SEASONAL ADJUSTMENT WHEN BOTH DETERMINISTIC AND STOCHASTIC SEASONALITY ARE PRESENT

David A. Pierce
Federal Reserve System

INTRODUCTION

The adjustment of economic and social time series for seasonal variation has been and continues to be a subject of much attention. Numerous strategies and procedures have been developed, their underlying assumptions and properties scrutinized, and their effects on other aspects of the series so adjusted both theoretically and empirically explored. Debate continues on how best to model seasonality, what constraints (information) should be imposed (utilized) in seasonally adjusting series, and whether one should attempt seasonal adjustment at all. Against this background, most procedures currently in use take relatively little account of such issues as the origin of seasonality, the series' relationship to other seasonal series, or (at least explicitly) the purpose of the seasonal adjustment. As such, these procedures are descriptive or empirical procedures, univariate in nature and oriented toward the statistical characteristics either of the particular series or of a group of series that experience has found to possess similar seasonal characteristics.

The development of procedures more explicitly oriented toward the causes of seasonality and the purposes of seasonal adjustment should be the ultimate aim of seasonal adjustment research. However, it is felt that, for many series, there are likely to be sufficient difficulties with structural approaches that empirical or unstructured procedures will doubtless remain important; thus, this paper is concerned primarily with methodology for seasonally adjusting a series, based on little additional information beyond that contained within the series, although the framework can often be extended in a straightforward manner.

Current descriptive seasonal adjustment procedures tend to fall into one or the other of two categories, the (ratio to) moving average methods and the regression methods. It will be seen in the second section that the latter are optimal for series where seasonality is deterministic, i.e., capable of prediction without error from previous months' and years' seasonal, and the former are appropriate for series where seasonality is *stochastic*, i.e., representable as a stationary or nonstationary stochastic process. Thus, virtually all of the procedures in current use explicitly or implicitly assume that a series' seasonality is either deterministic or stochastic, but seldom both. In addition, the specific filters in the moving average procedures are generally chosen with, at best, a limited examination of the stochastic properties of the series; thus, they are frequently suboptimal, even for stochastic seasonality.

This paper represents an attempt to synthesize these two classes of approaches. The next section develops the general model combining those underlying the regression and the moving average seasonal adjustment procedures. Frequently, such a model is unidentified (several models in this class are equally compatible with the data); the third section sets forth some properties for seasonal adjustment procedures, which are felt to be desirable ones and which, in any event, remove this nonunique problem.

One of the problems in assessing a series' seasonality is how to handle other systematic effects, i.e., trend, and the fourth section is devoted to this question. The fourth, fifth, and sixth sections set forth the seasonal adjustment procedure, including the identification of the deterministic/stochastic model appropriate for the given series and the estimation of its parameters. Tests of seasonality are also presented, since the adjustment procedure is obviously influenced by whether the empirical evidence is most compatible with the hypothesis of no seasonality, deterministic of stochastic seasonality only, or both.

The seventh, eighth, and ninth sections illustrate the procedure, and the tenth section presents some further discussion and conclusions. The development refers primarily to monthly data, but extensions to seasonality of a period other than 12 are straightforward.

SEASONAL ADJUSTMENT MODELS

Underlying any seasonal adjustment procedure is, at least implicitly, a set of assertions regarding the generation of the seasonal and nonseasonal parts of the series to be adjusted, i.e., a model for the series. What is represented in such a model is a decomposition of the series into its seasonal and nonseasonal constituents. Additionally, the nonseasonal part is generally separated into trend (trend cycle) and irregular. At the risk of oversimplification [2; 7;

Views expressed are not necessarily related to those of the Federal Reserve System. Helpful comments were provided by Arnold Zellner, Agustin Maravall, Robert Engle, and both discussants (in addition to their prepared texts).

10; 18], the schemes for representing this decomposition have been of two basic types. The multiplicative seasonal model for a time series $\{Y_t\}$ is

$$Y_t=P_tS_tE_t \qquad (1)$$

where P_t, S_t, and E_t are, respectively, the trend cycle, seasonal, and irregular factors of Y_t, all at time t. Many economic series exhibit exponential growth, and, for these, the multiplicative model is most appropriate. For other series, however, an additive model may be more suitable. In fact, the additive model may be derived from the multiplicative model by taking logarithms. If $y_t=\log Y_t$, $p_t=\log P_t$, etc., then (1) becomes

$$y_t=p_t+s_t+e_t \qquad (2)$$

which is the additive seasonal model. The term s_t is the seasonal component of y_t. Of course, in many cases, $\{y_t\}$ will be the actual series, rather than the logarithm of a multiplicatively generated series.

