{n-h}]g(L_n)$ where L_n is the lag matrix of dimension n , i.e., it has ones on the first sub-diagonal and zeroes

elsewhere. Here 1 denotes an identity matrix. We are principally interested in $\Delta(\delta)$, $\Delta(\delta^S)$ and $\Delta(\delta^N)$, which we write as Δ , Δ_S and Δ_N for short. Thus

$$W = \Delta Y, \quad U = \Delta_S S, \quad V = \Delta_N N;$$

where W , U , V , S , and N are column vectors like Y . To express

$$W_t = \delta^N(B)U_t + \delta^S(B)V_t. \quad (1)$$

in matrix form we need some additional notation. Let $\underline{\Delta}_N = [0 \ 1_{n-d}] \delta^N(L_{n-d_S})$ and $\underline{\Delta}_S = [0 \ 1_{n-d}] \delta^S(L_{n-d_N})$, which have the same form as Δ_N and Δ_S , but of reduced dimension. Then one can easily show (see Lemma 1 of McElroy and Sutcliffe (2006)) that

$$\Delta = \underline{\Delta}_N \Delta_S = \underline{\Delta}_S \Delta_N. \quad (2)$$

This is because $\underline{\Delta}_N \Delta_S = [0 \ 1_{n-d}] \delta^N(L_{n-d_S}) [0 \ 1_{n-d_S}] \delta^S(L_n)$, which equals $[0 \ 1_{n-d}] \delta^N(L_n) \delta^S(L_n) = \Delta$. Then we can write down the matrix version of (1):

$$W = \underline{\Delta}_N U + \underline{\Delta}_S V. \quad (3)$$

We also adopt the following notation: Γ_X denotes the covariance matrix of a random vector X , and for any square integrable (possibly complex) function f , $\Gamma(f)$ is the corresponding covariance matrix with jk th entry

$$\Gamma_{jk}(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda) e^{i\lambda(j-k)} d\lambda.$$

Hence if X is stationary with associated spectral density f , then $\Gamma_X = \Gamma(f)$. At this point we can define the MSE optimal matrix formulas of McElroy (2008a) for the signal extraction estimate, which we repeat for convenience.

$$M = \Delta'_N \Gamma_V^{-1} \Delta_N + \Delta'_S \Gamma_U^{-1} \Delta_S; \quad (4)$$

$$F_* = M^{-1} \Delta'_N \Gamma_V^{-1} \Delta_N, \quad (5)$$

where F_* is the MSE optimal filter matrix and M is the error covariance matrix. There are a few other concepts that we need as well. The components in (3) are essentially “over-differenced”, and will be referred to as ∂U and ∂V respectively, as a short-hand. Then they have covariance matrices

$$\begin{aligned}\Gamma_{\partial U} &= \underline{\Delta}_N \Gamma_U \underline{\Delta}'_N; \\ \Gamma_{\partial V} &= \underline{\Delta}_S \Gamma_V \underline{\Delta}'_S.\end{aligned}$$

Finally, we will need to consider the matrix square root of a given symmetric positive-definite matrix (Golub and van Loan, 1996, p.149). (This is not the same thing as the Cholesky decomposition.) Given such a symmetric positive-definite matrix A , its singular value decomposition takes the form $A = QDQ'$ with Q orthogonal and D diagonal with positive entries. Then $A^{1/2} = QD^{1/2}Q'$ by definition, and satisfies $A^{1/2}A^{1/2} = A$. Moreover this square root is symmetric and has inverse $QD^{-1/2}Q'$, which will be denoted $A^{-1/2}$.

3.2 The Filter Formulas

A signal extraction filter F should reduce the noise process to stationarity, and this condition can be expressed as $F = G \Delta_N$ for some matrix G . In order to avoid nonstationarity in the error process $FY - S$, we likewise require $1 - F = H \Delta_S$ for some matrix H , where 1 is the $n \times n$ identity matrix. These two criteria together will be called the *signal extraction conditions*:

$$F = G \Delta_N; \tag{6}$$

$$1 - F = H \Delta_S. \tag{7}$$

We also say that a signal estimate FY is *dynamic replicating* if the estimate has the same nonstationary differencing operator as S , and $\Delta_S FY$ has the same covariance structure as $\Delta_S S$. Since $\Delta_S FY = \Delta_S \hat{S} = \hat{U}$ should hold true if F is sensibly constructed, we can parse this condition as

$$\Gamma_{\hat{U}} = \Gamma_U. \tag{8}$$

Then under (8) the signal estimate \widehat{S} will have the exact same dynamics – i.e., same nonstationary differencing operator and same covariance matrix for its differences – as S , which is why we call it dynamic replicating. A further desirable quality is that F be centro-symmetric, i.e., $F_{ij} = F_{n-i+1, n-j+1}$ (see Dagum and Luati (2004) and McElroy (2008a)). In particular, the centro-symmetry property implies that the asymmetric filters corresponding to the first and last rows of F are reverses of one another, and the middle filter given by the central row of F (when n is odd) is a symmetric sequence.

There is no unique filter matrix with these properties (6), (7), (8). Moreover, when the noise process is non-stationary the problem has no solution at all, as the result below shows.

Proposition *Consider a filter matrix such that (6) and (7) hold. Then (8) holds iff $\delta^N(B) = 1$, i.e., the noise process is stationary.*

Since (6) and (7) are non-negotiable conditions for filtering, the Proposition indicates that we should relax the condition of dynamic replication. We therefore introduce the concept of *dynamic matching*, which states that the signal estimate has the same nonstationary differencing operator as S , and ΔFY has the same covariance structure as $\Delta S = \underline{\Delta}_N U$. In other words,

$$\underline{\Delta}_N \Gamma \widehat{U} \underline{\Delta}'_N = \underline{\Delta}_N \Gamma_U \underline{\Delta}'_N. \quad (9)$$

Note that (8) implies (9), but not vice versa. They are, of course, the same condition when $\delta^N(B) = 1$. We use the term “matching” rather than “replicating” to signify the weakened condition on the estimate’s dynamics. Then dynamic-matching is the right concept, in the sense that filter matrices F exist such that (6), (7) and (9) hold; this follows from the calculations in the proof of the Proposition. However, these conditions are not enough to guarantee uniqueness; there are actually many possible filter matrices. In order to obtain a unique filter matrix, we add an additional condition that will also yield centro-symmetry.

