

# Some Consideration of Seasonal Adjustment Variances\*

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## Abstract

We consider alternative definitions of seasonal adjustment variances for both model-based and X-11 type seasonal adjustments. In particular, when we account for sampling error components of time series we can consider either the variance of the estimation error for the nonseasonal component of the true series, or the variance of the estimation error for the seasonally adjusted series (observed series minus estimated seasonal component). The magnitudes of these variances can be quite different. We examine the relative contributions of various error components to seasonal adjustment variances, and the effects on model-based seasonal adjustment variances of uncertainty about model parameters. The considerations are illustrated with several examples.

**Keywords:** sampling error, canonical decomposition, forecast extension

## 1. Introduction

The desire for variances of seasonally adjusted data has a long history. It was mentioned in the report of the “Gordon commission” (President’s Committee to Appraise Employment and Unemployment Statistics 1962), predating the appearance of the popular X-11 seasonal adjustment program (Shiskin, Young, and Musgrave 1967). The development of methods of model-based seasonal adjustment (e.g., by Burman 1980; Hillmer and Tiao 1982; Harvey 1989; and others), and of corresponding computer software such as SEATS (Gomez and Maravall 1997), would seem to present opportunities for producing variances of seasonal adjustments, since these follow directly from signal extraction calculations for the models. Approaches to develop variances for X-11 type seasonal adjustments have also been proposed by Wolter and Monsour (1981), Pfeiffermann (1994), and Bell and Kramer (1999). Despite these developments, questions still remain about how to produce variances for seasonally adjusted data.

This paper reviews some of the issues that arise

for producing seasonal adjustment variances. Section 2 provides mathematical background and also makes the important point that, when sampling error is present in the observed time series, there are two choices of what to estimate in seasonal adjustment, and hence two possible definitions of seasonal adjustment error and of seasonal adjustment variances. Section 3 examines contributions to seasonal adjustment error from different perspectives, including (i) contributions from each of the components of the observed series (seasonal, nonseasonal, sampling error), (ii) contributions from forecast extension error, and (iii) contributions from uncertainty about regression and other model parameters. Examples illustrate that when sampling error is present its contribution to seasonal adjustment error can be quite important, and that the contribution to seasonal adjustment variances from parameter uncertainty can be quite erratic over time. Section 4 reviews different approaches to X-11 seasonal adjustment variances, noting how these approaches differ in regard to what components of error are accounted for. Section 5 summarizes the conclusions.

Examples presented here are intended to be illustrative of the issues discussed, and should not be interpreted as saying anything definitive about the particular time series used, many of which involve data that have since been revised. The series serve mainly to provide models of the type used in practice, which are used here only to illustrate various mathematical results. In most cases the results are presented as standard deviations rather than variances to provide more interpretable numbers.

## 2. Mathematical Background

Seasonal adjustment involves estimating components of an observed time series. The decomposition we generally consider here is:

$$y_t = S_t + N_t + e_t, \quad (1)$$

where  $y_t$  is the observed time series,  $S_t$  and  $N_t$  are the *seasonal* and *nonseasonal* components, and  $e_t$  is the *sampling error*. We shall say more about  $e_t$  shortly. We assume that the components in (1) are orthogonal, i.e., uncorrelated with each other at all leads and lags. Sometimes we shall further decom-

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pose the nonseasonal component as

$$N_t = T_t + I_t,$$

where  $T_t$  and  $I_t$  are the *trend* and *irregular* components, also assumed orthogonal to each other and to  $S_t$  and  $e_t$ . Often  $y_t$  will be the logarithms of original survey estimates, implying a log-additive decomposition, equivalent to a multiplicative decomposition on the original (unlogged) scale. In this case the standard deviations of the component estimates may be interpreted in percentage terms, approximating coefficients of variation (CVs) on the original scale.

In (1) we leave the range of the time index,  $t$ , unspecified. Most expressions given apply to the theoretical cases of a semi-infinite sample ( $t = \dots - 1, 0, 1, \dots, n$ ) or a doubly infinite sample ( $t = 0, \pm 1, \pm 2, \dots$ ), though actual calculations done here are only for the practically relevant case of a finite sample ( $t = 1, \dots, n$ ).

The inclusion of the sampling error component  $e_t$  in (1) is appropriate when the  $y_t$  are estimates from a repeated sample survey. We then view  $y_t$  as estimating corresponding population quantities  $Y_t$  that would be obtained if a complete census were conducted each time period, and so write

$$y_t = Y_t + e_t. \quad (2)$$

(We often think of  $Y_t$  as the unobserved “true” series, though this ignores *nonsampling errors* in the survey estimates  $y_t$  such as response and non-response errors, survey frame coverage errors, and so on.) Time series modeling and seasonal adjustment often ignores sampling error, implicitly assuming that the “true” series  $Y_t$  is observed. Though some economic time series are not subject to sampling error (import and export statistics being typical examples), many economic time series do contain significant sampling error components. Since (1) and (2) imply that  $S_t$  and  $N_t$  are the seasonal and non-seasonal components of  $Y_t$ , not of  $y_t$ , when sampling error is present it should be recognized as in (1) and (2), and accounted for when developing seasonal adjustment variances. The references to variances for X-11 seasonal adjustments cited in the Introduction all take sampling error into account in their analyses. Bell and Otto (1992) and Bell (2004) consider sampling error in the context of model-based seasonal adjustment.

Let  $\omega_N(B) = \sum_j \omega_{N,j} B^j$  denote a linear filter to be applied to  $y_t$  to estimate  $N_t$ , where  $B$  is the backshift operator ( $By_t = y_{t-1}$ ). We leave the limits of the summation unspecified to allow the expressions to apply for any given time  $t$ , and for the

finite, semi-infinite, and doubly infinite sample cases mentioned above. We write the estimator of  $N_t$  as

$$\hat{N}_t = \omega_N(B)y_t = \sum_j \omega_{N,j} y_{t-j}. \quad (3)$$

We consider only such linear estimators, although since model-based seasonal adjustment filters involve estimated model parameters, this makes their component estimators actually nonlinear functions of the data  $y_t$ . We shall use  $\hat{N}_t$  as a generic notation for an estimate of  $N_t$ , that is, we won't generally add to the notation of  $\hat{N}_t$  or  $\omega_N(B)$  anything to indicate the specific model or filter being used. The nature of the specific filter and estimator  $\hat{N}_t$  being discussed at any point should be clear from the accompanying text. Similarly, we let  $\omega_S(B) = \sum_j \omega_{S,j} B^j$  denote the linear filter used to estimate  $S_t$ , that is

$$\hat{S}_t = \omega_S(B)y_t = \sum_j \omega_{S,j} y_{t-j}. \quad (4)$$

We can alternatively view seasonal adjustment as the estimation of  $N_t$ , or as the estimation and removal of  $S_t$ , i.e., as the estimation of  $A_t = y_t - S_t$ . In the latter case the seasonally adjusted series,  $\hat{A}_t$ , is

$$\hat{A}_t = y_t - \hat{S}_t = [1 - \omega_S(B)]y_t. \quad (5)$$

We now consider errors in these estimates of  $N_t$ ,  $S_t$ , and  $A_t$ .