The seasonal adjustment process consists of obtaining estimates $\hat{S}_t$ or $\hat{s}_t$ of the seasonal factors or components and computing

$$Y_t^{(SA)}=Y_t/\hat{S}_t \qquad (3)$$

or

$$y_t^{(SA)}=y_t-\hat{s}_t \qquad (4)$$

which are the seasonally adjusted series. Of course, the crux of the matter is the first of these steps, after which the second is trivial. This section first describes two general classes of procedures currently in use for estimating the seasonal component s_t (symmetric filtering and regression) and the models (stochastic and deterministic) for which each procedure is appropriate. Then, the two are combined to form the seasonal adjustment model for this paper.

Moving Average Procedures and Stochastic Seasonality

Perhaps the most common seasonal adjustment procedures currently in use are those in which the seasonal component (we adopt the additive framework with the proviso that it applies to the logarithms of a multiplicatively generated series) is estimated via a symmetric moving average or filter

$$\hat{s}_t = \sum_{j=-n}^{n} \nu_j y_{t-j} \qquad (\nu_{-i}=\nu_i)$$
$$= \nu(B)y_t \qquad (5)$$

where

$$\nu(B) = \sum_{j=-n}^{n} \nu_j B^j$$

is a polynomial in positive and negative powers of the lag operator B (defined by $B^j y_t=y_{t-j}$). A simple example of this would be

$$\hat{s}_t=0.1y_{t-12}+0.2y_t+0.1y_{t+12}=0.1(1+B^{12})(1+B^{-12})y_t$$

The seasonally adjusted series

$$y_t^{(SA)}=y_t-\hat{s}_t=[1-\nu(B)]y_t \qquad (6)$$

is also the result of a symmetric filter applied to the series.

The Census Bureau X–11 seasonal adjustment procedure [36] is one of the most widely used procedures of this type, although it possesses other features as well (e.g., for treating outliers).

The moving averages or filters in procedures, such as X–11, have generally been designed to treat patterns of seasonality and trend commonly observed, particularly to handle changing seasonal patterns but apparently without conscious reference to a model or class of models assumed to represent the series. Nevertheless, it is probably more than coincidence that seasonal adjustment procedures of the form (6) are optimal ones (in the sense of minimizing $E(\hat{s}_t-s_t)^2$) for one of the most successfully employed classes of models for analyzing and forecasting economic time series, the autoregressive-integrated moving average (ARIMA) time series models ([4]; see also [34; 43; 44; 45].)

The general ARIMA model is of the form

$$\phi(B)\Delta(B)y_t=\theta(B)a_t$$

or

$$\Delta(B)y_t=\psi(B)a_t \qquad (7)$$

where $\phi(B)$ and $\theta(B)$ are polynomials of degree p and q in nonnegative powers of B with zeros outside the unit circle, $\psi(B)=\phi^{-1}(B)\theta(B)$, and a_t is white noise. If $\Delta(B)=1$, equation (7) represents the stationary autoregressive moving average (ARMA) model of order p, q—

$$y_t=\sum_{j=1}^{p} \phi_j y_{t-j}-\sum_{j=1}^{q} \theta_j a_{t-j}+a_t$$

The polynomial $\Delta(B)$ is a difference operator, given most generally by

$$\Delta(B)=\prod_{i=1}^{m} \Delta_{k_i}^{d_i} \qquad (8)$$

where $\Delta_h=1-B^h$. For example, a common seasonal model [4] is obtained by setting $\phi(B)=1$, $\theta(B)=(1-\theta B)(1-\Theta B^{12})$, $m=2$, $d_1=d_2=1$, $k_1=1$, and $k_2=12$, obtaining

$$\Delta\Delta_{12}y_t=(1-B)(1-B^{12})y_t=(1-\theta B)(1-\Theta B^{12})a_t$$

Assume that in (2) the trend and seasonal components are each generated by models of this form