We suppose that fixed mean effects exist corresponding to the signal and noise, so that we can write the observed data as $Z = Y + X\beta = S + X_S\beta_S +$

$N + X_N\beta_N$, where the regression matrix X breaks into sets of columns X_S and X_N corresponding to signal and noise mean effects, respectively, with associated parameter vectors β_S and β_N . Naturally these effects are annihilated by the corresponding differencing operators, so that $\Delta_S X_S$ and $\Delta_N X_N$ are both zero matrices. Our viewpoint is now that Y , S , N are mean zero stochastic processes, with the fixed effects describing the expectations of Z , signal, and noise (respectively). Then

$$FZ = X_S\beta_S + FY. \quad (10)$$

so long as (6) and (7) hold. It may be of interest to extract the fixed effect portion of the signal estimate FZ , namely $X_S\beta_S$ (the mean zero portion is FY). Suppose we utilize the weighted least squares (WLS) estimate given by $\tilde{\beta}_S = (X'_S D^{-1} X_S)^{-1} X'_S D^{-1} FZ$, where D is some invertible matrix. Then since Δ_N annihilates X_S , by (10) we know that $\tilde{\beta}_S$ is the same if defined in terms of Y rather than Z . Now if working with unfiltered data Z rather than FZ , we have the WLS estimate

$$\hat{\beta}_S = (X'_S D^{-1} X_S)^{-1} X'_S D^{-1} Z = \beta_S + (X'_S D^{-1} X_S)^{-1} X'_S D^{-1} [Y + X_N\beta_N].$$

Note that in order to eliminate the noise mean functions, we must have $X'_S D^{-1} X_N = 0$. Due to the structure of the regressor functions, one can show that D^{-1} must have the form

$$D^{-1} = \Delta'_N P \Delta_N + \Delta'_S Q \Delta_S,$$

for symmetric matrices P and Q . For any such D , the difference in the estimates, when based on the optimal filter matrix F_* , is

$$\begin{aligned} \tilde{\beta}_S - \hat{\beta}_S &= (X'_S D^{-1} X_S)^{-1} X'_S D^{-1} (FZ - Z) \\ &= - (X'_S D^{-1} X_S)^{-1} X'_S \Delta'_N P \Gamma_V \underline{\Delta}'_S \Gamma_W^{-1} \Delta Y. \end{aligned}$$

This will be zero, for all Y , iff $X'_S \Delta'_N P \Gamma_V \underline{\Delta}'_S = 0$. This relation is implied by $P = \Gamma_V^{-1}$, but this is not necessary. It follows from this discussion that $\tilde{\beta}_S = \hat{\beta}_S$ when using the optimal filter F_* and doing WLS with the matrix $D = M^{-1}$

given by (4). We will say that a filter matrix F is *compatible* with WLS regression estimates based on a matrix D iff the resulting regression parameter estimates are the same whether or not they are based on raw or filtered data, i.e., $\tilde{\beta}_S = \widehat{\beta}_S$. This is a reasonable property to require of our filters, and will also result in a unique – up to choice of a certain orthogonal matrix – filter.

Theorem *Let F be a filter matrix that is compatible with WLS estimates based on M^{-1} , such that (6), (7), and (9) hold. Then the filter is uniquely given, up to an indeterminate orthogonal matrix R of dimension $n - d$, by*

$$\begin{aligned} F &= M^{-1} (\Delta'_N \Gamma_V^{-1} \Delta_N - \Delta' \Gamma_{\partial V}^{-1} J \Delta); \\ J &= 1 - \Gamma_W \Gamma_{\partial U}^{-1/2} R \Gamma_W^{-1/2}. \end{aligned}$$

Also F is centro-symmetric iff R is. F is dynamic-replicating, i.e., (8) holds, iff the noise is stationary. In any case, the error covariance matrix is

$$M^{-1} + M^{-1} (\Delta' \Gamma_{\partial V}^{-1} J \Gamma_W J' \Gamma_{\partial V}^{-1} \Delta) M^{-1}. \quad (11)$$

Remark It is shown in McElroy (2008a) that the minimal mean square error signal extraction filter has error covariance matrix M^{-1} ; hence the second term of (11) represents the additional error that results from the dynamic matching approach. We also see from the formula for F that the DM filter equals the MSE optimal filter F_* minus a matrix term that fully differenced the data.

For implementation of these results, one first obtains the component models using either a decomposition or structural approach. The requisite Δ and Γ matrices are then easily formed from the differencing, AR and MA polynomials. Then it is a simple matter to form the filter matrix F from the Theorem, utilizing singular value decompositions to compute the requisite matrix square roots. Repeating this procedure for every desired signal, we obtain dynamic matching estimates for all the components of interest. For example, if the data has seasonal (S), trend (T), and irregular (I) components, then

$$Y = \tilde{S} + \tilde{T} + \tilde{I} + E,$$

where \widetilde{S} denotes the DM estimate of S , etc. Here E is a remainder component—unlike with WK smoothing, the filter matrices do not sum up to the identity matrix. However, E will have mean zero, since each of the errors $\widetilde{S} - S$, $\widetilde{T} - T$, and $\widetilde{I} - I$ do. For the application of seasonal adjustment, we see three possible ways of defining a seasonally adjusted component:

$$Y - \widetilde{S}, \quad \widetilde{T} + \widetilde{I}, \quad \widetilde{T + I}.$$

Only with the first definition do the estimated components for seasonal and nonseasonal sum to Y , and only the last estimate will have the desired dynamic-matching properties, in general. In this case, the remainder can be lumped in with the seasonal as an undesirable portion of the series, i.e., we propose to publish $\widetilde{T + I}$ as the seasonal adjustment and $Y - \widetilde{T + I}$ as the seasonal. This preserves additivity of the components, which is important for statistical agencies. Of course $Y - \widetilde{T + I}$ is not a DM estimate of seasonality, but nevertheless it will capture the main nonstationary seasonal patterns (since it equals $(1 - F)Y$, and we may apply (7)).