From (1) and (3), the error,  $\hat{\varepsilon}_t^N$ , in using  $\hat{N}_t$  to estimate  $N_t$  is

$$\begin{aligned} \hat{\varepsilon}_t^N &= N_t - \omega_N(B)[S_t + N_t + e_t] \\ &= [1 - \omega_N(B)]N_t - \omega_N(B)S_t - \omega_N(B)e_t. \end{aligned} \quad (6)$$

From the orthogonality of the components, the variance of  $\hat{\varepsilon}_t^N$  is

$$\begin{aligned} \text{Var}(\hat{\varepsilon}_t^N) &= \text{Var}\{[1 - \omega_N(B)]N_t\} \\ &\quad + \text{Var}[\omega_N(B)S_t] + \text{Var}[\omega_N(B)e_t]. \end{aligned} \quad (7)$$

The terms in (7) can be computed, in principle, if models for the components  $N_t$ ,  $S_t$ , and  $e_t$  are known (subject to the terms in (7) not requiring differencing, a qualification discussed shortly.) Knowledge of component models is the basis for doing model-based seasonal adjustment although, as noted above, the sampling error component is often ignored. Component models are not generally available when doing X-11 type seasonal adjustment.

Analogously, the error in using  $\hat{S}_t$  from (4) to estimate  $S_t$  is

$$\hat{\varepsilon}_t^S = [1 - \omega_S(B)]S_t - \omega_S(B)N_t - \omega_S(B)e_t \quad (8)$$

and the variance of this error is

$$\begin{aligned} \text{Var}(\hat{\varepsilon}_t^S) &= \text{Var}\{[1 - \omega_S(B)]S_t\} \\ &+ \text{Var}[\omega_S(B)N_t] + \text{Var}[\omega_S(B)e_t]. \end{aligned} \quad (9)$$

In addition, the error in the seasonally adjusted series,  $\hat{A}_t$ , is  $\hat{\varepsilon}_t^A = (y_t - S_t) - (y_t - \hat{S}_t) = -\hat{\varepsilon}_t^S$ , and so the variance of the error in  $\hat{A}_t$  equals the variance of the error in the estimate of the seasonal component, that is,  $\text{Var}(\hat{\varepsilon}_t^A) = \text{Var}(\hat{\varepsilon}_t^S)$ .

Considering (6)–(9), we note the following points:

- If there is no sampling error in the series  $y_t$  then  $\omega_S(B) + \omega_N(B) = 1$ , the terms involving  $e_t$  drop out of (6)–(9), and we see that in this case  $\hat{\varepsilon}_t^N = \omega_S(B)N_t - \omega_N(B)S_t = -\hat{\varepsilon}_t^S = \hat{\varepsilon}_t^A$ , also implying that  $\text{Var}(\hat{\varepsilon}_t^N) = \text{Var}(\hat{\varepsilon}_t^S) = \text{Var}(\hat{\varepsilon}_t^A)$ .
- As long as the filters are constructed so that  $\omega_S(B) + \omega_N(B) = 1$ , i.e., by assuming no sampling error, then we can substitute  $\omega_S(B)$  for  $1 - \omega_N(B)$  in (6) and (7), and we can substitute  $\omega_N(B)$  for  $1 - \omega_S(B)$  in (8) and (9). But if  $y_t$  really does contain sampling error, then (6) and (7) still differ from (minus) (8) and (9) due to the terms involving  $e_t$ .

In general, when sampling error is present  $\hat{\varepsilon}_t^N \neq \hat{\varepsilon}_t^A$ , and so  $\text{Var}(\hat{\varepsilon}_t^N) \neq \text{Var}(\hat{\varepsilon}_t^S) = \text{Var}(\hat{\varepsilon}_t^A)$ . In fact, the difference between these can be substantial, as is illustrated by the following example.

**Example 1.** We use an example from Bell (2004) that involves a model fitted to a monthly time series of (logarithms of) U.S. retail sales of drinking places. This series had fairly significant sampling error, with an average sampling CV over the stretch of data used of about 5.1 percent. The following model was estimated for  $y_t = Y_t + e_t$ :

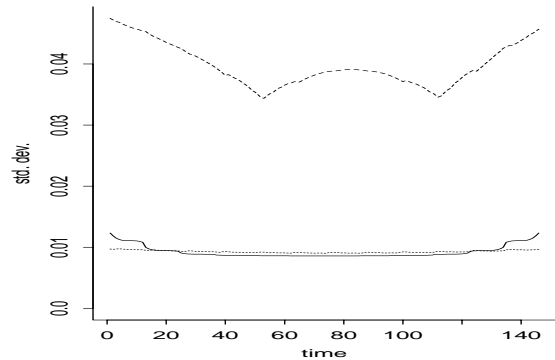
$$\begin{aligned} (1 - B)(1 - B^{12})[Y_t - x_t'\beta] \\ = (1 - \theta_1 B)(1 - \theta_{12} B^{12})b_t \end{aligned} \quad (10)$$

$$\begin{aligned} (1 - .75B)(1 - \phi^3 B^3)(1 - \Phi B^{12})e_t \\ = (1 - \eta B)c_t. \end{aligned} \quad (11)$$

where  $b_t$  and  $c_t$  are white noise, and the parameter estimates are (omitting  $\hat{\beta}$  for brevity)  $\hat{\theta}_1 = .23$ ,  $\hat{\theta}_{12} = .88$ ,  $\hat{\sigma}_b^2 = 3.97 \times 10^{-4}$ ,  $\hat{\phi}^3 = .66$ ,  $\hat{\Phi} = .71$ ,  $\hat{\eta} = -.13$ , and  $\hat{\sigma}_c^2 = 0.93 \times 10^{-4}$ . The parameter estimates for (11) were obtained by approximating directly estimated sampling error autocovariances. (The value .75 in the AR(1) operator is known from

the nature of the “composite” survey estimator used to produce the  $y_t$ .) The parameter estimates for (10) were obtained by fitting the component model for  $y_t$  by maximum likelihood using the REGCMPNT program (Bell 2004) and holding the parameters of (11) fixed. The regression variables  $x_t$  account for trading-day and length-of-month variation in the series (Bell and Hillmer 1983). Apart from the regression effects, the model for  $Y_t$  is the well-known “airline model” (Box and Jenkins 1976). The sampling error model takes into account features of the sample design and estimation (although these have changed in recent years). Apart from the high level of sampling error, the most notable features of the sampling error model are the substantial nonseasonal autocorrelation (coming from the  $(1 - .75B)(1 - \phi^3 B^3)$  factors) and the substantial seasonal autocorrelation coming from the large value of  $\hat{\Phi}$ . Another point worth noting is that the sample for the retail trade survey is independently redrawn about every five years, resulting in a break in the covariance structure of  $e_t$  when this happens. This was accounted for by the REGCMPNT program when implementing the model (11).

