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Seasonal Heteroskedasticity in Time Series Data: Modeling, Estimation, and Testing

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Abstract

Seasonal heteroskedasticity refers to regular changes in variability over the calendar year. Models for two different forms of seasonal heteroskedasticity were recently proposed by Proietti and by Bell. We examine use of likelihood ratio tests with the models to test for the presence of seasonal heteroskedasticity, and use of model comparison statistics (AIC) to compare the models and to search among alternative patterns of seasonal heteroskedasticity. We apply the models and tests to U.S. Census Bureau monthly time series of housing starts and building permits.

KEYWORDS: seasonal adjustment, trend, unobserved component.

JEL classification: C22, C51, C82

1 Introduction

Seasonal heteroskedasticity is defined by regular changes in variability over the calendar year. In this paper we examine approaches to modeling two different forms of seasonal heteroskedasticity: the seasonal specific models introduced recently by Proietti (2004), and an extension of the airline model proposed by Bell (2004). We examine use of likelihood ratio tests with the models to test for the presence of seasonal heteroskedasticity, and use of model comparison statistics (AIC) to compare the models and to search among alternative patterns of seasonal heteroskedasticity. We apply the models and tests to U.S. Census Bureau monthly time series of housing starts and building permits. For these time series there is a clear reason to expect seasonal heteroskedasticity – the potential effects of weather on activity surrounding new construction.

Seasonal heteroskedasticity can be thought of as a particular form of periodic behavior. Two general types of periodic models are the periodic autoregressive-moving average models studied in Tiao and Grupe (1980), and the form-free seasonal effects models of West and Harrison (1989). A key feature of the latter is the use of a multivariate model for a complete set of processes, one for each month, that is defined at all time points, though only the process corresponding to the calendar month of observation is observed at each time point. The required hidden components are then easily handled in the state space form of the model. Proietti's model is of this type. Bell (2004), in contrast, starts with the popular "airline model" of Box and Jenkins (1976), but augments it with an additional white noise component with seasonally heteroskedastic variance. Bell's model is thus more related to those of Tiao and Grupe (1980), and could be thought of as a simple extension to ARIMA component models of their approach.

While Bell's model differs fundamentally from Proietti's model, it suggests a modification of Proietti's model to allow for a seasonally heteroskedastic irregular component, so we consider this third model as well. Proietti (2004, p. 2) noted this possibility but did not pursue it.

In most economic applications the limited length of available data sets would raise concerns about whether modeling a fully general pattern of seasonal heteroskedasticity – different variances for each of the 12 months – would involve too many variances to estimate. Therefore, in this paper, for all three models considered we focus on the case where there are only two distinct variances, leading to what can be called "high variance months" and "low variance months." This addresses the concern about "too many parameters," but poses the challenge of determining which months fall in the high and low variance groups. This can be based on any available prior knowledge about which months are likely to have higher variance, or on empirical evidence. We take the latter approach here by developing an algorithm analogous to forward selection stepwise regression for selecting the high and low variance months.

Proietti (1998) discussed some earlier models for seasonal heteroskedasticity and developed an extension to Harvey's (1989) basic structural model (BSM) using a heteroskedastic seasonal component which generalizes the seasonal component of Harrison and Stevens (1976). Proietti (2004, p. 5) notes how his seasonal specific levels model reduces to a variant of his earlier model when certain constraints are imposed on the parameters. Tripodis and Penzer (2007) also used this heteroskedastic extension of the BSM, as well as a variant with a seasonally heteroskedastic irregular (analogous to our third model). For both these models Tripodis and Penzer considered the case where only one month has a different variance from the others, a particular case of the form of seasonal heteroskedasticity that we consider here.

The paper proceeds as follows. Section 2 presents the three models for seasonal heteroskedasticity: the seasonal specific levels model of Proietti (2004), the ARIMA plus seasonal noise model of Bell (2004), and the modified form of Proietti's model with a seasonally heteroskedastic irregular component. Section 3 presents our algorithm for determining which months should be regarded as high variance and which as low variance, and applies it to time series of regional housing starts and building permits from the U.S. Census Bureau. Section 4 examines, through simulations, the behavior of likelihood ratio tests for the presence of seasonal heteroskedasticity in the contexts of the three models. Section 5 then applies these likelihood ratio tests to the time series of regional housing starts and building permits. Finally, Section 6 offers conclusions.

2 Models for seasonal heteroskedasticity

This section presents the three time series models designed to capture seasonal heteroskedasticity: those of Proietti (2004) and Bell (2004), and the seasonal noise form of Proietti's model.

2.1 Seasonal specific levels model (Proietti 2004)

Here we review the model proposed in Proietti (2004), where heteroskedastic movements are specified in a set of level equations that involve separate processes for the different calendar months. We refer to this as the seasonal specific levels model. Since the applications we consider involve monthly data, "month" and "season" are used interchangeably in what follows.

Given a seasonal time series y_t observed at time points $t = 1, ..., T$, the model is

$$y_t = \mathbf{z}_t' \boldsymbol{\mu}_t + \varepsilon_t, \qquad \varepsilon_t \sim i.i.d.(0, \sigma_\varepsilon^2), \quad t = 1, ..., T \tag{1}$$

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t + \mathbf{i}\beta_t + \mathbf{i}\eta_t + \boldsymbol{\eta}_t^*, \qquad \boldsymbol{\mu}_t = (\mu_{1t}, ..., \mu_{st})' \tag{2}$$

$$\eta_t \sim i.i.d.(0, \sigma_\eta^2), \qquad \boldsymbol{\eta}_t^* = (\eta_{1t}^*, ..., \eta_{st}^*)', \qquad \eta_{jt}^* \sim i.i.d.(0, \sigma_{\eta*,j}^2)$$

$$\beta_{t+1} = \beta_t + \zeta_t, \qquad \zeta_t \sim i.i.d.(0, \sigma_{\boldsymbol{\zeta}}^2)$$

where $\mathbf{i}$ is an $s \times 1$ vector of ones, with s denoting the number of seasons in a year, that is $s = 12$ for monthly data. In the observation equation (1), $\mathbf{z}_t$ is an $s \times 1$ selection vector that has a one in the position $j = 1 + (t-1) \bmod s$ and zeroes elsewhere. Thus, $\mathbf{z}_t'\boldsymbol{\mu}_t$ picks off the element of $\boldsymbol{\mu}_t$ corresponding to the month of time point t, and this is the only part of $\boldsymbol{\mu}_t$ that directly affects the observation y_t. The observation at time t is then the sum of the appropriate μ_{jt} and the noise term ε_t.