$$\Delta_p(B)p_t=\psi_p(B)\xi_t,\ \mathrm{Var}(\xi_t)=\sigma_\xi^2 \qquad (9)$$

$$\Delta_s(B)s_t=\psi_s(B)\epsilon_t,\ \mathrm{Var}(\epsilon_t)=\sigma^2_\epsilon \quad (10)$$

with ξ_t, ϵ_t, and e_t white noise. Then, it is known [17] that y_t is of form (7). Suppose temporarily that y_t is stationary, i.e., all difference operators are unity; thus, y_t has the representation [17]

$$y_t\equiv\psi_p(B)\xi_t+\psi_s(B)\epsilon_t+e_t=\ \psi(B)a_t \quad (11)$$

For this additive ARMA model, it is shown in [43] that the estimate $\hat{s}_t$ that minimizes $E(\hat{s}_t-s_t)^2$ is

$$\hat{s}_t=\frac{\sigma^2_\epsilon\psi_s(B)\psi_s(F)}{\sigma^2_a\psi(B)\psi(F)}\,y_t \quad (12)$$

where $F=B^{-1}$ is the forward-shift operator ($F^jy_t=y_{t+j}$). The numerator and denominator of (12) are the covariance generating functions [4, p. 49] of the unobserved seasonal component s_t and the observable series y_t. Result (12), in fact, holds for any absolutely convergent $\psi_p(B)$ and $\psi_s(B)$, i.e., for s_t and p_t stationary linear processes (not necessarily ARMA).

This result has been generalized in [6] to nonstationary ARIMA series, where $\Delta(B)\neq 1$ in (7), i.e., series that are stationary only after suitable differencing. If $\Delta_p(B)$ and $\Delta_s(B)$ are the minimal difference operators such that $\Delta_p(B)p_t$ and $\Delta_s(B)s_t$ are stationary, then the minimal difference operator $\Delta(B)$ that renders y_t stationary is the least common multiple of $\Delta_p(B)$ and $\Delta_s(B)$. The appropriate version of the symmetric filter (12) may be viewed in either of two ways. First, differencing y_t gives, in place of (11),[1]

$$\Delta(B)y_t=\Delta_s(B)\psi_p(B)\xi_t+\Delta_p(B)\psi_s(B)\epsilon_t+\Delta_p(B)\Delta_s(B)e_t$$
$$=\psi(B)a_t \quad (13)$$

The required adjustment to (12) is, thus, to replace $\psi_s(z)$ by $\Delta_p(z)\psi_s(z)$, where z represents B or F. This filter, applied to $\Delta(B)y_t$, gives a minimum MSE estimate of $\Delta(B)s_t$; applied to y_t, it produces a corresponding estimate of s_t. Alternatively, this result can be thought of as arising through a limiting process: If the differencing operators are approximated by autoregressive operators with roots close to 1, the autoregressive approximation to Δ_s will cancel in the covariance generating functions of s_t and y_t, resulting in (12) (again with $\Delta_p(z)\psi_s(z)$ replacing $\psi_s(z)$).

As an illustration, assume that

$$\Delta_s(B)=\sum_{j=o}^{11}B^j\ \psi_s(B)=1$$

$$\Delta_p(B)=(1-B)^2\ \psi_p(B)=1$$

so that, noting $(1-B)\Delta_s(B)=\Delta_{12}$,

$$\Delta(B)=\Delta\Delta_{12},\ \Delta_p(B)p_t=\xi_t,\ \Delta_s(B)s_t=\epsilon_t$$

whence

$$\Delta\Delta_{12}y_t=\theta(B)a_t \quad (14)$$

where $\theta(B)$ is of order 13. Then (12) is

$$\hat{s}_t=\frac{(1-B)^2(1-F)^2}{\theta(B)\theta(F)}y_t$$

It is important to note that equation (12) is a symmetric linear filter, applied to the series y_t; i.e., it is a special case of equation (5) that, as noted, is the defining characteristic of moving average seasonal adjustment procedures, such as X–11. It is, in this sense, that these procedures are optimal adjustment procedures for stochastic seasonality, i.e., for extracting seasonal components generated by stochastic (ARIMA) models.