Figure 1 shows signal extraction standard deviations,  $[\text{Var}(\hat{\varepsilon}_t^A)]^{.5} = [\text{Var}(\hat{\varepsilon}_t^S)]^{.5}$  (dotted curve) and  $[\text{Var}(\hat{\varepsilon}_t^N)]^{.5}$  (dashed curve), obtained from a canonical decomposition (Hillmer and Tiao 1982) of the model (10) for  $Y_t$  to get the decomposition  $y_t = S_t + N_t + e_t$ . For simplicity, these results ignore parameter estimation error. We see that the values of  $[\text{Var}(\hat{\varepsilon}_t^N)]^{.5}$  are about four times as large as those of  $[\text{Var}(\hat{\varepsilon}_t^A)]^{.5}$ . This result is typical, and will be seen later in other examples, though the magnitude of the difference depends on the magnitude of the sampling error in the series (in relation to the amount of variation in the true series  $Y_t$ ). The two local minima in the graph of  $[\text{Var}(\hat{\varepsilon}_t^N)]^{.5}$  result from the covariance break in  $e_t$  due to the redrawing of the sample, and



**Figure 1.** Signal extraction standard deviations for Example 1.

are not characteristic of other examples. The solid curve is what results for  $[\text{Var}(\hat{\varepsilon}_t^A)]^{-.5} = [\text{Var}(\hat{\varepsilon}_t^N)]^{-.5}$  if we fit model (10) ignoring the sampling error (i.e., behaving as if the series had no sampling error), and then do the canonical decomposition. We see that, for this example, these results are in reasonable agreement with those of  $[\text{Var}(\hat{\varepsilon}_t^A)]^{-.5}$  from the complete model with sampling error, but they therefore greatly understate  $[\text{Var}(\hat{\varepsilon}_t^N)]^{-.5}$ .

To actually compute seasonal adjustment variances when a model-based method is used, and we make the usual assumption that we have the correct model (so our estimators are assumed optimal), we do not need to use expressions such as (7) and (9) — it is easier to do the calculations more directly. One approach is to put the model in state-space form and use the Kalman filter and an associated smoother. This is discussed for the non-stationary case (appropriate to models used in seasonal adjustment) by Kohn and Ansley (1987) and Bell and Hillmer (1991). Alternatively, we can use matrix results developed by Bell and Hillmer (1988), simplified formulas for which are given by McElroy (2005).

To compute variances for an X-11 seasonal adjustment one could use equations (7) and (9) given the X-11 filter and given models for the components. (To deal with X-11 with forecast extension one can compute the filter that results from convoluting the X-11 filter with the forecast extension process as pointed out by Pfeiffermann, Morry, and Wong (1995).) Section 4 notes, however, that the proposed approaches to X-11 variances all ignore some terms from (7) and (9).

The qualification noted above is that to compute  $\text{Var}(\hat{\varepsilon}_t^N)$  and  $\text{Var}(\hat{\varepsilon}_t^A)$ , whether from (7) and (9) or more directly, we need the time series  $\hat{\varepsilon}_t^N$  and  $\hat{\varepsilon}_t^A$  to be stationary, or at least we need that they not require differencing. This can be examined using (6) and (8). Models used in seasonal adjustment typically assume that the components  $S_t$  and  $T_t$ , and hence  $N_t$ , are nonstationary, but are made stationary (considering the monthly case for concreteness) by taking  $U(B)S_t \equiv (1 + B + \dots + B^{11})S_t$  and  $(1 - B)^d N_t$ , where typically  $d = 1$  or  $2$ . From (6) we see that for  $\hat{\varepsilon}_t^N$  to be stationary  $\omega_N(B)$  must then contain  $U(B)$  as a factor, and  $1 - \omega_N(B)$  must contain  $(1 - B)^d$ . This is equivalent to requiring  $\omega_S(B)$  to contain  $(1 - B)^d$  and  $1 - \omega_S(B)$  to contain  $U(B)$  (whether or not sampling error is present), and so the same conditions are required for  $\hat{\varepsilon}_t^A$  to be stationary. These conditions will hold in all the examples we consider here. The conditions hold generally for model-based methods assuming the model

actually used has the correct (or at least has sufficient) differencing. These conditions can also be shown to hold for X-11 symmetric filters, and for X-11 filters with full forecast extension, again assuming that the model used for forecast extension has sufficient differencing. (Note: Full forecast extension means enough forecasts and backcasts are appended to the ends of the series so that the symmetric X-11 filter can be applied to the extended series. Partial forecast extension is any forecast extension short of this.) The conditions do not always hold for X-11 asymmetric filters, or X-11 filters with partial forecast extension because these filters  $\omega_N(B)$  reproduce only constant polynomials (not linear polynomials in  $t$ ). This means that the corresponding  $1 - \omega_N(B) = \omega_S(B)$  will contain only  $1 - B$ , not  $(1 - B)^2$ , so the conditions will fail if  $d = 2$  in the model for  $N_t$ . Section 4 notes, however, that the proposed general approaches to X-11 variances all ignore the term  $\omega_S(B)T_t$  in (6) and (8).

## 2.1 Comments on Sampling Error and Seasonal Adjustment

Seasonal adjustment practice has yet to deal explicitly with the presence of sampling error components in many of the time series that are seasonally adjusted. Partly this is due to the fact that empirical methods of seasonal adjustment, such as X-11 and its successors, are not based on statistical models and were not designed to explicitly recognize sampling error in the construction of filters. (They essentially assume no sampling error.) Also, though sampling error components can be readily added to models used for seasonal adjustment (Bell and Otto 1992, Bell 2004), available software for doing model-based seasonal adjustment, such as SEATS, does not allow for this. Finally, explicit accounting for sampling error in seasonal adjustment generally requires that one have available estimates of sampling error autocovariances over time to use in building a time series model for  $e_t$ . While estimates of sampling error variances are routinely produced by statistical agencies, for many economic surveys estimates of sampling error autocovariances are unavailable, or are produced only occasionally as part of special studies. Thus, information to permit an explicit accounting for sampling error in seasonal adjustment is often lacking.

The failure to explicitly account for sampling error in seasonal adjustment practice has important implications for developing seasonal adjustment variances. Since statistical agencies generally regard as essential the provision of sampling error variances to indicate statistical uncertainty in the unadjusted

data, developing seasonal adjustment variances that ignore the contribution of sampling error seems unacceptable. Thus, while model-based seasonal adjustment methods easily provide error variances of the component estimates, the failure of current software to allow for sampling error components in models is a severe limitation as far as seasonal adjustment variances are concerned. The approaches to variances for X-11 type seasonal adjustments that are cited in the Introduction and discussed in Section 4 all recognize the importance of explicitly accounting for sampling error and do so. These methods are all limited to some extent, however, in that they ignore the contributions to seasonal adjustment error from some of the other components ( $S_t$ ,  $T_t$ , and  $I_t$ ). This is discussed further in Section 4. Also, the X-11 filters were not themselves designed to account for sampling error.