One further point is that a choice of the orthogonal matrix R in Theorem must be determined, such that it is centro-symmetric. For simplicity we choose the identity matrix in all our applications below. In Section 4 below, we will compute DM seasonal adjustments by the following procedure:

1. Fit a model to Y
2. Obtain the component models for seasonal, trend, and irregular (e.g., via canonical decomposition)
3. Compute F from Theorem, and the DM seasonal adjustment is $\widetilde{T + I} = FY$ using $R = 1_{n-d}$.

4. COMPARISON OF METHODOLOGIES

We now proceed to evaluate the DM filtering methodology of Section 3, in comparison to the traditional WK approach (which is also explicitly described

in Section 3). First, we look at 88 time series published by the U.S. Census Bureau, and apply both methods to seasonally adjust each series. Comparisons are then made between the lag 12 sample autocorrelations. Second, we examine revision variances for both methods on all 88 time series. Thirdly, we plot the signal extraction MSE curves for both methods applied to two series, m00190 and x3, as an illustration. Finally, we examine the gain and phase delay properties of concurrent WK and DM seasonal adjustment filters for a particular Airline process.

Let us begin with a discussion of the data. Since it was our intention to fix a particular model – due to its frequent identification in X-12-ARIMA as well as its pedagogical appeal, we have chosen the Airline model – we began with a large suite of series, performed automatic model selection, and retained only those with a common model. Beginning with 19 Foreign Trade Import series (1989.1 through 2001.11), 20 Foreign Trade Export series (1989.1 through 2001.11), 10 Retail series (1992.1 through 2007.12), 10 Housing Starts series (1992.1 through 2006.12), and 87 Manufacturing Value of Shipments series (1992.1 through 2009.11), a subset of 88 series were best fit with an Airline model – these are listed in Tables 1 and 2. Many of these 88 required a log transformation and/or correction for regression effects (e.g., additive outliers, level shifts, trading day, and holiday effects). Five of the 88 series had a seasonal moving average parameter extremely close to unity – though X-12-ARIMA preferred the Airline model to a reduced model with seasonal regressors. See Findley et al. (1998) for background concepts.

So for each of the 88 regression-adjusted series, we proceeded with application of the WK and DM seasonal adjustment procedures (end of Section 3), taking a log transformation first if needed. For an example of what these adjustments look like, along with the accompanying MSE curves, see Figures 3 and 4. We summarize the overall patterns of residual annual cyclicity in Tables 1 and 2: the lag 12 sample autocorrelation for the twice-differenced seasonal adjustments are reported, along with the corresponding critical value, which is computed in the manner described in Section 2. Significant autocorrelations are in bold font. Now by reckoning up the number of significant WK seasonal adjustments, and for how many of these the corresponding DM seasonal

Table 2 Results for Manufacturing Series

Series	WK acf	DM acf	Crit	DM/WK	Rev
u11avs	-0.247	-0.049	0.180	0.20	0.57
u11bvs	-0.197	0.041	0.164	0.21	0.55
u11cvs	-0.091	0.002	0.177	0.02	0.57
u11svs	-0.189	-0.023	0.180	0.12	0.56
u12avs	-0.215	0.026	0.182	0.12	0.59
u12bvs	-0.118	0.009	0.186	0.07	0.57
u14svs	-0.064	0.068	0.176	1.06	0.54
u15svs	-0.240	-0.068	0.177	0.28	0.58
u16svs	-0.306	-0.055	0.178	0.18	0.60
u21svs	-0.170	-0.051	0.164	0.30	0.57
u22avs	-0.050	0.143	0.164	2.83	0.59
u24svs	-0.117	0.003	0.165	0.02	0.59
u25bvs	-0.096	0.122	0.185	1.27	0.61
u25svs	-0.121	0.000	0.164	0.00	0.59
u26svs	-0.140	0.070	0.180	0.50	0.57
u31cvs	-0.182	-0.010	0.178	0.06	0.56
u33avs	-0.235	-0.079	0.176	0.33	0.57
u33dvs	-0.105	0.033	0.178	0.31	0.56
u33gvs	-0.099	0.047	0.179	0.47	0.55
u33ivs	-0.270	-0.100	0.182	0.37	0.56
u34jvs	-0.056	0.103	0.184	1.84	0.57
u35avs	-0.167	0.038	0.184	0.23	0.59
u35bvs	-0.274	-0.087	0.184	0.32	0.65
u35cvs	-0.284	0.030	0.185	0.11	0.61
u35dvs	-0.197	0.019	0.185	0.10	0.61
u35svs	-0.232	-0.011	0.183	0.05	0.59
u36avs	-0.140	-0.015	0.182	0.11	0.57
u36bvs	-0.161	0.005	0.181	0.03	0.56
u36svs	-0.119	0.032	0.182	0.27	0.57
u36zvs	0.007	0.060	0.185	8.23	0.63
u37svs	-0.281	-0.084	0.176	0.30	0.56
uanmvs	-0.246	-0.045	0.164	0.18	0.55
ubtpvs	-0.141	-0.005	0.180	0.04	0.58
ucdgvvs	-0.204	-0.024	0.180	0.12	0.56
ucmsvs	-0.353	-0.129	0.170	0.37	0.55
ucngvs	-0.226	-0.033	0.179	0.15	0.57