For such filtering procedures to be optimal, however, their form (equation (5)) must correspond to the one (equation (12)) implied by the model generating the series. For example, in [5], it is found that the X–11 procedure is consistent with a model for y_t of form (14), with $\theta(B)$ a moving average operator of degree 24 in B (though closely approximated by an operator of degree 13). That economic time series are often well represented by ARIMA models of approximately this form is probably one reason that X–11 has done quite well for numerous series. However, series with a much different ARIMA representation have been found to be poorly adjusted by X–11 [6; 29]. In particular, series for which seasonal differencing is inappropriate (series where nonstationarity is deterministic and not stochastic, so that the differenced series is not invertible) can be overadjusted by X–11.

Regression Procedures and Deterministic Seasonality

The second major class of seasonal adjustment procedures is the regression procedures in which the seasonal and nonseasonal components are assumed explanable through a linear regression model. In (2), if

$$p_t=\sum_{i=1}^{I}\alpha_ic_{it} \quad (15)$$

$$s_t=\sum_{j=1}^{J}\beta_jd_{jt} \quad (16)$$

and e_t is white noise, then s_t and p_t are estimated for a sample series $y=(y_1,\ \ldots,\ y_n)'$ through a model of the form

$$y=C\alpha+D\beta+e$$

obtained by substituting (15) and (16) into (2). The elements $\{d_{jt}\}$ of D are most often periodic variables (sines and cosines or seasonal dummy variables) and

[1] Equation (13) assumes that the least common multiple of $\Delta_s(B)$ and $\Delta_p(B)$ is their product, i.e., they are relatively prime. When this is not the case, let $\Delta_c(B)$ denote the common factors, let $\Delta^*_s(B)=\Delta_s(B)/\Delta_c(B)$, and $\Delta^*_p(B)=\Delta_p(B)/\Delta_c(B)$. Then, in (13), $\Delta_s(B)$ and $\Delta_p(B)$ contain asterisks, and their product is replaced by $\Delta^*_p(B)\Delta^*_s(B)\Delta_c(B)$. In the ensuing text, it is $\Delta^*_p(z)\psi_s(z)$ that replaces $\psi_s(z)$.

interactions of those with powers of a time variable to capture a changing seasonal pattern. The trend variables $\{c_{jt}\}$ are generally powers of the time variable, though any nonseasonal influence can be included in this way.

Equation (16) defines deterministic seasonal (and trend) components. The essential feature of such as component is that, given knowledge of the model, it is predictable without error; by contrast, the innovation ϵ_t in the stochastic seasonal (10) is not consistently estimable, even with perfect knowledge of the model.

Ideally, C should be orthogonal to D; otherwise, an ambiguity occurs concerning the definition of the seasonal. (There are analogous problems with stochastic seasonal models, and both sets of issues are taken up in the third section.) The simplest example of a deterministic seasonal, and one we shall find can go a long way toward seasonally adjusting a series, is the fixed periodic function

$$s_t=\sum_{j=1}^{12} \beta_j d_{jt}=\beta_t \quad (\beta_t=\beta_{t\pm 12k},\ k=1, 2, \ldots) \tag{17}$$

where $d_{1t}, \ldots, d_{12t}$ are seasonal dummy variables and $\sum_{j=1}^{12} \beta_j=0$. For any given year, the seasonal component for January is β_1, for February β_2, etc. A flexible regression method, which allows for changing deterministic seasonality, is that of Stephenson and Farr [35].

Regression methods of seasonal adjustment received an impetus from the work of Lovell [22], who noted that they possess a number of properties not enjoyed by the symmetric filtering procedures currently in use. However, some of this adverse comparison is due to the use of the same symmetric filters for series with differing stochastic properties; for example, the procedure of choosing an optimal filter (in the sense discussed in the first subsection) for the given series would dictate using the identity filter for an already adjusted series; thus, such a procedure would be idempotent. Indeed the only criterion for deciding whether to use a regression or a moving average procedure should be whether the seasonal component is generated by (10) or by (16), and the raison d'être of this paper is that both may very well be involved.