### 3. Contributions to Seasonal Adjustment Error

In this section we examine contributions to seasonal adjustment error, and hence to seasonal adjustment variances, from a few different perspectives.

#### 3.1 Contributions from the components

Equations (6) and (8) express the errors  $\hat{\varepsilon}_t^N$  and  $\hat{\varepsilon}_t^A = -\hat{\varepsilon}_t^S$  (the latter after multiplying equation (8) by  $-1$ ) as sums of contributions from the components  $S_t$ ,  $N_t$ , and  $e_t$ . The corresponding equations (7) and (9) for  $\text{Var}(\hat{\varepsilon}_t^N)$  and  $\text{Var}(\hat{\varepsilon}_t^A)$  can thus be viewed as variance decompositions. If we compute the individual terms in (7) and (9) we can thus examine the relative importance of the component contributions to the seasonal adjustment error, whether defined as  $\hat{\varepsilon}_t^N$  or  $\hat{\varepsilon}_t^A$ .

**Example 2.** Figure 2 shows such results for two series. For both series the models used for  $Y_t$  were like that of (10), though without any regression effects. Details of the models are omitted. Canonical decomposition of the model for  $Y_t$  yielded models for the seasonal and nonseasonal components,  $S_t$  and  $N_t$ . Models for the sampling errors are discussed briefly below. The graphs show variances, not standard deviations, to illustrate the additivity of the variance components. As before, parameter estimation error is ignored for simplicity.

The top two plots in Figure 2 show results for the logarithms of monthly U.S. total housing starts over about a 12-year period as estimated in the Census Bureau's Survey of Construction. As this is a

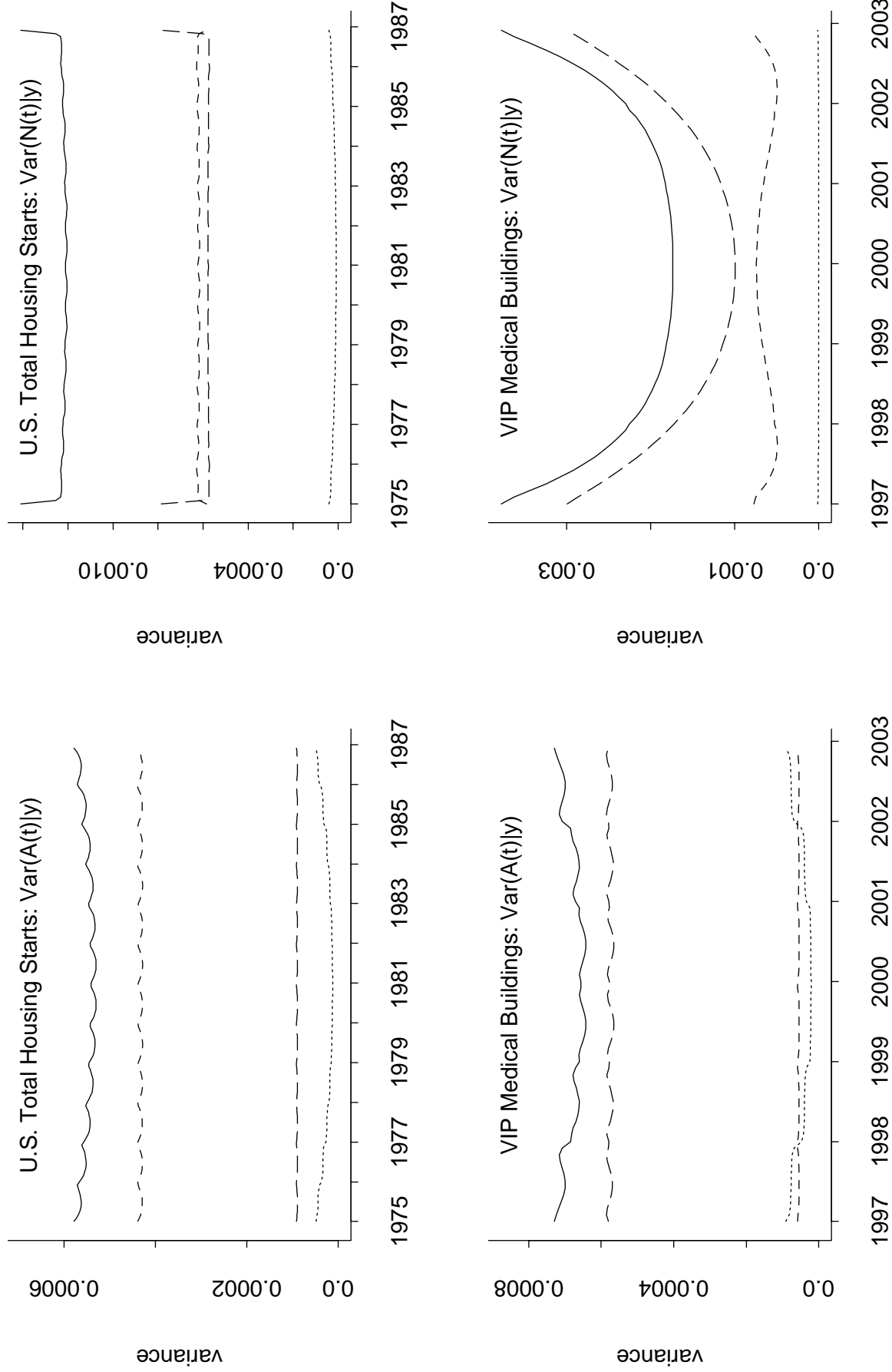
very aggregate series the level of sampling error is low, with an average sampling CV of around 2 percent. Although an MA(2) model was used for  $e_t$ , the two MA parameters were quite small, reflecting little autocorrelation in the sampling errors. The true series  $Y_t$  is not particularly stable, containing a large amount of time series variation. The top left plot in Figure 2 shows the resulting  $\text{Var}(\hat{\varepsilon}_t^A)$  (solid curve) and the contributions to this from  $S_t$  (dotted curve),  $N_t$  (dashed curve), and  $e_t$  (long dashed curve). We see that the largest contribution to  $\text{Var}(\hat{\varepsilon}_t^A)$  comes from  $N_t$ , with a relatively small contribution from the sampling error  $e_t$ , and even less from the seasonal  $S_t$ . The top right plot shows corresponding results decomposing  $\text{Var}(\hat{\varepsilon}_t^N)$ . First, comparing the vertical scales of the two graphs, we note that  $\text{Var}(\hat{\varepsilon}_t^N)$  is much larger than  $\text{Var}(\hat{\varepsilon}_t^A)$ , something noted for the example of Figure 1. Here, in a series with lower sampling error,  $\text{Var}(\hat{\varepsilon}_t^N)$  is about twice as large as  $\text{Var}(\hat{\varepsilon}_t^A)$ . Further examining the top right plot, we see the contributions to  $\text{Var}(\hat{\varepsilon}_t^N)$  from  $N_t$  and  $e_t$  are similar, though that for  $e_t$  increases near the ends of the series. Again, the contribution from  $S_t$  is quite small.