Table 2 Results for Manufacturing Series (Continued)

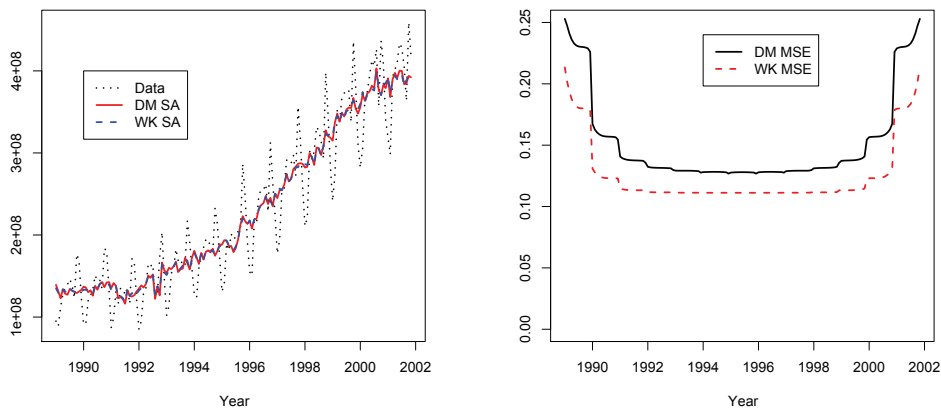
Series	WK acf	DM acf	Crit	DM/WK	Rev
ucogvs	-0.316	-0.086	0.182	0.27	0.59
udapvs	-0.128	-0.040	0.185	0.31	0.58
umtmvs	-0.305	-0.017	0.176	0.06	0.58
umvpvs	-0.189	-0.036	0.181	0.19	0.57
umxdvs	-0.307	-0.012	0.174	0.04	0.58
unapvs	0.097	0.097	0.185	1.00	1.00
uodgvs	-0.246	0.027	0.177	0.11	0.57
utgpvs	-0.298	-0.056	0.184	0.19	0.60

Note: For Manufacturing series, the lag 12 sample autocorrelation of twice-differenced seasonal adjustments from both the WK and DM methodologies is given, along with the critical value associated with a 5 percent rejection rate (see Section 2 for discussion). The absolute ratio of the correlations is given in the fifth column. The sixth column gives the ratio of DM revision variance to WK revision variance, for the respective seasonal adjustments (see Section 4).

adjustments are significant, we can get an overall idea of the empirical performance of the methods. We summarize over all 88 series, although results were not remarkably different if we restrict to just Manufacturing or just Foreign Trade.

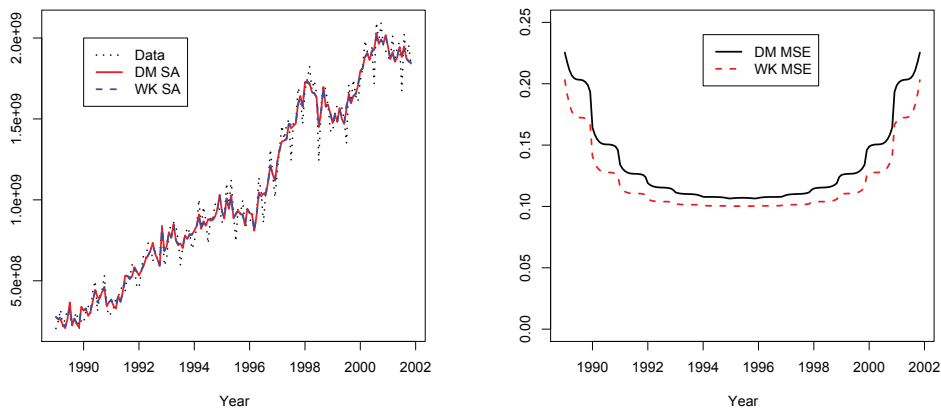
We find that 46 out of 88 series had significant annual cyclicity in the WK SAs. Out of these 46 series, 40 did not have significant annual cyclicity in the DM SAs. That is, 87 percent of the “problematic” series were “fixed” by the DM procedure. Six of the 46 series were not “fixed” by the DM procedure, although the overall magnitude of lag 12 autocorrelation tended to decrease. Of the other 42 series for which there was no significant annual cyclicity in their WK SAs, none of them had significant annual cyclicity in their DM SAs; so the DM procedure never made things worse than the WK procedure.

However, because our null hypothesis – that the WK estimate’s γ_{12} is *not* significantly different from the γ_{12} for the true component – is known *a priori* to be false, some readers may find formal testing to be unhelpful in gauging the empirical prevalence of residual annual cyclicity. Another approach is to just chart the ratios of $\widehat{\gamma}_{12}$ for the WK and DM estimates. So we also assess



Note: The left panel displays the m00190 series with both the DM and WK seasonal adjustments. The right panel displays the MSE curves from both the DM and WK methods, in units of the innovation variance. The fitted model was a (0.63, 0.42) Airline model.

Figure 3 Seasonal Adjustments and MSE Curves for the m00190 Series



Note: The left panel displays the x3 series with both the DM and WK seasonal adjustments. The right panel displays the MSE curves from both the DM and WK methods, in units of the innovation variance. The fitted model was a (0.36, 0.62) Airline model.

Figure 4 Seasonal Adjustments and MSE Curves for the x3 Series

performance by taking the absolute ratio of lag 12 sample autocorrelations for the DM and WK SAs, which is reported in Tables 1 and 2. Values less than one favor the DM method. This generally happened, with exceptions falling into two classes. In two cases the WK autocorrelation was indeed much less, but both correlations were close to zero and non-significant. In five other cases the autocorrelations were exactly the same for the two methods – this arose when the seasonal moving average parameter was close to unity. Here, the value of the parameters makes the matrices F_* and F almost identical, so that the SAs are really the same. The average of all the ratios was 0.608 – overall, the DM gives a 39 percent reduction in lag 12 sample autocorrelation over the WK method. Moreover, except for the seven cases described above, DM always reduced the magnitude of $\widehat{\gamma}_{12}$ as compared to WK.