Combined Procedures

If deterministic and stochastic trend and seasonality are both present, the trend and seasonal components of the observable series y_t can be written

$$p_t=\sum \alpha_i c_{it}+\psi_p(B)\xi_t=p_{1t}+p_{2t} \tag{18}$$

$$s_t=\sum \beta_j d_{jt}+\psi_s(B)\epsilon_t=s_{1t}+s_{2t} \tag{19}$$

The operators $\psi_p(B)$ and $\psi_s(B)$, which satisfy

$$\psi_p(B)\Delta_p(B)=\psi_p(B),\ \psi_s(B)\Delta_s(B)=\psi_s(B)$$

are nonconvergent unless $\Delta_p(B)$ or $\Delta_s(B)$ are unity; however, we may take them to be finite with suitable initial conditions on the p_2 and s_2 series. (See [4] for a further discussion of this point.) The combined model is then

$$\begin{aligned} y_t&=\sum \alpha_i c_{it}+\sum \beta_j d_{jt}+\psi_p(B)\xi_t+\psi_s(B)\epsilon_t+e_t \\ &=p_{1t}+s_{1t}+p_{2t}+s_{2t}+e_t \end{aligned} \tag{20}$$

This equation is the general model of this paper. The first and second terms on the right-hand side of (20) are the deterministic component (trend plus seasonal) of y_t; the remaining terms

$$u_t=p_{2t}+s_{2t}+e_t \tag{21}$$

are the stochastic component. A series displays deterministic seasonality if $s_{1t}=\sum \beta_j d_{jt}$ is nonzero; it possesses stochastic seasonality if $s_{2t}=\psi_s(B)\epsilon_t$ is nonzero.

It is often convenient to work with the differenced series $\Delta(B)y_t$, which has a stochastic component that is stationary. We write this as

$$\begin{aligned} \Delta(B)y_t=&\sum \alpha_i c_{it}^*+\sum \beta_j d_{jt}^*+\psi_p^*(B)\xi_t \\ &+\psi_s^*(B)\epsilon_t+\Delta(B)e_t \\ &=p_t^*+s_t^*+e_t^*=p_{1t}^*+s_{1t}^*+u_t^* \end{aligned} \tag{22}$$

where $c_{it}^*=\Delta(B)c_{it}$, $s_t^*=\Delta(B)s_t=s_{1t}^*+s_{2t}^*$, etc. Of course, differencing often changes the nature of the deterministic trend and seasonal variables; in particular, seasonal differencing eliminates a fixed periodic mean.

Since it is important to estimate the deterministic trend and seasonal components with a stationary residual, we will frequently be working with the differenced form (22). The recoverability of (20) from (22), particularly of s_t from s_t^*, is investigated in the fourth section, but it first must be ensured that a model of this form is identified.

SEASONAL MODEL IDENTIFICATION

The development leading up to the model in the third subsection for determininistic and stochastic seasonality assumed that the seasonal and trend component models were uniquely determined; this is seldom possible to do, given no information other than the time series $\{y_t, -\infty<t<\infty\}$. Secondly, given only a finite segment $y_1, \ldots, y_n$ of this series, the model and its components (if determined) are only estimable with a degree of error; thus, it is impossible to discriminate between theoretically incompatible models for the series. Both of these sources of ambiguity reflect a failure of seasonal model identification, identification in the economic sense [12] for the former and in the sense of Box and Jenkins [4] for the latter. Indeed, the difference between these two usages of the term "identification" is, in practice, less clear-cut than, conceptually, one emphasizing the class of models themselves and the other, their relation to the data for which they are postulated.

It is necessary, in these situations, to restrict the class

(20) of models so that the seasonal component of a series can be determined, theoretically and empirically. Often, restrictions are provided by the nature of the problem or by specific information; special events or nonrecurring changes can often be accounted for in this manner. Additionally, several authors [2; 11; 21; 31; 39; 41] have examined multivariate approaches in which seasonality, in one series, is identified through interrelationships. Certainly, available information should be used, and it can often be incorporated into the present procedures in a straightforward manner. The problem here, as elsewhere, is that a consensus on this theory is lacking. One person prefers to define trend or cyclic effects in one way, another differently. In multivariate approaches, there are probably as many varieties of variables y, z, . . . relevant to seasonally adjusting a variable x, and as many varieties of plausible specifications of relationships among and between their components, all essentially compatible with the data, since there are social scientists (economists, statisticians, etc.) to specify these variables and relationships. This situation is evidently a general one in econometric modelling, where a variety of specifications, including a purely autoregressive equation, are all compatible with the data and all have comparable predictive power. (See [28].)