The bottom two plots in Figure 2 show corresponding results for a rather short (6 years) series of logs of estimates of value of construction put in place (VIP) for the category of medical buildings. In contrast to the U.S. total housing starts series, this series has substantial sampling error, with an average sampling CV somewhere around 9 percent. An AR(1) model was used for  $e_t$  with an AR parameter of about .8. As before, comparing the vertical scales of the two graphs shows that  $\text{Var}(\hat{\varepsilon}_t^N)$  (bottom right plot) is much larger than  $\text{Var}(\hat{\varepsilon}_t^A)$  (bottom left plot), here about 3-5 times as large. In both graphs the contribution from the sampling error is the largest, dominating the decompositions except near the middle of the series for  $\text{Var}(\hat{\varepsilon}_t^N)$ , where the contribution from  $N_t$  becomes almost as important. Again, the seasonal makes very little contribution.

We can generalize a few conclusions from these two examples. First, when sampling error is present in the observed series,  $\text{Var}(\hat{\varepsilon}_t^N)$  tends to be much larger than  $\text{Var}(\hat{\varepsilon}_t^A)$  (recall that when there is no sampling error these two are the same.) Second, even with relatively low levels of sampling error, we see  $e_t$  can make an important contribution, particularly to  $\text{Var}(\hat{\varepsilon}_t^N)$ . Moderate to high sampling error can make a dominant contribution to seasonal adjustment variances. In light of these results, the common practice of *not* taking explicit account of sampling error in seasonal adjustment

Fig. 2 Seasonal adjustment variances and component contributions for Example 2

solid =  $\text{Var}(A(t)|y)$  or  $\text{Var}(N(t)|y)$ , long dash =  $e(t)$ , dot =  $S(t)$ , dash =  $N(t)$



seems problematic for developing seasonal adjustment variances, however defined. Third, we see that the seasonal component tends to make relatively little contribution to  $\text{Var}(\hat{\varepsilon}_t^N)$  and  $\text{Var}(\hat{\varepsilon}_t^A)$ .

### 3.2 Contributions from forecast extension error

One approach to dealing with the problem of computing seasonal adjustments near the end of a time series is to extend a series  $y_t$  observed for  $t = 1, \dots, n$  with sufficient forecasts and backcasts to apply the symmetric filter that would be used if sufficient data were available. The calculations of model-based seasonal adjustment can be viewed in this way, and if the correct model is used for the forecast extension this will produce the optimal (minimum mean squared error) results (Cleveland 1972, Bell 1984a). The same approach minimizes mean squared revisions in X-11 seasonal adjustment (Pierce 1980), though since X-11 symmetric filters are finite, only a finite number of forecasts and backcasts are required. This idea was implemented in the X-11-ARIMA program (Dagum 1975), although with the forecast and backcast extension generally limited to just one year, which, for X-11 filters, is less than full forecast extension. Here we consider only full forecast and backcast extension for reasons noted in Section 2.

Let  $\hat{N}_t$ ,  $\hat{S}_t$ , and  $\hat{A}_t$  denote the “final” estimators of the respective components obtained with “sufficient data” (possibly infinite data) to apply the symmetric filters. Let  $\tilde{N}_t$ ,  $\tilde{S}_t$ , and  $\tilde{A}_t$  denote estimators obtained by applying the symmetric filters to the finite observed time series  $y_t$  extended with optimal forecasts and backcasts. The error in the estimator  $\tilde{N}_t$ ,  $\tilde{\varepsilon}_t^N = N_t - \tilde{N}_t$ , can be written

$$\begin{aligned}\tilde{\varepsilon}_t^N &= (N_t - \hat{N}_t) + (\hat{N}_t - \tilde{N}_t) \\ &= \hat{\varepsilon}_t^N + (\hat{N}_t - \tilde{N}_t).\end{aligned}\quad (12)$$

The first term in (12),  $\hat{\varepsilon}_t^N$ , is the error in the final estimator. The second term in (12), the difference between the final estimator and the estimator obtained with the forecast and backcast extended series, depends on the forecast and backcast errors, as noted by Pierce (1980). From (12), the variance of the error  $\tilde{\varepsilon}_t^N$  is

$$\begin{aligned}\text{Var}(\tilde{\varepsilon}_t^N) &= \text{Var}(\hat{\varepsilon}_t^N) + \text{Var}(\hat{N}_t - \tilde{N}_t) \\ &\quad + 2 \times \text{Cov}(\hat{\varepsilon}_t^N, \hat{N}_t - \tilde{N}_t).\end{aligned}\quad (13)$$

An analogous expression holds for  $\text{Var}(\tilde{\varepsilon}_t^S) = \text{Var}(\hat{\varepsilon}_t^A)$ .

If  $\hat{N}_t$  is the optimal final estimator of  $N_t$ , as is usually assumed under a model-based approach,

then  $\hat{\varepsilon}_t^N$  and  $\hat{N}_t - \tilde{N}_t$  are uncorrelated, and the covariance term drops out of (13). This is because the error in  $\hat{N}_t$  is uncorrelated with the observed data, and  $\hat{N}_t$  and  $\tilde{N}_t$  are both linear functions of the data. (Actually, Bell (1984a) notes that for the optimal estimator  $\hat{\varepsilon}_t^N$  is uncorrelated with the differenced data, but this is sufficient here since  $\hat{N}_t - \tilde{N}_t$  turns out to be a linear function of the differenced data.) In this case  $\text{Var}(\tilde{\varepsilon}_t^N)$  equals the variance of the error in the final estimator ( $\text{Var}(\hat{\varepsilon}_t^N)$ ) plus  $\text{Var}(\hat{N}_t - \tilde{N}_t)$ , implying that  $\text{Var}(\tilde{\varepsilon}_t^N) \geq \text{Var}(\hat{\varepsilon}_t^N)$ . Since  $\text{Var}(\hat{N}_t - \tilde{N}_t)$  generally increases as  $t$  approaches the ends of the series, so does  $\text{Var}(\tilde{\varepsilon}_t^N)$ . (Exceptions can occur for models that are not time homogenous, e.g., with a sampling error component whose variance changes over time.) The characteristic shape of signal extraction variances increasing towards the ends of the series can be seen in many examples in the literature.

If  $\hat{N}_t$  is not the optimal final estimator of  $N_t$ , then the covariance term in (13) is needed. Bell and Kramer (1999) show how to compute (13) for their approach to X-11 seasonal adjustment. They also show an example where the increase near the ends from the  $\text{Var}(\hat{N}_t - \tilde{N}_t)$  term is partially offset by the contribution of  $2 \times \text{Cov}(\hat{\varepsilon}_t^N, \hat{N}_t - \tilde{N}_t)$ , which turns out to be negative near the ends of the series (it is zero in the middle of the series). Thus, these seasonal adjustment variances increase less at the ends of the series than might be expected, a phenomenon that has been observed in other examples.