We make one other observation here. In our empirical work, we experimented with what happens when the DM and WK methods are utilized with parameter values other than the mles. This really tended to ruin the DM method's matching properties, as we would expect from the Theorem; the resulting autocorrelations then did not improve measurably over those from the WK method. This indicates that correct model specification and good fit – such as provided by mles – is required for the DM method to be practically useful.

Now let us consider the issue of revisions. A signal extraction revision is defined as an update to a previous estimate when new data becomes available (see McElroy and Gagnon (2008) for discussion). If new data have lots of pertinent information about the signal, we can expect large revisions – this should be viewed favorably, since we are improving on an obsolete estimate, although statistical agencies tend to be nervous when revisions are large and are aware that they bewilder many users. Application of a WK filter always decreases the series' variance, because the gain function is bounded between 0 and 1. Therefore they tend to smooth series, which indicates that we can expect substantial revisions in practice. The DM approach does less smoothing: in AW it is shown that its gain (for the case of stationary bi-infinite series) is the square root of that of the WK filter, resulting in a larger number. Hence, we should expect revisions to be smaller for the DM method, which is indeed the case in practice.

Ultimately, preliminary signal extraction estimates are revised to a final estimate. This could be viewed asymptotically as the output of a bi-infinite filter that uses the maximum possible set of information. Note that AW derives a frequency domain expression for a dynamic-replicating filter for stationary bi-infinite time series; the extension to the non-stationary case is currently being studied by the author. Now the final dynamic-replicating estimator discussed in AW differs substantially from the final WK estimator – the former’s frequency response function being the square root of the latter’s. We emphasize that our discussion of revisions is in terms of preliminary to final estimates (though for finite samples), for *each* method, and these final estimators are indeed different.

To fix ideas in our study of revisions, we consider estimating the non-seasonal component at a given time t based on the 120 present and past observations (i.e., a concurrent estimate of the signal). This initial estimate is updated 12 months later; the new estimate is based on the asymmetric filter of length 132, that uses 12 future observations (i.e., future to time t , which is fixed, but now available to us) and 120 present and past observations. In this manner revisions can be computed – for either the DM or WK SAs – by windowing through each series, taking as many windows of length 132 as can be contained. We call the average of the square of these revisions the revision variance (this is an empirical quantity; theoretical variances could be calculated for the DM filter by extension of the formulas in McElroy and Gagnon (2008), which handle the WK case). Note that we have fixed the model and mles for the entire data span, so this revision variance is “in-sample”. Of course, results will depend upon this choice, as well as the window size (120) and revision lead (12), and this tempers our findings accordingly. However, the revision variance pattern in the final column of Tables 1 and 2 is remarkable, none of the ratio being greater than unity. The average of the ratios over the 88 series is 0.624, a 38 percent reduction of the WK revision variance. For those agencies favoring small revisions to SAs, the DM method may be appealing.

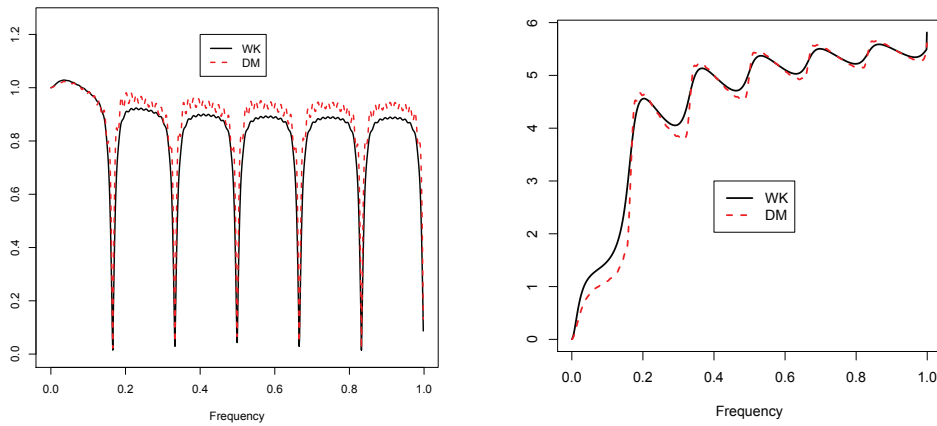
However, as in all things, there is a tradeoff to using the DM method. The WK method minimizes MSE, so the signal extraction error is higher with the DM method. Its MSE can be computed using the Theorem; see the Remark of Section 3. We plot these MSE curves for both methods, for the series m00190

and x_3 , in Figures 3 and 4. The irregular convex shape is a well-known phenomenon; higher error is present at the sample boundaries due to increased uncertainty in concurrent filtering, whereas the oscillations arise from the finite-sample aspect of the model (Bell, 2005; McElroy, 2008a). Comparing the minimum MSEs for the WK and DM methods, and taking the ratio of the former to the latter, we find that the WK MSE is 86.9 percent of the DM MSE for $m00190$, whereas the number is 93.7 percent for the x_3 series. We can also compare MSEs at the boundary of the sample – corresponding to concurrent estimators – and obtain the percentages 84.4 and 90.1 respectively. This increase in the DM MSE can be weighed against the decrease in revision variance.

A final comparison between the methods is afforded by comparing gain and phase delay plots for concurrent filters. As discussed in Findley and Martin (2006), (squared) gain plots can be used to assess how a filter attenuates the variance of a stationary time series at various frequencies, whereas the phase delay gives information about how much lag a filter induces on a stationary time series at each frequency. We refer the reader to Findley and Martin (2006) for details on definitions; note that we construct the *continuous* phase delay function, which can be done by introducing a \pm sign to the gain function. When we plot the corresponding gains, we take the absolute value (these functions would be squared to understand their impact on variances). Graphs are provided (Figures 5 and 6) for the (0.63, 0.42) Airline model of $m00190$ and the (0.36, 0.62) Airline model of x_3 , with concurrent filters arising from the respective series (so length 155) analyzed for both the WK and DM methods.