What seems needed, therefore, is something of a fallback position, a principle of insufficient reason; thus, in the absence of (1) a clear, unequivocal consensus concerning prior knowledge of the series—its decomposition, its relation to causal economic or physical phenomena—and (2) of a clear indication in the data, we will take the most direct and simple route possible, given only the information on the series itself over a finite time span.[2] This is, in fact, done separately by the regression and symmetric filtering procedures discussed in the second section. By adopting a few principles that, we will argue, are reasonable ones for any seasonal adjustment procedure to possess, the identifiability problems are resolved in a unique manner. These assumptions or properties of the procedures are stated and briefly discussed in this section and, in some cases, returned to later in this paper.

Assumption 1

Assumption 1 is that a zero-mean, fixed, periodic function where the period of 1 year is part of the seasonal component. The essence of seasonality would seem to include, at least, those phenomena that recur regularly year after year. This is a deterministic seasonal effect, one captured by 12 monthly dummy variables as in equation (17) or, equivalently, a linear combination of sines and cosines at the seasonal frequencies. It is not intended to preclude the awareness of other influences in special situations; e.g., if one knows that, in five successive Januarys, the money supply would have been $500 million higher, except for a policy of restraint, then, in effect, one can take this into account. Otherwise, however, the presumption is that if a phenomenon is periodic (annual, including annual harmonics), then it is seasonal. One consequence of this is that a deterministic trend is orthogonal to this deterministic seasonal component; this is in line with [22] but not [20] or [35].

Assumption 2

Assumption 2 is that a changing seasonal pattern is stochastic. This is the most arbitrary of our assumptions, since a sufficiently flexible regression method, such as [38], can probably cope with many changing seasonal patterns about as well as symmetric filtering procedures. It is in the area of moving seasonality that the identification problem is at its worst, and our choice is based on the proven effectiveness of moving average procedures such as X–11 and stochastic (ARIMA) models for representing economic series in many applications. This restriction could probably be relaxed somewhat with a sufficiently long-time series available.

Assumption 3

Assumption 3 is that the seasonal component is preserved under a nonseasonal linear filter. For example, if s_t is the seasonal component of y_t, then $(1-\nu B)s_t$ is the seasonal component of $(1-\nu B)y_t$. This property, which enables us to handle adequately the stochastic trend of most series (see the following section), seems basic to moving average seasonal adjustment procedures, although it evidently has not received much attention. We would want, for example, the change in a seasonally adjusted series to be the seasonally adjusted change in the series. This is actually an extension of the sum-preserving property [22] to certain linear combinations of lagged values of the series. Specifically excluded are seasonal filters, such as $(1-\nu B^{12})$, that intrinsically change the nature of the series seasonality.

Assumption 4

Assumption 4 is that, among all decompositions into stochastic seasonal and nonseasonal components satisfy the previous assumptions, the decomposition chosen is the one that minimizes the variance of the seasonal component. The goal is to extract no more than necessary from the series in order to remove its seasonality.

The adjustment procedure will be examined in the next three sections. By way of preview, the first two assumptions allow us to regress a suitable transformation of the series on seasonal dummies plus deterministic trend variables, thus, estimating the deterministic (fixed) seasonal. The residuals from this regression are consistent estimates

[2] This rationale, vis-à-vis structural seasonal models, is, in several respects, analogous to the rationale for final equations vis-à-vis structural econometric models. (For example, see [28; 31; 32; 41; 46].) The correspondence is discussed further in the tenth section.

of the (possibly differenced) stochastic component of the series. The third assumption enables us to filter this component into a form that is seasonal plus white noise, the filter (of low order) eliminating the stochastic trend. Thus, we have a two-, rather than a three-, component additive ARMA model, which can be decomposed by (12), having specified the seasonal component model using the fourth assumption.

DETRENDING

Frequently, whenever seasonality has been mentioned, trend has also; e.g., the model for the series to be adjusted contains trend as well as seasonal terms in both its deterministic and its stochastic components. The reason is that the influence of trend on seasonality can be very strong. This influence is twofold: Methods of estimation of a given seasonal component can be strongly affected by unaccounted-for trend, and, often, the specification of the seasonal component itself changes as a result of trend specification.