The state equation (2) specifies the evolution of the vector of monthly levels, $\boldsymbol{\mu}_t$. At time t, each of the s elements of $\boldsymbol{\mu}_t$ is subject to a shared level disturbance η_t and to an idiosyncratic level disturbance η_{jt}^* that is uncorrelated with η_t and with η_{kt}^* for $k \neq j$. The η_{jt}^* have variances $\sigma_{\eta*,j}^2$ which depend, in general, on the season j, and so are the source of seasonal heteroskedasticity in the model. If, however, $\sigma_{\eta*,j}^2$ is constant over $j = 1, \ldots, 12$, then the model becomes homoskedastic. The elements of $\boldsymbol{\mu}_t$ are incremented at time t by a common slope β_t, whose disturbance ζ_t is assumed uncorrelated with the other disturbances in the model.

Proietti (2004) considers various alternative versions of the structure given in equation (2). These include a restricted version that sets $\beta_t = 0$, a more general version that allows seasonal heteroskedasticity in β_t, a version with a circular correlation structure between the η_{jt}^*, and a multivariate extension. However, he appears to suggest the model given by equation (2) as the most common variant.

2.2 Airline model with seasonal noise (Bell 2004)

Bell (2004) starts with a standard seasonal time series model but extends it to include seasonally heteroskedastic noise. Specifically, he extends the airline model as follows:

$$y_t = Y_t + h_t \varepsilon_t \tag{3}$$

$$(1-B)(1-B^{12})Y_t = (1-\theta B)(1-\theta_{12}B^{12})a_t \tag{4}$$

$$a_t \sim i.i.d.(0, \sigma_a^2), \qquad \varepsilon_t \sim i.i.d.(0, \sigma_\varepsilon^2)$$

For the most general model of seasonally heteroskedasticity we could set $\sigma_\varepsilon^2 = 1$ and thus make h_t the standard deviations of the additive noise, which would follow some seasonal pattern. Bell (2004), however, considers just the simpler case of high and low variance months by letting h_t be 1 for the high variance months and zero otherwise. Thus, σ_ε^2 becomes additional irregular variance added in only for the high variance months. One can then think of base irregular variation as embedded in the Y_t component, say via the canonical seasonal plus trend plus irregular decomposition of Hillmer and Tiao (1982).

2.3 Seasonal specific irregular model

Bell's model suggests an easy modification to Proietti's model to use a heteroskedastic irregular rather than a heteroskedastic level series. To do this we set $\sigma^2_{\eta*,j} = \sigma^2_{\eta*}$, a constant value for all j, and make the irregular in (1) heteroskedastic. The model then becomes

$$y_t = \mathbf{z}_t' \boldsymbol{\mu}_t + \varepsilon^*_{jt}, \qquad \varepsilon^*_{jt} \sim i.i.d.(0, \sigma^2_{\varepsilon*,j}), \qquad t = 1, ..., T \tag{5}$$

The random shocks ε^*_{jt} are still assumed uncorrelated across seasons, and their variance depends only on the season index $j = 1 + (t-1) \bmod s$. Equation (2) still applies for $\boldsymbol{\mu}_t$, but with the homoskedasticity constraint on the variances of the η^*_{jt}.

For heteroskedastic seasonal variation linked to weather or other factors with seasonal, but otherwise temporary, effects, models (3) and (5) have some appeal since the heteroskedasticity comes from the irregular component.

3 Determining the pattern of seasonal heteroskedasticity

We will consider the heteroskedastic models of Section 2 in the case for which months are classified into two groups with different variances. For Proietti's model (1) we can label the variances for the two groups $\sigma^2_{\eta * I}$ and $\sigma^2_{\eta * II}$ with no constraint about which of these two is the larger. Similarly for the modified version (5) we can write $\sigma^2_{\epsilon * I}$ and $\sigma^2_{\epsilon * II}$ for the irregular variances of the two groups. For these two models reversing the assignment of the months between groups I and II does not change the model. For example, the model for which January is assigned to group I and the remaining months are all assigned to group II is equivalent to the model with January assigned to group II and the remaining months all assigned to group I. There are thus $2^{11} = 2{,}048$ possible groupings of the months for these models. The labels "high variance months" and "low variance months" can be assigned to the two groups according to the estimated variances, e.g., for the model (1), according to whether $\hat{\sigma}^2_{\eta * I} \geq \hat{\sigma}^2_{\eta * II}$ or $\hat{\sigma}^2_{\eta * I} < \hat{\sigma}^2_{\eta * II}$.

For the airline plus seasonal noise model (3), the months with $h_t = 1$ are necessarily the "high variance months" since $\sigma^2_{\varepsilon} \geq 0$. For example, if $h_t = 1$ only for January then January is the only high variance month, and the remaining months are low variance. Note that this model is not equivalent to the model where $h_t = 1$ for all months except January. There are thus $2^{12} = 4{,}096$ possible groupings of the months for this model, though for half of these groupings the estimated model is likely to reduce to the homoskedastic model. For example, if the model for which $h_t = 1$ only for January yields $\hat{\sigma}^2_{\varepsilon} > 0$, then it seems likely (though we have no mathematical proof that this is always true), that the model for which $h_t = 1$ for all months except January will yield $\hat{\sigma}^2_{\varepsilon} = 0$.

For all three models we first need to specify the grouping of the months. In some cases the grouping may be suggested by prior knowledge or previous analyses, but in other cases it may need to be determined empirically. Even if prior knowledge suggests a possible grouping, an empirical search over alternative groupings may be needed to refine or confirm the prior grouping. Therefore, in Section 3.1 we present an algorithm for empirically selecting the best fitting grouping of months. As it is not feasible to search over the full set of 2,048 possible groupings for models (1) and (5), or 4,096 for model (3), the algorithm employs a search strategy to approximate the best fitting grouping without checking all possibilities. The search strategy is somewhat analogous to forward selection stepwise regression. In Section 3.2 we apply the algorithm to the time series of building permits and housing starts.

Previous papers (Proietti (1998, 2004), Tripodis and Penzer (2007)) have determined reduced parameterizations of seasonal heteroskedasticity in less formal ways, though in most of their applications the months with potentially different variances were fairly obvious.

3.1 Algorithm for grouping months

We now describe a simple algorithm that attempts to approximate the best fitting grouping of months. The algorithm uses the Akaike Information Criterion (AIC, Akaike 1974) to compare model fits, where the AIC is defined as $\text{AIC} = -2\log\hat{L} + 2k$, where $\log\hat{L}$ is the maximized log-likelihood and k is the number of model parameters. However, for any of our three model forms, k will be constant over all models being compared that actually involve two groups, that is, k will be different only for the homoskedastic model. So, except for comparisons with the homoskedastic model, the AIC comparisons reduce to comparisons of differences in $2\log\hat{L}$.

The search algorithm proceeds as follows. First, the homoskedastic model is estimated, and the AIC recorded. Second, the algorithm runs a number of iterations. At each iteration, the algorithm starts with January, switches its grouping, fits the resulting model, and preserves the switch if this decreases the AIC. Then, the algorithm moves on to February, switches its grouping, fits this model, and preserves the switch if this decreases the AIC. The algorithm then moves through each calendar month until December is reached, which ends the iteration. If any calendar months have switched groups, then the next iteration starts again with January. Otherwise, if no calendar months have switched groups, then the algorithm stops.