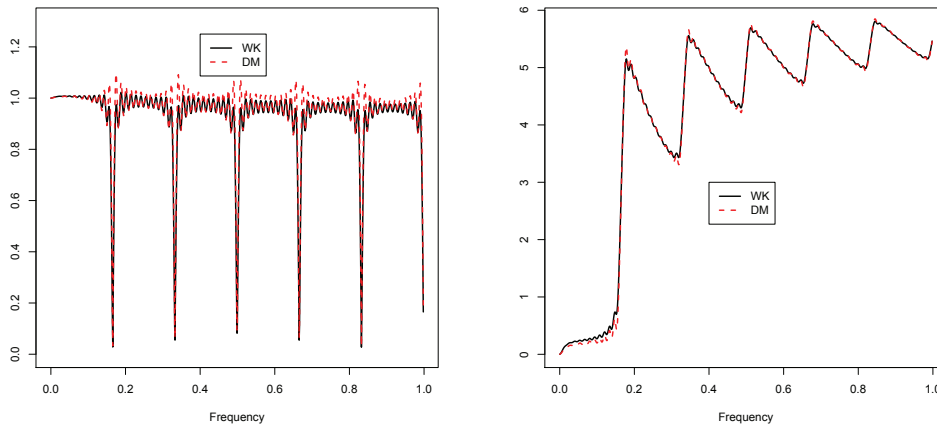
The first gain plot (Figure 5) has the characteristic “nose” in the low-pass band, a known feature with low-pass (e.g., trend or seasonal adjustment) concurrent filters (cf. Findley and Martin, 2006). The dips at seasonal frequencies serve to annihilate poles in the data pseudo-spectrum. Note that the DM filter has narrower dips, which facilitates dynamic matching. Also, the higher humps of the DM filter indicate that less smoothing of the middle and high frequencies occurs, as compared to the WK – this is consistent with the output in Figures 3 and 4. The oscillations in the humps are a known finite-sample phenomenon (Findley and Martin, 2006).

The phase delay plots are to be read as follows: at each frequency, the corresponding ordinate tells you how much the corresponding harmonic in the data



Note: The left panel displays the gain functions for the concurrent filters used on the m00190 series, for both the WK and DM methods. The right panel displays the two phase delay functions for the concurrent filter used on the m00190 series, for both the WK and DM methods.

Figure 5 Gain and Phase Delay Functions for the m00190 Series



Note: The left panel displays the gain functions for the concurrent filters used on the x3 series, for both the WK and DM methods. The right panel displays the two phase delay functions for the concurrent filter used on the x3 series, for both the WK and DM methods.

Figure 6 Gain and Phase Delay Functions for the x3 Series

is delayed in time by the action of the filter. This amount of delay is between 0 and 6 months, in these cases. Now it is desirable that a concurrent low-pass filter have less phase delay in its low-pass band, since this allows for timeliness and reduced bias in trends and non-seasonal components. Although the overall contours are similar in both Figures 5 and 6, the DM filter has less phase delay in the low-pass band (roughly speaking, frequencies between 0 and 0.1 or 0.2, say). This is an encouraging and intuitive result: since the WK method does more smoothing, we can expect it to have greater phase delay in the low frequency band.

5. CONCLUSION

This paper's empirical work shows the presence of annual cyclicity in model-based seasonal adjustments arising from the Wiener-Kolmogorov (WK) filtering methodology (Tables 1 and 2). Although this phenomenon has been known to exist for more than four decades, our methods of quantifying it are new and easily applied in practice (the Bartlett confidence thresholds of Section 2). More importantly, we have presented an extension of the AW methodology to finite samples drawn from non-stationary time series (the Theorem of Section 3), called the dynamic matching (DM) approach. This represents a non-trivial extension of the AW idea. Our extensive empirical study of 88 economic time series shows that DM improves upon the WK approach, in terms of reducing the presence of annual cyclicity in adjusted series (Section 4). However, DM seasonal adjustments have higher MSE. This is the principal tradeoff.

Of course there are other issues to assess in a comparison of methods. We have hit upon some of the topics of most interest to practitioners – namely, revision variance and phase delay (Section 4 and Figures 5 and 6). It might also be asked: what of the implementation of the DM method versus the WK? And what of computational speed? The direct encoding of F in the Theorem requires virtually no additional effort beyond that of F_* in (5), if one adopts a matrix-based approach to signal extraction. This is also far easier than implementing a state space algorithm; our R code is available upon request from the author. As for speed, the required matrix inversions are extremely fast for series

of moderate length (i.e., less than 30 years of data), those commonly encountered in seasonal adjustment practice.

Another feature of the DM method is that estimates of signal and noise do not aggregate to the original process, as it does with the WK method. It is a common phenomenon that quantities no longer satisfy innate aggregation relations after the application of a statistical procedure – this has spurred many efforts in the time series literature on benchmarking and reconciliation. The enforcing of aggregation relations necessarily destroys the statistical properties enjoyed by the original estimates (except in the rare case that these properties are invariant to the reconciliation procedure). One may not retain both the DM property and the additivity of signal and noise estimates, except in the special case that true signal and noise are decoupled – i.e., the product of their pseudo-spectra is identically zero. This case does not arise in ARIMA-modeling of time series, though the WK approach would be dynamic matching.

We have written this paper with an audience of seasonal adjustment practitioners in mind. We have attempted to clearly lay out the pros and cons of both the DM and WK methods in an honest manner. The AW philosophy may be appealing to some, the benefits of reduced annual cyclicity and revisions outweighing the increased MSE; others may prefer to retain the classical WK approach instead. However, there is little cost to examining both seasonal adjustments; one can view the lag 12 sample autocovariance of the differenced seasonal adjustments, assess against the Bartlett-based significance levels, and then decide upon which method to utilize.