### 3.3 Contributions from error in estimating model parameters

We begin with contributions from error in estimating regression parameters assuming that all other model parameters are known. As in (10) the regression effects in the model are  $x_t'\beta$ , where  $x_t$  is known and  $\beta$  is the vector of regression parameters to be estimated. As discussed in Bell (1984b), some of the regression effects may be assigned to the seasonal component and others to the nonseasonal component. Partition  $x_t'$  as  $[x_{St}', x_{Nt}']$  and  $\beta$  as  $[\beta_S', \beta_N']'$  so that  $x_t'\beta = x_{St}'\beta_S + x_{Nt}'\beta_N$ , where  $x_{St}'\beta_S$  are the regression effects assigned to the seasonal and  $x_{Nt}'\beta_N$  are the regression effects assigned to the nonseasonal component. We assume that none of the regression effects are assigned to the sampling error component. We expand the decomposition (1) to include the regression effects by writing

$$\begin{aligned}y_t &= S_t^x + N_t^x + e_t \\ &= (x_{St}'\beta_S + S_t) + (x_{Nt}'\beta_N + N_t) + e_t,\end{aligned}\quad (14)$$

where  $S_t$  and  $N_t$  remain the seasonal and nonseasonal components without regression effects. If we

knew the regression parameters  $\beta$ , then to estimate  $N_t^x$  in (14) we could subtract  $x_t'\beta$  from  $y_t$ , apply the filter  $\omega_N(B)$  to the result, and add back the regression effects  $x_{Nt}'\beta_N$  that are part of  $N_t^x$ . In practice we do this with  $\beta$  replaced by an estimate  $\hat{\beta}$ . We denote these two estimators of  $N_t^x$  by  $\hat{N}_t^x$  and  $\bar{N}_t^x$ , respectively. They are written explicitly as

$$\hat{N}_t^x = x_{Nt}'\beta_N + \omega_N(B)[y_t - x_t'\beta], \quad (15)$$

$$\bar{N}_t^x = x_{Nt}'\hat{\beta}_N + \omega_N(B)[y_t - x_t'\hat{\beta}]. \quad (16)$$

The error in the infeasible estimator,  $\hat{N}_t^x$ , of  $N_t^x$ , can be seen to be just  $\hat{\varepsilon}_t^N$  given by (6). To assess the error  $\bar{\varepsilon}_t^N = N_t - \bar{N}_t^x$  we write

$$\bar{\varepsilon}_t^N = (N_t - \hat{N}_t^x) + (\hat{N}_t^x - \bar{N}_t^x) \quad (17)$$

$$= \hat{\varepsilon}_t^N + \{[0 \quad x_{Nt}'] - \omega_N(B)x_t'\}(\beta - \hat{\beta}). \quad (18)$$

Let  $d'_{Nt} = [0 \quad x_{Nt}'] - \omega_N(B)x_t'$ . Having computed  $d_{Nt}$ , and assuming that we also have available  $\text{Var}(\hat{\beta})$ , the variance-covariance matrix of the estimated regression parameters, then from (18) the variance of  $\bar{\varepsilon}_t^N$  is

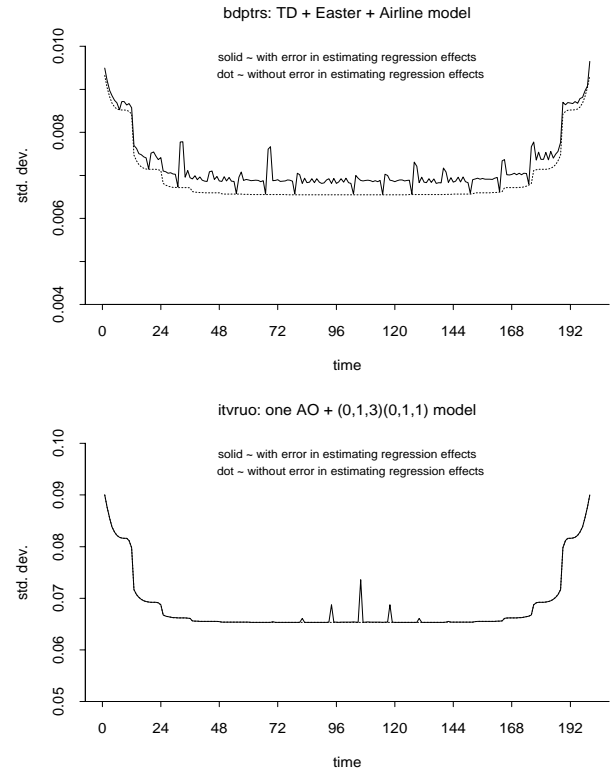
$$\begin{aligned} \text{Var}(\bar{\varepsilon}_t^N) &= \text{Var}(\hat{\varepsilon}_t^N) + d'_{Nt}\text{Var}(\hat{\beta})d_{Nt} \\ &\quad - 2 \times d'_{Nt} \times \text{Cov}(\hat{\beta}, \hat{\varepsilon}_t^N). \end{aligned} \quad (19)$$

$\text{Var}(\hat{\varepsilon}_t^N)$  can be computed as discussed previously. If  $\omega_N(B)$  is a model-based filter that is assumed optimal for estimating  $N_t$ , then, as in the previous subsection,  $\hat{\varepsilon}_t^N$  is orthogonal to the data and so  $\text{Cov}(\hat{\beta}, \hat{\varepsilon}_t^N) = 0$  assuming that, as is generally the case,  $\hat{\beta}$  is a linear function of the data. So in this case the covariance term drops out of (19). But if  $\omega_N(B)$  is not assumed to be an optimal signal extraction filter, say if it is an X-11 filter, then the covariance term is present. Bell and Kramer (1999) show how to compute something analogous to  $\text{Cov}(\hat{\beta}, \hat{\varepsilon}_t^N)$ . Although they actually replace  $\hat{\varepsilon}_t^N$  by a different “error” corresponding to a particular definition of the estimand for X-11 seasonal adjustment, the approach they use also readily applies to calculating  $\text{Cov}(\hat{\beta}, \hat{\varepsilon}_t^N)$ .

Note that to compute  $d_{Nt}$  we must filter each regression variable  $x_{it}$  in the vector  $x_t' = [x_{1t}, \dots, x_{rt}]$  by the filter  $\omega_N(B)$ . If the filtering actually involves forecast and backcast extension as discussed in the preceding section, this would be applied to the regression variables as well, using the same model that would be applied to forecast and backcast extend the series  $y_t - x_t'\beta$ . In the model-based context the calculation of  $d_{Nt}$  can be accomplished with the Kalman filter and a smoother as pointed out by Kohn and Ansley (1985). Bell and Hillmer (1988) show how a

matrix approach can also be used. These references also note how in fitting the model to the data  $\text{Var}(\hat{\beta})$  can be obtained from generalized least squares results.