We estimate the models by maximum likelihood (ML) using state space methods. Given a feasible parameter vector, the likelihood function is evaluated from the prediction error decomposition obtained from the Kalman filter, see Harvey (1989). The parameter estimates are computed by optimizing over the likelihood surface in each case. To do the calculations for the results given below, we used programs written in the Ox language (Doornik 1999), which included the Ssfpack library of state space functions (Koopman, et. al 1999).

3.2 Application

The set of time series we consider, listed in Table 1, are monthly estimates of the numbers of building permits issued, and also estimates of the numbers of total housing starts, for the four regions of the U.S. (NE = Northeast, MW = Midwest, SO = South, and WE = West). The data are from the U.S. Census Bureau. (Source and reliability information for these series is available from www.census.gov/const/www/newresconstdoc.html.) The observation period for building permits is January 1959 to March 2006, while for housing starts it is January 1964 to March 2006. However, because we are interested later (Section 4) in testing for the presence of seasonal heteroskedasticity, for each series the sample period is divided into prior and testing periods. In making this division, our aim is to have a long enough series for testing, while leaving enough observations at the beginning to give a representative prior selection. The permits series are thus split into the two samples, 1959:1 to 1984:12 (prior) and 1985:1 to 2006:3 (testing), and the housing starts series are split into the two samples, 1964:1 to 1984:12 (prior) and 1985:1 to 2006:3 (testing).

Prior to applying the three models, each series was logged and then adjusted for trading-day effects via a fitted RegARIMA model (Bell and Hillmer 1983) using the X-12-ARIMA program (Findley, Monsell, Bell, Otto, and Chen 1998). It is the logged trading-day adjusted series that are modeled in this paper.

Tables 1-3 show the results of applying the grouping algorithm to the permits and housing starts series with our three models. The second column in each table shows the group of months determined by the algorithm to have higher variance. The third column shows $[\Delta\text{AIC}]_1$, the difference in AIC between the selected heteroskedastic model and the homoskedastic model. This gives an indication of evidence of heteroskedasticity (larger AIC differences imply stronger evidence of heteroskedasticity), but since the specific patterns of heteroskedasticity were determined by searching over the various month groupings to find the best fitting model, the results can be misleading. (A more objective decision on whether heteroskedasticity is present comes from application of the likelihood ratio tests to the models selected here for the series but fitted over their testing samples in Section 5.) Some indication of how definitive the groupings are is given by the fourth and fifth columns of the tables. These show the changes from the best fitting grouping that yield the second best fitting grouping, and the corresponding change in AIC, $[\Delta\text{AIC}]_2$. As noted earlier all these comparisons involve models with the same number of parameters, hence the $[\Delta\text{AIC}]_2$ are just differences in $2\log\hat{L}$.

Table 1. Calendar month groups for the building permits and housing starts time series: seasonal specific levels model

Series	High Variance Months	$[\Delta\text{AIC}]_1$	Second Best	$[\Delta\text{AIC}]_2$
Permits, NE	Jan, Feb, Dec	−9.3	−Dec	0.13
Permits, MW	Jan, Feb, Dec	−28.8	+Mar	4.10
Permits, SO	Jan, Mar, May, Jun, Sep, Dec	−2.8	−May, −Jan	0.08
Permits, WE	Jan, Apr, Jun, Jul, Sep, Dec	−4.9	+Mar	0.71
Starts, NE	Jan, Feb	−39.2	+Sep	6.53
Starts, MW	Jan, Feb	−58.1	+Dec	9.78
Starts, SO	Jan, Feb	−0.2	−Jan	2.18
Starts, WE	Jan, Dec	−7.3	−Dec	0.96

Note: The second column shows, for each series, the months assigned higher variability in the selected model, and the third column shows the AIC difference from the homoskedastic model. The fourth column shows the change in grouping from the best to the second best model, with the associated change in AIC given in the fifth column.

Table 2. Calendar month groups for the building permits and housing starts time series: airline plus seasonal noise model

Series	High Variance Months	$[\Delta\text{AIC}]_1$	Second Best	$[\Delta\text{AIC}]_2$
Permits, NE	Jan, Feb, Mar, Dec	−32.6	−Mar	0.11
Permits, MW	Jan, Feb, Mar, Dec	−33.1	−Mar, −Dec	3.38
Permits, SO	Mar, Jul, Sep, Dec	−12.0	+Apr	2.62
Permits, WE	Jan, Mar, Apr, Jun to Aug, Nov, Dec	−10.9	−Jul	0.39
Starts, NE	Jan, Feb	−39.0	+Sep	2.33
Starts, MW	Jan, Feb, Dec	−58.7	−Dec	0.00
Starts, SO	Jan, Feb, Mar, Dec	−10.9	+Jun	0.12
Starts, WE	Jan, Feb, Apr, Aug, Nov, Dec	−21.9	+Oct, −Nov	1.20

Note: The definitions of entries are the same as in Table 1.

Table 3. Calendar month groups for the building permits and housing starts time series: seasonal specific irregular model

Series	High Variance Months	$[\Delta\text{AIC}]_1$	Second Best	$[\Delta\text{AIC}]_2$
Permits, NE	Jan, Feb, Dec	−32.1	+Mar	0.02
Permits, MW	Jan, Feb, Mar, Dec	−32.5	−Mar	1.28
Permits, SO	Mar, Jul, Sep, Dec	−12.9	+Apr	2.31
Permits, WE	Jan, Mar, Apr, Jun to Aug, Nov, Dec	−11.8	−Jul	1.02
Starts, NE	Jan, Feb	−38.8	+Sep	2.29
Starts, MW	Jan, Feb	−56.2	+May	8.27
Starts, SO	Jan, Feb, Mar, Dec	−10.9	+Jun	0.12
Starts, WE	Jan, Feb, Apr, Aug, Nov, Dec	−20.8	+Oct, −Nov	0.27

Note: The definitions of entries are the same as in Table 1.

We first consider results for the Northeast and Midwest series. In all cases $[\Delta\text{AIC}]_1$ is negative and in nearly all cases (Northeast permits for the seasonal specific levels model being perhaps the sole exception) it is large in magnitude, so the heteroskedastic model is strongly preferred over the homoskedastic model. In a few cases (e.g., Midwest starts with either the seasonal specific levels or seasonal specific irregular models) the $[\Delta\text{AIC}]_2$ values are large so that the grouping appears decisive. In other cases the value of $[\Delta\text{AIC}]_2$ is small, so that the model with the alternative grouping fits about as well. For example, for Midwest starts with the airline plus seasonal noise model $[\Delta\text{AIC}]_2 \approx 0$, with the second best grouping obtained by removing December from the high variance months. This suggests that the amount of variability in December may lie between that of the high and low variance groups. To handle this the model might be extended by setting $h_t = .5$ for December.