APPENDIX 1 PROOF OF THE PROPOSITION

By (6) and (7), we have $\Delta_S F = \Delta_S G \Delta_N = (1 - \Delta_S H) \Delta_S$. It follows from (4) and results in McElroy (2008a) that $\Delta_S M^{-1} \Delta'_N = \Gamma_U \underline{\Delta}'_N \Gamma_W^{-1} \underline{\Delta}_S \Gamma_V$; also $\Delta_N M^{-1} \Delta'_N$ is invertible with inverse $\Gamma_V^{-1} + \underline{\Delta}'_S \Gamma_{\partial U}^{-1} \underline{\Delta}_S$. Then if we multiply $\Delta_S G \Delta_N = (1 - \Delta_S H) \Delta_S$ on both sides by $M^{-1} \Delta'_N (\Delta_N M^{-1} \Delta'_N)^{-1}$, we obtain

$$\begin{aligned} \Delta_S G &= (1 - \Delta_S H) \Delta_S M^{-1} \Delta'_N (\Delta_N M^{-1} \Delta'_N)^{-1} \\ &= (1 - \Delta_S H) \Gamma_U \underline{\Delta}'_N \Gamma_{\partial U}^{-1} \underline{\Delta}_S, \end{aligned}$$

which implies that $\Delta_S F = B \Delta$ for a matrix $B = (1 - \Delta_S H) \Gamma_U \underline{\Delta}'_N \Gamma_{\partial U}^{-1}$, i.e., the signal-differenced filter matrix first does $\delta(B)$ -differencing on the data. We also note that $\Delta_S M^{-1} \Delta'_S$ is invertible with inverse $\Gamma_U^{-1} + \underline{\Delta}'_N \Gamma_{\partial V}^{-1} \underline{\Delta}_N$. Thus a similar calculation to that given above will yield

$$\begin{aligned} \Delta_N H &= (1 - \Delta_N G) \Delta_N M^{-1} \Delta'_S (\Delta_S M^{-1} \Delta'_S)^{-1} \\ &= (1 - \Delta_N G) \Gamma_V \underline{\Delta}'_S \Gamma_{\partial V}^{-1} \underline{\Delta}_N, \end{aligned}$$

so that $\Delta_N(1 - F) = C \Delta$ with $C = (1 - \Delta_N G) \Gamma_V \underline{\Delta}'_S \Gamma_{\partial V}^{-1}$. So $\Delta = \Delta F + \Delta(1 - F) = \underline{\Delta}_N B \Delta + \underline{\Delta}_S C \Delta$, and multiplying this by $\Delta'(\Delta \Delta')^{-1}$ we obtain

$$\underline{\Delta}_N B + \underline{\Delta}_S C = 1. \tag{A.1}$$

Then we obtain $\Delta_N F = \Delta_N - C \Delta = (1 - C \underline{\Delta}_S) \Delta_N$, and it follows from (A.1) and (5) that

$$\begin{aligned} M F &= \Delta'_N \Gamma_V^{-1} (1 - C \underline{\Delta}_S) \Delta_N + \Delta'_S \Gamma_U^{-1} B \Delta; \\ F &= F_* + M^{-1} (\Delta'_S \Gamma_U^{-1} B - \Delta'_N \Gamma_V^{-1} C) \Delta. \end{aligned}$$

Now the estimate of the differenced signal is $\widehat{U} = \Delta_S \widehat{S} = \Delta_S F Y = B W$. Hence $\Gamma_{\widehat{U}} = B \Gamma_W B'$, and B is $n - d_S \times n - d$ dimensional. Now (8) holds iff $\Gamma_U = B \Gamma_W B'$, which has no solution unless $d_N = 0$, in which case B is square and equals $\Gamma_U^{1/2} R \Gamma_W^{-1/2}$ for some orthogonal matrix R . Proceeding under the assumption that $d_N = 0$, note that all occurrences of Δ_N can be replaced by identity matrices, and also V can be replaced by N (likewise Γ_N substitutes for Γ_V , and Δ_S replaces $\underline{\Delta}_S$, etc.). In this case we obtain

$$F = F_* + M^{-1} \left(\Delta'_S \Gamma_U^{-1/2} R \Gamma_W^{-1/2} - \Gamma_N^{-1} C \right) \Delta,$$

where $\Delta_S C = 1 - B$. Then any choice of R and C (e.g., $C = \Delta'_S (\Delta_S \Delta'_S)^{-1} (1 - B)$) yields a filter matrix with the desired properties.

APPENDIX 2 PROOF OF THE THEOREM

From the proof of the Proposition we know that under the stated conditions (6) and (7) the matrix has the form

$$F = F_* + M^{-1} \left(\Delta'_S \Gamma_U^{-1} B - \Delta'_N \Gamma_V^{-1} C \right) \Delta.$$

But the compatibility condition (for both signal and noise) requires that $0 = X'_N \Delta'_S \Gamma_U^{-1} B$ and $0 = X'_S \Delta'_N \Gamma_V^{-1} C$, which – due to the fact that $X = [X_S \ X_N]$ and $\Delta_N X_N = 0 = \Delta_S X_S$ – is equivalent to $X' A = 0$, where $A = \Delta'_S \Gamma_U^{-1} B - \Delta'_N \Gamma_V^{-1} C$. By linear independence of the columns of X , and the fact that they all lie in the null space of Δ (cf. the proof of Lemma 2 of McElroy and Sutcliffe (2006)), we must have A' equal to a linear combination of Δ , or in other words

$$A = \Delta'_S \Gamma_U^{-1} B - \Delta'_N \Gamma_V^{-1} C = \Delta' K, \tag{A.2}$$