**Example 3.** Figure 3 illustrates these results for model-based seasonal adjustment variances for two series. The graphs shown compare  $\text{Var}(\hat{\varepsilon}_t^N)$ , the variance when regression and other model parameters are assumed known (dotted curves), with  $\text{Var}(\bar{\varepsilon}_t^N)$  from (19) (solid curves). Since these results assume the true model is known, the covariance term drops out of (19) and  $\text{Var}(\bar{\varepsilon}_t^N) \geq \text{Var}(\hat{\varepsilon}_t^N)$ . The top graph shows results for a time series (U.S. retail sales of department stores) with negligible sampling error, and following a model as in (10) with  $x_t$  including variables as in Bell and Hillmer (1983) to account for trading-day and Easter holiday effects. The erratic fluctuations in the solid curve are due to the effects of error in estimating the trading-day parameters, though these effects essentially vanish in non-leap-year Februaries (the points at which the solid and dotted curves essentially coincide). The occasional larger variance increases (e.g., at time points 31-32 and 67-68) result from error in estimating the Easter



**Figure 3.** Seasonal adjustment standard deviations with and without accounting for error in estimating regression effects (Example 3)



holiday parameter. These larger increases occur only in March and April, with the magnitude of the increase in a given year depending on the date of Easter for that year. The bottom graph shows results for another time series (unfilled orders of U.S. radio and television manufacturers) assumed to follow a model similar to (10), also without a sampling error component, but with the nonseasonal MA(1) of the ARIMA model replaced by an MA(3), and with  $x_t$  accounting for a single additive outlier (AO) at  $t = 106$ . The solid curve shows the effects of uncertainty in the estimate of the AO effect, with the biggest variance increase at time 106, and decaying increases before and after this time point at yearly intervals. (Note: These results assume that a single AO is known to be present at time 106 and do not reflect uncertainty in the detection of outliers.)

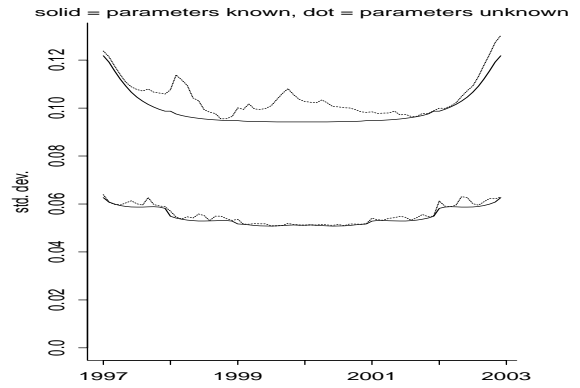
Bell and Kramer (1999) provide an example illustrating analogous results for X-11 seasonal adjustment variances.

Two approaches could be used to account for the contribution to signal extraction error from uncertainty about other model parameters. One approach would write an expression analogous to (17) but with  $\tilde{N}_t^x$  replaced by the estimator of  $N_t$  based on estimated values for all model parameters, and then use asymptotic or simulation results to approximate the variance of the term analogous to the second term in (17). See Ansley and Kohn (1986) for a general discussion. Instead, here we pursue a Bayesian approach as in Bell and Otto (1992). To summarize this approach let  $\eta$  indicate model parameters apart from the regression parameters  $\beta$ , and assume that for any given value of  $\eta$  we can produce the assumed optimal estimator,  $\tilde{N}_t^x = E(N_t^x | \eta, y)$ , where  $y = [y_1, \dots, y_n]'$  is the observed data. Assume that we can also produce the conditional variance,  $\text{Var}(N_t^x | \eta, y) = \text{Var}(\tilde{\varepsilon}_t^N | \eta)$ , from (19), omitting the covariance term since  $\tilde{N}_t^x$  is assumed optimal. Then the desired posterior variance of  $N_t^x$  can be obtained from

$$\begin{aligned} \text{Var}(N_t^x | y) &= E_{\eta|y} \{ \text{Var}(N_t^x | \eta, y) \} \\ &\quad + \text{Var}_{\eta|y} \{ E(N_t^x | \eta, y) \}, \end{aligned} \quad (20)$$

where  $E_{\eta|y}$  and  $\text{Var}_{\eta|y}$  denote the expectation and variance taken with respect to the marginal posterior distribution of the parameters,  $p(\eta|y)$ . Given simulations  $\eta_i$  from  $p(\eta|y)$  or an approximation thereto, for  $i = 1, \dots, M$  for some large  $M$ , we can calculate  $\text{Var}(N_t^x | \eta_i, y)$  and  $E(N_t^x | \eta_i, y)$  for each  $\eta_i$ , and compute the sample mean and variance over the simulations of these results as an approximation to (20).

**Example 4.** Figure 4 provides an example illustrating the effects of uncertainty about other model parameters. This example uses a model developed for a time series of value of construction put in place for other education buildings. In this example the “other model parameters” are the MA and innovation variance parameters of the airline model as in (10), along with the innovation variance of the sampling error model. There are no regression effects in the model. The sampling error model used was AR(1), but the AR(1) parameter was estimated fairly precisely and so was held fixed. We see that the effect of uncertainty about the “other model parameters,” as reflected by the difference between the dashed and solid curves, is erratic over time, and sometimes of consequence. We also notice that  $\text{Var}(N_t|y) = \text{Var}(\tilde{\varepsilon}_t^N)$  is again much larger (3 to 4 times larger) than  $\text{Var}(A_t|y) = \text{Var}(\tilde{\varepsilon}_t^A)$ .



**Figure 4.** Seasonal adjustment standard deviations for Example 4: upper curves show  $\text{StdDev}[N_t|y]$ , lower curves show  $\text{StdDev}[A_t|y]$ .

#### 4. Variances for X-11 Seasonal Adjustments

We now compare various approaches that have been proposed for producing variances of X-11 seasonal adjustments. We focus only on additive decompositions (typically applied to logged data) so the X-11 linear filters are used. The approaches are compared by examining what contributions to seasonal adjustment error are being recognized. Since the X-11 filters do not explicitly recognize sampling error, we have  $\omega_S(B) + \omega_N(B) = 1$ , resulting in the following simplifications to formulas (6) and (8), with  $N_t$  replaced by  $T_t + I_t$ , and with (8) multiplied by  $-1$  to express the error in  $\hat{A}_t = y_t - \hat{S}_t$ :

$$\begin{aligned} \hat{\varepsilon}_t^N &= -\omega_N(B)S_t + \omega_S(B)T_t \\ &\quad + \omega_S(B)I_t - \omega_N(B)e_t \end{aligned} \quad (21)$$

$$\begin{aligned}\hat{\varepsilon}_t^A &= -\omega_N(B)S_t + \omega_S(B)T_t \\ &\quad + \omega_S(B)I_t + \omega_S(B)e_t.\end{aligned}\quad (22)$$

Since the X-11 filters do not recognize sampling error, it is not immediately obvious whether X-11 seasonal adjustment should be regarded as estimating  $N_t$  or  $A_t = y_t - S_t$ , so it is not obvious whether the error in X-11 seasonal adjustment should be regarded as  $\hat{\varepsilon}_t^N$  given by (21) or  $\hat{\varepsilon}_t^A$  given by (22). We thus examine the proposed approaches in reference to both (21) and (22).