Notice that the grouping results for the Northeast and Midwest series suggest a simple pattern of higher variance in winter. For a given model there is some variation across these series in which months comprise the "winter group," and for a given series there are some differences in the grouping across the three models. The pattern of higher variance in winter is plausibly due to effects of unusually bad or unusually good winter weather on construction activity.

For the South and West series, in contrast, $[\Delta\text{AIC}]_1$, while always negative, is smaller in magnitude than for the Northeast and Midwest series. Also, in most of these cases, particularly for the permits series, the month groupings for the South and West series do not suggest a simple explanation (such as higher variance in winter), as the groupings appear somewhat random. Given that the month groupings were selected to minimize the AICs for the heteroskedastic models, there is thus some doubt about whether these results suggest heteroskedasticity is really present. (As noted earlier, we shall revisit this issue in Section 5.) The $[\Delta\text{AIC}]_2$ values for the South and and West series are generally small, suggesting that their month groupings are not very precisely determined.

Finally, for any of these eight series there is some consistency between the groupings from the airline plus seasonal noise and the seasonal specific irregular models, with the grouping from the seasonal specific levels model more often showing differences.

4 Finite sample behavior of likelihood ratio tests for seasonal heteroskedasticity

Given a grouping of months, as discussed in the previous section, likelihood ratio tests can be applied to the three models under consideration to test for the presence of seasonal heteroskedasticity. For the seasonal specific levels model (1) the hypothesis to be tested is H_0: $\sigma^2_{\eta * I} = \sigma^2_{\eta * II}$, and the corresponding hypothesis for the seasonal specific irregular model (5) is H_0: $\sigma^2_{\epsilon * I} = \sigma^2_{\epsilon * II}$. For the airline plus seasonal noise model (3) the hypothesis to be tested is H_0: $\sigma^2_{\epsilon *} = 0$. For any of these models, denoting the maximized likelihood from estimation of the model under the null hypothesis (homoskedastic model) as $\hat{L}_0$, and that from the unrestricted model as $\hat{L}$, respectively, the likelihood-ratio test statistic is $LR = -2(\log \hat{L}_0 - \log \hat{L})$. For all three models the hypotheses to be tested imply one restriction on the model parameters, so that standard asymptotic results for ML estimates would suggest comparing LR to a critical value from a chi-squared distribution with one degree of freedom. In this section we use simulation to examine such approximations to the null distributions of these LR tests in time series with specified lengths (10 or 20 years), month groupings, and values of the model parameters.

One issue with the finite sample performance of the LR tests is that there may be positive probability of the ML estimates satisfying the homoskedasticity constraint, thus yielding zero for the LR test statistic, and potentially affecting its finite sample distribution. For the two versions of Proietti's model a homoskedastic estimated model would arise from getting $\hat{\sigma}^2_{\eta * I} = \hat{\sigma}^2_{\eta * II} = 0$, or $\hat{\sigma}^2_{\epsilon * I} = \hat{\sigma}^2_{\epsilon * II} = 0$, respectively, since estimates of zero for variance components are fairly common in component models. (See Tanaka (1996, Section 8.7) and Shephard (1993).) Assuming that the true values of these variances are positive this problem should disappear as $T \to \infty$, but it may affect the finite sample distribution of LR.

The situation for Bell's model (3) is somewhat different. The homoskedastic version of this model has $\sigma^2_{\epsilon} = 0$, which is on the boundary of the parameter space. Potential problems from this could be avoided by allowing negative values for σ^2_{ϵ}, which would lower the irregular variance in the months with $h_t = 1$ (using the approach noted in Section 2.2 of decomposing the model for Y_t into seasonal plus trend plus irregular components. We then need to constrain σ^2_{ϵ} from becoming so negative that the pseudo-spectrum of Y_t takes on negative values at some frequency.) A simple way to fit this alternative model is to fit both the original model (3) and the corresponding model that reverses the month groupings, and then pick the model with the highest maximized likelihood. The resulting two-sided LR test of equality of variance for the two groups of months avoids the boundary value problem, facilitating the usual large-sample χ_1^2 approximation to the null distribution. However, when we perform the LR test by fitting only model (3) with the constraint $\sigma_\epsilon^2 \geq 0$, considerations of symmetry suggest that, when the null hypothesis of homoskedasticity is true, we would have $\Pr(\hat{\sigma}_\epsilon^2 = 0) = 0.5$ (at least asymptotically and when the month grouping assigns six months to each of the low and high variance groups). Since $\hat{\sigma}_\epsilon^2 = 0 \Rightarrow LR = 0$, to take this into account we assume that the null distribution of LR can be approximated by a distribution with $\Pr(LR = 0) = 0.5$ and with 0.5 times a χ_1^2 distribution for the probability distribution over $LR > 0$. Thus, to perform a 5% test with model (3) we would compare LR to the 10% critical value from the χ_1^2 distribution, which is 2.71. This is consistent with general results presented by Self and Liang (1987) for the case of independent observations. For testing whether a single parameter is on the boundary with other (nuisance) parameters away from the boundary (their Case 5), they obtain the $\frac{1}{2}I(LR = 0) + \frac{1}{2}\chi_1^2$ asymptotic distribution.

We used simulation to assess the null distribution of LR for each of the models. For a given model we simulated a large number J of series from the null (homoskedastic) model, estimated both the null and heteroskedastic models by ML, and computed LR. We then examined the distribution of LR across the simulations. The iterations to maximize the likelihood of a model started with initial values of parameters set to the true values – those used to generate the simulated series. To help ensure that we had reached the global, and not a local, maximum of the likelihood, for each model the maximzation was repeated with two alternative sets of initial values different from the true values, and the fit with the largest likelihood value was used. Occasionally use of the alternative starting values led to an improvement in log-likelihood, and though the difference was usually relatively small, this strategy helped guarantee that the strict global optimum was computed in each case.

Tripodis and Penzer (2007) used simulations to study the size and power of likelihood ratio tests of seasonal heteroskedasticity for Proietti's (1998) basic structural model with the heteroskedastic extension of Harrison and Stevens's (1976) seasonal, and for the corresponding model with a seasonally heteroskedastic irregular. They restricted consideration to the case of quarterly series with the first quarter having a variance different from the rest. For both models they found reasonable sizes for the likelihood ratio tests with the asymptotic χ_1^2 approximation. The tests had good power in most cases when the variance differences were large and the series were of moderate length (20 years) or longer, but not surprisingly had low power when the variation in the heteroskedastic component was low relative to the total variation in the other components. Their simulation results are difficult to compare to ours below because of differences in the model forms and their use of quarterly series.