for some matrix K . Applying $\Delta_S M^{-1}$ to the left hand side of (A.2) then yields (after much algebra) $B = \Gamma_U \underline{\Delta}'_N (K + \Gamma_{\partial V}^{-1} \underline{\Delta}_S C)$. Likewise, applying $\Delta_N M^{-1}$ to the left hand side of (A.2) produces $C = -\Gamma_V \underline{\Delta}'_S (K - \Gamma_{\partial U}^{-1} \underline{\Delta}_N B)$. This shows that $B = \Gamma_U \underline{\Delta}'_N Q$ and $C = \Gamma_V \underline{\Delta}'_S P$ for some matrices Q and P . Then under condition (9) we obtain $\Gamma_{\partial U} = \Gamma_{\partial U} Q \Gamma_W Q' \Gamma_{\partial U}$, and by the singular

value decomposition we obtain $Q = \Gamma_{\partial U}^{-1/2} R \Gamma_W^{-1/2}$ for some orthogonal R . So $B = \Gamma_U \underline{\Delta}'_N \Gamma_{\partial U}^{-1/2} R \Gamma_W^{-1/2}$, and from (A.1) we obtain $C = \Gamma_V \underline{\Delta}'_S \Gamma_{\partial V}^{-1} (1 - \Gamma_{\partial U} Q)$. Now plugging these formulas in and simplifying yields the stated expression for F .

To prove centro-symmetry, define the transverse-transpose of a square matrix A to be A^* with jk th entry $A_{n-k+1, n-j+1}$. Then A is centro-symmetric iff $A' = A^*$. Now from the definition of the matrix square root and the elementary properties $(AB)^* = B^* A^*$ and $A^{*'} = A^*$, we find that $(A^{1/2})^* = (A^*)^{1/2}$. It follows that if A is centro-symmetric, so is $A^{1/2}$. Let \mathcal{C} denote the space of invertible centro-symmetric matrices. By the properties discussed in McElroy (2008a), we have the following closure properties of this space: $A, A + B \in \mathcal{C} \Rightarrow B \in \mathcal{C}$; $A, AB \in \mathcal{C} \Rightarrow B \in \mathcal{C}$. Then it suffices to demonstrate centro-symmetry of $\Delta' \Gamma_{\partial V}^{-1} J \Delta$. So let $\tilde{\Delta}$ be shorthand for $\delta(L_n)$, noting that $[0 \ 1] \tilde{\Delta} = \Delta$. Then

$$\Delta' \Gamma_{\partial V}^{-1} J \Delta = \tilde{\Delta}' \begin{bmatrix} 0 & 0 \\ 0 & \Gamma_{\partial V}^{-1} (1 - \Gamma_W \Gamma_{\partial U}^{-1/2} R \Gamma_W^{-1/2}) \end{bmatrix} \tilde{\Delta}.$$

Applying the operator $*$ yields

$$\begin{aligned} \tilde{\Delta}^* & \begin{bmatrix} (1 - \Gamma_W^{-1/2} R^* \Gamma_{\partial U}^{-1/2} \Gamma_W) \Gamma_{\partial V}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tilde{\Delta}^{*'} \\ & = \Delta' (1 - \Gamma_W^{-1/2} R^* \Gamma_{\partial U}^{-1/2} \Gamma_W) \Gamma_{\partial V}^{-1} \Delta. \end{aligned}$$

But so long as $R \in \mathcal{C}$, this is the transpose of the original expression, which establishes the centro-symmetry of $\Delta' \Gamma_{\partial V}^{-1} J \Delta$, and hence $F \in \mathcal{C}$. Finally, we compute the signal extraction error covariance matrix. Note that $F = F_* + M^{-1} \Delta' \Gamma_{\partial V}^{-1} J \Delta$, so that the error process is

$$\epsilon = FY - S = (F_* - 1)S + F_*N + M^{-1} \Delta' \Gamma_{\partial V}^{-1} JW.$$

This equals the error process for the optimal matrix filter F_* – which is orthogonal to W (this is the optimality property) – plus a second term. Hence the covariance matrix at once is shown to be given by (11).

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以模型為基礎 降低殘差季節自我相關度的季節調整

Tucker McElroy*

Center for Statistical Research and Methodology
U.S. Census Bureau

關鍵詞: ARIMA、季節性、訊號抽取、Wiener-kolmogorov

JEL 分類代號: C01

* 聯繫作者: Tucker McElroy, Center for Statistical Research and Methodology, U.S. Census Bureau, 4600 Silver Hill Road, Washington, DC 20233。電話: 301-763-3227; 傳真: 301-763-8399; E-mail: tucker.s.mcelroy@census.gov。

摘 要

當擴展 Wiener-Kolmogorov (WK) 訊號抽出濾器到非平穩數序列及干擾時, 它對於高斯過程具有均方誤差 (MSE) 最小的特性。然而, WK 訊號估計的隨機動態性質卻經常與目標過程大相逕庭。使用這廣為周知的濾器, 它可能會在季節調整後時間序列的譜函數中產生凹點。這些凹點符合落後 12 期的負自我相關 (或負季節自我相關), 亦即存在全年隨機週期的現象。所謂的「平方根」WK 濾器是由 Wecker (1979) 在分析平穩的訊號和干擾數列時提出的, 它能確保訊號估計與原始的數列有相同的隨機動態, 亦即消除了譜凹點。這說明了一個不同的統計原理: 我們不只想要估計量能精準貼近目標值, 我們同時也希望估計量的動態也能很貼近目標的動態。MSE 標準忽略訊號抽出在這方面的問題, 然而「動態匹配」濾器雖然會增加額外的 MSE, 它考慮並解決這個問題。本文對於季節調整後序列發生負季節自我相關這個現象提出一個實證研究, 並提供符合理想訊號動態特徵且同時符合非平穩時間序列有限樣本下濾器的矩陣公式。我們將這些濾器應用到 88 個時間序列, 大幅降低出現負季節自我相關的頻率。