As noted in the Introduction, approaches to variances for X-11 seasonal adjustment have been proposed by Wolter and Monsour (1981), hereafter WM, Pfeffermann (1994), hereafter DP, and Bell and Kramer (1999), hereafter BK. WM and DP each proposed two approaches, which we denote as WM-1 and WM-2, and as DP-1 and DP-2, respectively. Table 1 summarizes these approaches in regard to the seasonal adjustment “error” whose variance is computed, and the extent to which forecast extension error is accounted for. The sign of the error term is essentially arbitrary, so for WM and DP the sign is changed to keep the error expressions given here consistent with each other. The second column of Table 1 shows that all the approaches are better thought of as approximating  $\text{Var}(\hat{\varepsilon}_t^N)$  than  $\text{Var}(\hat{\varepsilon}_t^A)$ , since they all take the sampling error contribution as  $-\omega_N(B)e_t$ , in agreement with (21). All the approaches ignore the contributions to error of  $-\omega_N(B)S_t$  and  $\omega_S(B)T_t$ , and WM-1 and BK also ignore the contribution of  $\omega_S(B)I_t$ . WM arrived at this approximation from a design-based perspective that recognized only sampling error. BK arrived at this error by defining the target of the seasonal adjustment as what one would get by applying  $\omega_N(B)$  (actually the final, symmetric version of  $\omega_N(B)$ ) to  $Y_t$ , the “true series” without sampling error.

**Table 1. Approaches to variances for X-11 seasonal adjustments**

Approach	seas. adj. error	accounting for forecast ext.
WM-1	$-\omega_N(B)e_t$	partial
WM-2	$\omega_N(B)I_t - \omega_N(B)e_t$	partial
DP-1	$\omega_S(B)I_t - \omega_N(B)e_t$	partial
DP-2	$\omega_N(B)I_t - \omega_N(B)e_t$	partial
BK	$-\omega_N(B)e_t$	full

Note: WM-1 and WM-2 are the two approaches of Wolter and Monsour (1981), DP-1 and DP-2 are the two approaches of Pfeffermann (1994), and BK is the approach of Bell and Kramer (1999).

WM-2 and DP-2 allow contributions from the irregular of  $\omega_N(B)I_t$ . This conforms with neither (21) nor (22), and for that reason we regard these approaches as essentially incorrect. WM arrived at this by considering the variance of  $\omega_N(B)[y_t - \hat{S}_t - \hat{T}_t]$ . Unfortunately, this approximates the variance of something like the seasonally adjusted estimator, not the variance of the error in the estimator. Hence, the wrong filter,  $\omega_N(B)$ , gets applied to  $I_t$ . DP arrived at DP-2 by considering the error in taking  $\omega_N(B)y_t$  as an estimator of  $T_t$ . The rationale for taking the error in an estimator of  $T_t$  as the “seasonal adjustment error” is unclear. We shall not consider WM-2 and DP-2 further.

The entries in the third column of Table 1, that labelled, “accounting for forecast ext.,” require more explanation. The entry “full” for BK means that the approach fully accounts for the contribution of forecast extension error using the approach discussed in Section 3.2 (although, as noted above, some component contributions to  $\text{Var}(\hat{\varepsilon}_t^N)$  in (13) are ignored.) The term “partial” for the other approaches means that some contribution of the forecast and backcast extension errors is ignored. WM and DP do their calculations with  $\omega_N(B)$  being the X-11 asymmetric filter actually applied at a given time point near the end of the time series. Pfeffermann, Morry, and Wong (1995) extended DP’s approach so  $\omega_N(B)$  could be an X-11 filter convoluted with forecast and backcast extension. (Actually, only one year of forecast and backcast extension was used, so this convolution was often with asymmetric X-11 filters, not the final symmetric filter.) When  $\omega_N(B)$  is a finite filter, symmetric or asymmetric, computing the variance of (21) and (22) will fully account for forecast extension error *only if no terms in (21) and (22) are omitted*. However, WM and DP do omit the terms  $-\omega_N(B)S_t$  and  $\omega_S(B)T_t$  from (21) and (22) so they are not fully accounting for the contribution of forecast extension error. WM-1 also omit  $\omega_S(B)I_t$ . These approaches assume that these omitted terms can be ignored not just when  $\omega_N(B)$  is a symmetric filter, but also when  $\omega_N(B)$  is an asymmetric filter (either an original asymmetric X-11 filter or the asymmetric filter resulting from forecast extension and application of X-11 symmetric filters). This assumption seems less tenable when  $\omega_N(B)$  is an asymmetric filter than when it is a symmetric filter. Note that, as discussed in Section 2, when  $\omega_N(B)$  is an X-11 asymmetric filter,  $\omega_S(B)T_t = [1 - \omega_N(B)]T_t$  won’t even be stationary if  $T_t$  requires two differences, as is common in some time series models. In such cases WM and DP’s approaches ignore a non-stationary contribution to seasonal adjustment er-

ror.

Two papers have calculated  $\text{Var}(\hat{\varepsilon}_t^N)$  accounting for all the terms in (21). Hausman and Watson (1985) did so for two unemployment series, using models that accounted for the sampling error in the observed series of survey estimates. Chu, Tiao, and Bell (2003), in unpublished work, compared model-based and X-11 filters (the latter with full forecast extension) via a comparison criteria that involved the mean squared error of both the model-based and X-11 estimators (for canonical decomposition of the airline model without sampling error.)

## 5. Conclusions

The following conclusions can be drawn from the analyses and results presented:

1. When sampling error is not present in the observed series then the error in the estimate of the nonseasonal component is the same as that of the seasonally adjusted series ( $\hat{\varepsilon}_t^N = \hat{\varepsilon}_t^A$ ), hence, the variances of the two are the same. When sampling error is present these two errors are not the same ( $\hat{\varepsilon}_t^N \neq \hat{\varepsilon}_t^A$ ), and their two variances can be quite different, with  $\text{Var}(\hat{\varepsilon}_t^N)$  typically being larger than  $\text{Var}(\hat{\varepsilon}_t^A)$ .
2. Contributions from the seasonal, nonseasonal, and sampling error components to seasonal adjustment variances can be calculated and compared. Often when sampling error is present, its contribution is quite important.
3. Uncertainty about regression parameters and other model parameters has erratic effects on seasonal adjustment variances.
4. Forecast extension error and parameter estimation error are uncorrelated with optimal symmetric seasonal adjustment error (i.e., from a known true model). This is not the case when the seasonal adjustment filter used is not optimal, that is, forecast extension error and parameter estimation error will be correlated with error in seasonal adjustments from suboptimal symmetric filters.
5. Proposed approaches to variances for X-11 seasonal adjustment are more consistent with defining  $\hat{\varepsilon}_t^N$  than  $\hat{\varepsilon}_t^A$  to be the seasonal adjustment error.

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