4.1 Finite sample size of the LR test for the seasonal specific levels model

To focus on the scale invariant dynamics of the model, we look at the variance parameters relative to the irregular variance, and so use the signal-noise ratio for level, $q_\eta = \sigma_\eta^2/\sigma_\epsilon^2$, and for the seasonal, $q_{\eta*} = \sigma_{\eta*}^2/\sigma_\epsilon^2$, by setting $\sigma_\epsilon^2 = 1$. We set $\sigma_\zeta^2 = 0$ (so $\beta_t = 0$) for simplicity. We considered model estimation results for the permits and starts series to determine a range of parameter values to use in the simulations, settling on the values $q_\eta = \{0.1, 10\}$ and $q_{\eta*} = \{0.002, 0.02, 0.2\}$. We simulated 20,000 time series for each of the six possible combinations of $(q_\eta, q_{\eta*})$ and for two series lengths: $T = 120$ (ten years of monthly data) and $T = 240$ (twenty years of monthly data). In applying the model and performing the LR test we assumed four patterns of month groupings with the following months assigned to the "high variance" group: January only, January and February only, January to April (first four), and January to June (first six). Results are presented for the four different month groupings in Table A.1 of the appendix.

The first three columns in the tables show the $(q_\eta, q_{\eta*})$ values and the series length. The remaining three columns show the results of the simulations: $\Pr(LR = 0)$; $X_{0.05}$, the actual 5% critical value (which can be compared to $\chi_1^2(.05) = 3.84$); and $\Pr(Reject)$, the actual size of the test when using the χ_1^2 critical value. Monte Carlo uncertainty in the estimates of $\Pr(LR = 0)$ and $\Pr(Reject)$ is reflected in standard errors computed from the binomial distribution of the number of "successes," which gives $\sqrt{p(1-p)/20,000}$ where p is the probability in question. For reference, with 20,000 trials these standard errors are approximately .001, .0015, .002, .003, and .0035 for p values of .02, .05, .1, .2, and .5, respectively.

Examining Table A.1 we see that the lowest value of $q_{\eta*}$, corresponding to the least amount of monthly idiosyncratic variation in the μ_{jt}, often leads to substantial values of $\Pr(LR = 0)$. These values decrease as

$q_{\eta *}$ increases and also as the length of the series, T, increases. Examining the values of Pr(*Reject*), we see the LR test is often undersized, and by substantial amounts for the cases where $\Pr(LR = 0)$ is substantial. The corresponding actual 5% critical values are thus often substantially lower than the χ^2_1 5% critical value of 3.84. The results are slightly better for the first four and first six month groupings than they are for the January only and January and February only groupings, though these differences are smaller than the differences across different values of $q_{\eta *}$. In fact, the magnitude of the latter differences makes it difficult to specify approximate size-adjusted critical values, since the appropriate critical value obviously depends strongly on $q_{\eta *}$, which in practice will be unknown. One could take estimated values of q_η and $q_{\eta *}$, along with the series length T, and interpolate in the tables to determine an approximate critical value. Or, one could do this by running another set of simulations for the specific estimated values of q_η and $q_{\eta *}$. But either of these approaches would be compromised by the inherent error in estimation of the model parameters. These considerations suggest that proper application of the LR test in finite samples for the seasonal specific levels model is difficult.

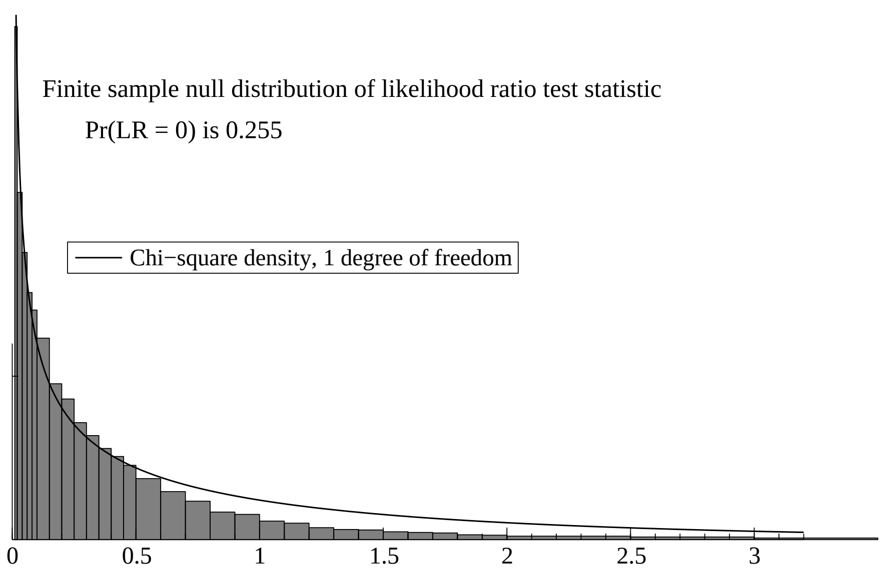

Figure 1: Distribution of LR test from Proietti's model for case $q_\eta = 0.1, q_{\eta *} = 0.002$ with $T = 120$ observations

More detail can be seen in Figure 1, which shows a histogram of simulations from the null distribution of LR for the January-only case when $q_\eta = 0.1, q_{\eta *} = 0.002$, and $T = 120$, along with an approximating χ^2_1 density. The actual 5% critical value of 1.60 (from Table 4.a) is well below the standard value of 3.84, and in the graph, the steeper decline of the density compared to the χ^2_1 is easily seen. (The bin widths of the histogram are narrower for values of LR closer to zero to provide more visual detail; thus, the heights of the rectangles are set so the areas of the rectangles all represent the relative frequencies from the simulations.)

4.2 Finite sample size of the LR Test for the airline plus seasonal noise model

Parameter values for the standard airline model are nearly always estimated to have positive values (e.g., Depoutot and Planas 1998). Considering this, and considering estimation results for model (3) for the building permits and housing starts series, we chose the following values for the nonseasonal and seasonal moving average coefficients for the simulation models: $\theta = 0.3, 0.6$, and $\theta_{12} = 0.5, 0.7, 0.9$. We set $\sigma_a^2 = 1$. For each of the six combinations of (θ, θ_{12}), and for each of the four different month groupings and two series lengths considered above, we simulated 20,000 time series from the airline model, this being the homoskedastic version of (3), i.e., the model with $\sigma_\epsilon^2 = 0$. We then computed the LR statistics for the simulated series, along with $\Pr(LR = 0)$, the actual 5% critical values, and the actual test sizes, $\Pr(\mathit{Reject})$. For reasons discussed above, the latter were obtained by checking if $LR > 2.71$, where 2.71 is the $\chi_1^2(.10)$ value. Results of this exercise are reported in Table A.2 of the appendix.