The subject of detrending has, therefore, received prominence in virtually all treatments of seasonal adjustment, and the present one is no exception. However, our point of departure from some such treatments is that we are interested only in eliminating the undesirable effects of trend on the definition and extraction of the seasonal component. In particular, it will be seen to be unnecessary to separate the stochastic trend from the irregular component in order to seasonally adjust a series. The treatment of stochastic trend in this section is, therefore, fundamentally different from the usual one, symmetric filtering (which can actually induce strong low-order autocorrelation—properly considered as trend). On the other hand, the deterministic detrending is, in principle, similar to the usual regression approaches, though, in practice, the stochastic detrending procedure (often involving first differencing the series or its logarithm) frequently eliminates deterministic trend as well. This effect is discussed in greater detail in the first subsection, following which the stochastic detrending (based on the filter-preserving property) is considered. This section concludes with a critique of detrending via symmetric filters.

First (Logarithmic) Differencing

A characteristic of many U.S. postwar economic time series is exponential growth; the rates of change of series, such as GNP, price indices, and many other measures of economic activity, tend to be more stable than the series themselves. Moreover, or perhaps, therefore, seasonality and trend have been most effectively measured as percentages or factors of the series, leading to multiplicative seasonal adjustment models. The additive model of this paper is appropriate for the logarithms of such series, seasonal factors being derived as exponentiated logarithmic seasonal components.

An additional characteristic of most U.S. economic series is high first-order autocorrelation, even after removal of a deterministic linear or exponential time trend, in the majority of cases consistent with the hypothesis of homogeneous nonstationarity [4] in the series levels (or logs). Series differencing is, thus, appropriate to achieve stability in the series, i.e., to achieve stationarity in the stochastic component of the series. When logging and differencing are both appropriate, the result is a close approximation to the monthly rate of change of the series.

While the decision to difference a series is based on the nature of its stochastic trend, differencing also eliminates a deterministic linear or exponential time trend. Moreover, the only deterministic component apparent in the logarithmic differences of many economic time series is a periodic mean (seasonal component plus constant mean). That is, if Y_t is the original series, (22) very often becomes

$$\Delta y_t = \Delta \log Y_t = \alpha + \sum_{j=1}^{12} \beta_j d_{jt}^* + u_t^* \qquad (23)$$

$$= \alpha + \sum_{j=1}^{12} \delta_j d_{jt} + u_t^* \qquad (24)$$

where u_t^* is the (differenced) stochastic component of the series, as in (22), and d_{jt}^* is the first difference of the seasonal dummy variable d_{jt},

$$\delta_j = \beta_j - \beta_{j-1},\ j=2, \ldots, 12$$
$$\delta_1 = \beta_1 - \beta_{12} \qquad (25)$$

Note that α is the slope of a linear time trend in y_t = log Y_t and that $e^{\alpha t}$ represents a multiplicative trend factor of Y_t itself. More complex trend variables can certainly be employed, e.g., higher order time trend, other economic variables or variables to capture the effects of special events. Other differencing patterns will also occur, and, of course, some series are stable in level form. But, logarithmic first differencing and the model (24) are the procedure and model that we have found appropriate for the majority of economic time series.[3]

Relationship Between Levels and Changes of a Deterministic Seasonal Component

Equation (25) gives $\{\delta_i\}$ readily from $\{\beta_j\}$; however, the inverse transformation is required in order to seasonally adjust the series levels, having estimated a seasonal

[3] This includes the ones analyzed in the seventh, eighth, and ninth sections and other series in [29]. However as noted previously a more general analysis would include formal investigations of whether another power transformation was appropriate to achieve homogeneity, e.g., as in [7], and whether stochastic trend in the homogeneous series is nonstationary. Quenouille [33, pp. 54–57] discusses the latter point in one of the earliest references to the concept of stochastic trend, and a test of the null hypothesis of nonstationarity in the form of a unit autoregressive root has been recently given by Dickey [9]. (See also [13].)

component such as in equation (24). The desired relationship is given by the following theorem.

Theorem 1—If a series of changes $\{\Delta y_t\}$ is given by (24) then the series $\{y_t\}$ itself has the representation

$$y_t = u_o + \alpha t + \sum_{j=1}^{12} \beta_j d_{jt} + u_t \tag{26}$$

where

$$\Delta u_t = u_t^*,\ u_t = \sum_{i=0}^{t-1} u_{t-i}^*$$

and

$$\beta_j = \sum_{i=1}^{j} \delta_i + \frac{1}{12}\sum_{i=1}^{12} i\delta_i,\ \sum_{j=1}^{12} \beta_j = 0 \tag{27}$$

That is, if Δy_t can be represented as the sum of a constant plus a fixed seasonal plus a stationary error term, then y_t itself can be represented as the sum of a constant, a linear time trend, a fixed seasonal, and an integrated error term.