Examining first the values of $\Pr(LR = 0)$ we notice that, for the models with the first three patterns of heteroskedasticity, these probabilities slightly exceed .5, but for the fourth pattern (January–July modeled as high variance) they are not significantly different from .5. (Note that twice the standard error of these entries is about .007.) So the postulated symmetry argument does appear to hold when there are six months in both the high and low variance groups, but it does not quite hold in finite samples when the high and low variance groups are of unequal size. Turning to $\Pr(\mathit{Reject})$, we see that the LR test appears to be very slightly undersized for the January only and the January and February only month groupings, about right for the first four grouping, and possibly slightly oversized for the first six grouping. (Twice the standard error of these entries is about .003.) But the deviations from the nominal 5% value are slight, and it appears that applying the test as suggested (using the $\chi_1^2(.10)$ critical value for an overall 5% test) will yield reasonable results. If desired, the critical value used could be refined by making reference to the $X_{0.05}$ entries in the tables, say by taking as critical values something like 2.4 for the January only grouping, 2.55 for the January and February only grouping, 2.71 (the standard value) for the first four grouping, and 2.8 for the first six grouping. This inference is aided by the fact that, in contrast to the seasonal specific levels model, for the airline plus seasonal noise model there is not much variation in the critical values across different values of the model parameters – there is just some slight variation across the different month groupings.

Figure 2 shows an example of an estimated density for the LR test with Bell's model. This result is typical across different parameter values. Thus, with $T = 120$ and for a single high variance month, the probability of getting zero for LR is slightly greater than one-half, and the density for values of $LR > 0$ appears well approximated by half of a chi-squared density with one degree of freedom.

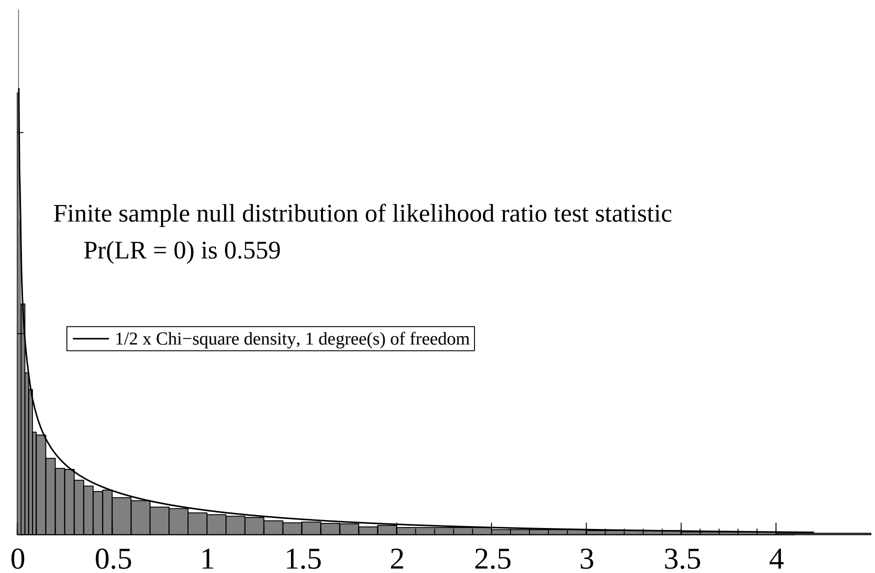

Figure 2: Estimated density of LR test statistic for airline plus seasonal noise model for certain parameter values. The estimates are based on $J = 20,000$ draws.

We also found that, for the parameter values used in our simulations, the null distribution of the LR test for the model (3) could be well-approximated by the null distribution of the corresponding LR test statistic for the case of independent observations. If there are T independent observations of which n_1 are $N(0, \sigma_1^2)$ and n_2 are $N(0, \sigma_2^2)$, then this LR statistic can be written as

$$LR_I = T[\log(\alpha F + 1 - \alpha) - \alpha \log(F)]$$

where $\alpha = n_1/T$, $F = \hat{\sigma}_1^2/\hat{\sigma}_2^2$, and $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ are the usual ML estimators of σ_1^2 and σ_2^2. Under the null hypothesis $\sigma_1^2 = \sigma_2^2$, F follows an F-distribution with (n_1, n_2) degrees of freedom, so the null distribution of

LR_I is easily simulated. Comparing results from 10 million such simulations of LR_I with the results in our tables, we noted that, for the January only and January and February only month groupings, the distribution of LR_I offered some improvement over the $\frac{1}{2}I(LR = 0) + \frac{1}{2}\chi_1^2$ approximation. Results for the first four and first six month groupings didn't yield an appreciable improvement. Further study may establish that using the distribution of LR_I could also yield some improvement for the first three month grouping and for shorter series lengths or parameter values not covered by our tables.

4.3 Finite sample size of the LR test for the seasonal specific irregular model

The null model for this case, (5), is the same as that for the seasonal specific levels model, thus, the same simulated series were used. The results are reported in Table A.3 of the appendix. We first note that in most cases $\Pr(LR = 0)$ is quite small, though when $q_{\eta^*} = .2$ and $T = 120$ it exceeds 2%. Next, we note that the actual test sizes are mostly close to 5%, though not quite as close as for the airline plus seasonal noise model. Size distortions occur mostly for the January only pattern and while these are fairly small they are not readily corrected since they depend on the parameter values. Overall, though, use of the $\chi^2(1)$ distribution to approximate the null distribution of LR does not seem unreasonable.

5 Application of LR tests and Model Comparisons

This section applies the LR tests developed in the previous section to the set of monthly regional building permit and housing starts time series under investigation. We also use AIC to compare, for each given series, the fits of the three (heteroskedastic) models. (As noted in Section 3.2, each series was logged and then adjusted for trading-day effects before applying the models analyzed here.) The three heteroskedastic models were then specified using the month groupings given in Tables 1-3. Recall that each series was split into two samples, a prior sample used in Section 3 for determining the month grouping, and a testing sample used here for computing the LR tests and comparing the model fits. For the building permits series the two samples were 1959:1 to 1984:12 (prior) and 1985:1 to 2006:3 (testing), and for the housing starts series the two samples were 1964:1 to 1984:12 (prior) and 1985:1 to 2006:3 (testing).

5.1 LR test results

Table 4 shows the LR test statistics for the eight series under study, for each of the three models. For the seasonal specific levels model and the seasonal specific irregular model these can be compared to the 5% χ_1^2 value of 3.84, though for the former model the size distortions shown in Table A.1 of the appendix imply that the distribution of the LR is not well-approximated by a χ_1^2. Nonetheless, Table 4 presents the LR values. For the airline plus seasonal noise model the LR test statistics for a 5% test can be compared to the 10% χ_1^2 value of 2.71, as discussed in Section 3.

Table 4. LR test statistics for seasonal heteroskedasticity for the building permits and housing starts time series

Series	Levels	Airline	Irregular
Permits, NE	23.6	27.0	28.6
Permits, MW	68.5	85.1	84.6
Permits, SO	0.053	0.000	0.007
Permits, WE	1.79	5.92	5.94
Starts, NE	16.5	21.7	24.7
Starts, MW	29.3	49.5	49.6
Starts, SO	20.1	11.0	12.4
Starts, WE	0.03	6.48	6.53

Note: The "Levels" column refers to the seasonal specific levels model (1), "Airline" to the airline plus seasonal noise model (3), and "Irregular" to the seasonal specific irregular model (5).

There is a good deal of agreement in the implications of the tests. For both permits and starts the test statistics are large and highly significant for both the Northeast and Midwest regions, the two regions for which we expect that unusually bad or unusually good winter weather would have significant effects on construction activity, leading to higher variance in the winter months. For the South and West regions the statistics are smaller, though several are significant. Only for building permits in the South are all the test statistics insignificant, and only for the seasonal specific levels model are most (three) of the statistics insignificant across the series for the South and West. The latter result may be partly due to the tendency seen in Table A.1 of the LR test to be undersized for this model in some cases.

Another interesting result in the table is that the values of the test statistics for the airline plus seasonal noise and for the seasonal specific irregular model are fairly similar. Given that both of these models allow for seasonal heteroskedasticity in the irregular, though the model forms differ in other ways, it is reassuring that inferences about seasonal heteroskedasticity do not seem to be affected much by the choice between these two different models. This is also consistent with the result noted in Section 3 that the selection of the month groupings was fairly similar for these two models.

5.2 Heteroskedastic model comparisons

Table 5 gives AIC values for comparing the fit of the three heteroskedastic models being investigated to our eight time series. For South Permits we also include AICs for the three homoskedastic models, since the results of Table 4 show that the LR tests do not reject any of these three models in favor of the heteroskedastic versions. The LR tests for West permits and starts for the seasonal specific levels model also fail to reject homoskedasticity, but as the performance of the LR tests for this model are questionable, and the LR tests with the other two models reject homoskedasticity for these two series, we do not pursue homoskedastic models further for these series.

Note that the validity of the AIC comparisons in Table 5 depends on the fact that all three models generally apply one seasonal and one nonseasonal difference to the series and no more, i.e., we take $(1-B)(1-B^s)y_t$. Exceptions occur for the component models if $\sigma^2_\zeta = 0$, or for the airline plus seasonal noise model if $\theta_{12} = 1$. These things do happen with the estimated models for a few of the cases considered here. In such cases application of $(1-B)(1-B^s)$ will "overdifference" the series, though since this does not invalidate the model, we include AIC comparisons for these cases. It is also worth noting that the seasonal specific levels and irregular models both involve 5 parameters, so that AIC comparisons between them reduce to log-likelihood comparisons, while the airline plus seasonal noise model contains 4 parameters, so its AIC "penalty term" is 2 less than those for the other two models.

In Table 5 the minimum AIC value for each series, indicating the preferred model, is shown in bold. Clearly the overall preferred model for these series is the airline plus seasonal noise model. Only for the Midwest building permits series is the seasonal specific irregular model preferred, though there is effectively a tie between it and the airline plus seasonal noise model for the Northeast housing starts series. The seasonal specific levels model is never preferred, and its AIC differences from the preferred model are mostly large (e.g., more than 10). Since this model and the seasonal specific irregular model differ only in regard to how the seasonal heteroskedasticity is modeled, it seems clear that, for these series, seasonal heteroskedasticity in the irregulars is preferred to seasonal heteroskedasticity in the month-to-month changes in the levels.

Table 5. AIC values for the three seasonal heteroskedastic models for the building permits and housing starts time series

Series	Levels	Airline	Irregular
Permits, NE	−381.3	**−391.0**	−386.3
Permits, MW	−393.6	−404.3	**−409.6**
Permits, SO (heteroskedatic)	−489.8	**−492.5**	−489.8
Permits, SO (homoskedatic)	−491.7	**−494.5**	−491.8
Permits, WE	−408.7	**−422.5**	−414.2
Starts, NE	−209.7	**−220.7**	**−220.7**
Starts, MW	−265.1	**−292.3**	−287.3
Starts, SO	−465.0	**−475.7**	−471.3
Starts, WE	−358.2	**−369.5**	−364.7

Note: The model with the minimum AIC value for a given series – the AIC preferred model – is indicated by its AIC value being in bold. The "Levels" column refers to the seasonal specific levels model (1), "Airline" to the airline plus seasonal noise model (3), and "Irregular" to the seasonal specific irregular model (5).

6 Conclusions

This paper considered alternative time series models for seasonal heteroskedasticity: the seasonal specific levels model of Proietti (2004), the airline plus seasonal noise model of Bell (2004), and a modification of Proietti's model that moves the seasonal heteroskedasticity to the irregular component. In this paper the heteroskedasticity in the models was limited to having two groups of months with different levels of variability. We presented an algorithm analogous to forward selection stepwise regression that uses AIC comparisons to determine, for a given series, which months to assign to each of the two groups. Then, given a grouping of the months, likelihood ratio tests can be used to test for the presence of seasonal heteroskedasticity. We used simulations to examine the finite sample distributions of such tests under the null hypothesis of homoskedasticity, finding appreciable size distortions in the seasonal specific levels model, but much better performance of the tests in the other two models. We also noted some differences in the distribution of the test statistics according to differences in the month grouping (e.g., a grouping with January alone distinct from the other months, versus a grouping with January through June in one group and July through December in the other).

We applied our month grouping algorithm and the likelihood ratio tests for seasonal heteroskedasticity to a set of U.S. Census Bureau regional construction series (building permits and housing starts). For these series we expect any seasonal increase in variance to be concentrated in the winter months. Results from the grouping algorithm were consistent with this expectation for the series for the Northeast and Midwest regions, the two regions most likely to show seasonal heteroskedasticity due to winter weather effects. The likelihood ratio tests then strongly confirmed the presence of seasonal heteroskedasticity in these series. (To avoid application of the grouping algorithm biasing the LR test results, we split the series into two parts, with the grouping algorithm applied to the first part and the LR test to the second.) While the tests also detected some evidence for seasonal heteroskedasticity in some of the series for the South and West regions, the evidence there was less conclusive. Finally, AIC comparisons over the three models considered favored the airline plus seasonal noise model overall for most of the series, while between the two versions of Proietti's model the model with the heteroskedastic irregular was strongly preferred.

Acknowledgements

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Appendix: Simulation results

The following tables contain estimates of critical values and size for the LR test for the three different models when testing for the four different monthly patterns of heteroskedasticity. The estimates are based on 20,000 simulated series for each case.

Table A.1 Seasonal specific levels model: likelihood ratio (LR) test for heteroskedasticity, simulations from null (homoskedastic) model

***T* = 120 observations**

		Months specified in the model as the "high variance" group											
		January only			**January and February**			**January – April**			**January – July**		
q_η	q_{η^*}	$Pr(0)$	$X_{.05}$	Pr(reject)	$Pr(0)$	$X_{.05}$	Pr(reject)	$Pr(0)$	$X_{.05}$	Pr(reject)	$Pr(0)$	$X_{.05}$	Pr(reject)
.100	.002	.253	1.60	.009	.245	1.83	.010	.210	2.35	.013	.203	2.44	.014
.100	.020	.016	2.38	.014	.014	3.17	.029	.011	3.70	.045	.011	3.82	.049
.100	.200	.004	4.06	.059	.004	4.41	.068	.003	4.38	.065	.003	4.30	.063
1.000	.002	.284	1.70	.012	.261	1.91	.011	.242	2.20	.013	.235	2.29	.012
1.000	.020	.037	2.27	.017	.030	2.87	.022	.027	3.42	.038	.025	3.74	.047
1.000	.200	.004	3.70	.045	.003	4.29	.065	.003	4.28	.065	.002	4.28	.062

***T* = 240 observations**

.100	.002	.066	1.97	.013	.062	2.46	.016	.051	3.03	.025	.049	3.17	.028
.100	.020	.002	3.28	.028	.001	4.00	.057	.002	4.21	.061	.002	4.08	.058
.100	.200	.002	4.14	.059	.002	4.14	.059	.001	4.02	.056	.002	4.12	.058
1.000	.002	.117	1.93	.013	.106	2.28	.015	.096	2.72	.018	.095	2.86	.021
1.000	.020	.002	2.89	.020	.001	3.78	.048	.001	4.09	.058	.001	4.19	.021
1.000	.200	.001	4.22	.062	.001	4.12	.058	.001	4.00	.054	.001	4.03	.056

Notes: $q_\eta = \sigma_\eta^2 / \sigma_\varepsilon^2$ and $q_{\eta^*} = \sigma_{\eta^*}^2 / \sigma_\varepsilon^2$ are the variance ratios for the homoskedastic seasonal specific levels model with $\sigma_\zeta^2 = 0$. The other entries give results from values of the LR test statistic for heteroskedasticity obtained from 20,000 simulated series: Pr(0) gives the estimated probability that LR is zero, $X_{.05}$ gives the empirically determined 5% critical value, and Pr(reject) gives the actual size of the test when the asymptotic χ_1^2 5% critical value (3.84) is used.

Table A.2 Airline plus seasonal noise model: likelihood ratio (LR) test for heteroskedasticity, simulations from null (homoskedastic) model

***T* = 120 observations**

		Months specified in the model as the "high variance" group											
		January only			**January and February**			**January – April**			**January – July**		
θ	θ_{12}	$Pr(0)$	$X_{.05}$	Pr(reject)	$Pr(0)$	$X_{.05}$	Pr(reject)	$Pr(0)$	$X_{.05}$	Pr(reject)	$Pr(0)$	$X_{.05}$	Pr(reject)
.3	.5	.562	2.36	.040	.547	2.54	.045	.520	2.71	.050	.505	2.77	.052
.3	.7	.555	2.39	.041	.539	2.53	.045	.513	2.80	.053	.501	2.91	.057
.3	.9	.553	2.45	.043	.532	2.63	.047	.514	2.88	.055	.501	3.02	.059
.6	.5	.560	2.41	.041	.537	2.55	.046	.517	2.65	.049	.504	2.93	.057
.6	.7	.558	2.33	.040	.533	2.57	.046	.514	2.68	.049	.503	2.77	.052
.6	.9	.554	2.49	.043	.533	2.59	.045	.515	2.65	.048	.503	2.76	.052

***T* = 240 observations**

.3	.5	.540	2.50	.044	.527	2.59	.046	.518	2.72	.050	.503	2.74	.051
.3	.7	.538	2.47	.043	.525	2.52	.045	.512	2.62	.048	.503	2.78	.052
.3	.9	.542	2.37	.040	.529	2.52	.045	.518	2.60	.046	.502	2.72	.050
.6	.5	.541	2.46	.042	.525	2.56	.046	.510	2.64	.048	.500	2.83	.055
.6	.7	.544	2.38	.041	.524	2.50	.044	.515	2.63	.048	.504	2.67	.049
.6	.9	.533	2.58	.046	.520	2.55	.046	.507	2.71	.050	.496	2.71	.050

Notes: θ and θ_{12} are the nonseasonal and seasonal moving average parameters for the homoskedastic airline model (without seasonal noise, i.e., $\sigma_\varepsilon^2 = 0.$) The other entries give results from values of the LR test statistic for heteroskedasticity obtained from 20,000 simulated series: Pr(0) gives the estimated probability that LR is zero, $X_{.05}$ gives the empirically determined 5% critical value, and Pr(reject) gives the actual size of the test when the asymptotic 5% critical value (2.71, obtained from the ½ + ½ × χ_1^2 distribution) is used.

Table A.3 Seasonal specific irregular model: likelihood ratio (LR) test for heteroskedasticity, simulations from null (homoskedastic) model

***T* = 120 observations**

		Months specified in the model as the "high variance" group											
		January only			**January and February**			**January – April**			**January – July**		
q_η	$q_{\eta*}$	*Pr*(*0*)	$X_{.05}$	*Pr*(reject)	*Pr*(*0*)	$X_{.05}$	*Pr*(reject)	*Pr*(*0*)	$X_{.05}$	*Pr*(reject)	*Pr*(*0*)	$X_{.05}$	*Pr*(reject)
.100	.002	.002	4.39	.068	.003	4.28	.063	.003	4.16	.060	.003	4.04	.056
.100	.020	.003	4.30	.065	.003	4.23	.061	.003	4.12	.059	.003	3.95	.053
.100	.200	.022	3.40	.035	.021	3.87	.051	.020	3.99	.055	.025	3.93	.052
1.000	.002	.002	3.66	.043	.001	4.10	.059	.001	4.11	.058	.001	4.09	.057
1.000	.020	.002	3.52	.039	.001	4.02	.055	.002	4.06	.056	.001	4.09	.057
1.000	.200	.011	2.89	.023	.008	3.47	.039	.007	3.76	.048	.006	3.86	.051

***T* = 240 observations**

.100	.002	.002	4.00	.055	.002	4.04	.056	.002	3.95	.053	.002	3.92	.052
.100	.020	.002	4.03	.056	.002	3.90	.052	.001	3.85	.050	.002	3.93	.053
.100	.200	.003	3.80	.049	.002	3.91	.052	.002	3.85	.051	.003	3.88	.051
1.000	.002	.001	4.01	.056	.001	3.97	.054	.001	3.97	.054	.001	3.94	.053
1.000	.020	.002	3.83	.050	.001	4.01	.055	.001	3.91	.052	.001	3.96	.053
1.000	.200	.007	3.27	.032	.005	3.72	.046	.004	3.87	.051	.005	3.91	.053

Notes: $q_\eta = \sigma^2_\eta / \sigma^2_\varepsilon$ and $q_{\eta*} = \sigma^2_{\eta*} / \sigma^2_\varepsilon$ are the variance ratios for the homoskedastic seasonal specific irregular model with $\sigma^2_\zeta = 0$. The other entries give results from values of the LR test statistic for heteroskedasticity obtained from 20,000 simulated series: Pr(*0*) gives the estimated probability that LR is zero, $X_{.05}$ gives the empirically determined 5% critical value, and *Pr*(reject) gives the actual size of the test when the asymptotic χ^2_1 5% critical value (3.84) is used.