The proof of the relationship between $\Delta y_t - \sum \delta_j d_{jt}$ and $y_t - \sum \beta_j d_{jt}$ is shown in [4, ch. 4]; it, therefore, suffices to investigate the relationship between $\sum \delta_j d_{jt}$ and $\sum \beta_j d_{jt}$. A general solution to (25) is of the form

$$\beta_1 = \delta_1 + c$$

$$\beta_2 = \delta_1 + \delta_2 + c$$

$$\vdots$$

$$\beta_{12} = \delta_1 + \delta_2 + \ldots + \delta_{12} + c$$

where c is an arbitrary constant. But, the presence of a constant term in (26) means that we can impose the constraint $\sum \beta_j = 0$. Thus, the constant c is uniquely determined, since

$$0 = 12\delta_1 + 11\delta_2 + \ldots + \delta_{12} + 12c$$

so that

$$c = -\frac{1}{12}\sum_{i=1}^{12}(13-i)\delta_i = \frac{1}{12}\sum_{i=1}^{12} i\delta_i$$

This gives β_j as in (27) and the theorem is proved. Higher order nonseasonal differences can be treated iteratively in this manner.

Stochastic Detrending: Preservation of the Seasonal Adjustment Under Nonseasonal Filters

The stochastic component of equation (22), e.g., of equation (24), is of the form

$$u_t^* = p_{2t}^* + s_{2t}^* + e_t^* \tag{28}$$

where the 2-subscript denotes the stochastic trend and seasonal and the asterisk denotes the respective component after differencing or stationarity inducing transformation. Noting that seasonal differencing is rarely necessary to remove seasonal nonstationarity [30], once a periodic mean is included in (24), the differencing operator $\Delta(B)$ [most often $\Delta(B) = 1 - B$] is in general a nonseasonal filter. (When this filter does involve seasonal differencing, a decomposition, such as equation (14), seems most appropriate; see [3; 5] for further discussion of this situation.) Thus, assumption 3 (see the second section) is that s_{2t}^* is the seasonal component of u_t^*, as s_{2t} is the seasonal component of u_t in (21). Moreover, p_{2t}^* and e_t^* are nonseasonal, so that, with $n_t^* = p_{2t}^* + e_t^*$, u_t^* is the sum

$$u_t^* = s_{2t}^* + n_t^* \tag{29}$$

of two components, seasonal (signal) and nonseasonal (noise).

Now, the stochastic properties of the stationary series u_t^* can be assessed via its sample autocorrelation function. In general, there will be a nonseasonal filter that will eliminate whatever low-order autocorrelation (trend) from u_t^* that was not eliminated previously from differencing the series; thus, only autocorrelation at seasonal lags (generally lags 12, 24, . . .; see the sixth section for further discussion) remains. If this filter is $h(B)$, i.e., $\tilde{u}_t = h(B)u_t^*$, etc., the relation

$$\tilde{u}_t = \tilde{s}_{2t} + \tilde{n}_t \tag{30}$$

will be such that $\tilde{n}_t$ is white noise and $\tilde{s}_{2t}$ is autocorrelated only at seasonal lags.

If the models for $\tilde{u}$ and $\tilde{s}_2$ are

$$\tilde{s}_{2t} = \tilde{\psi}_s(B)\epsilon_t,\ \tilde{u}_t = \tilde{\psi}_u(B)a_t$$

then $\tilde{s}_{2t}$ is estimated by (12), using these models and with $\tilde{u}_t$ replacing y_t. The seasonal component of u_t^* is then

$$s_{2t}^* = h^{-1}(B)\tilde{s}_{2t} \tag{31}$$

Alternatively, the symmetric filter

$$\frac{\sigma_\epsilon^2 \psi_s(B)\psi_s(F)}{\sigma_a^2 \psi_u(B)\psi_u(F)} \tag{32}$$

may be applied directly to u_t^* (rather than $\tilde{u}_t$) to estimate s_{2t}^*.

The final application of the filter-preserving principle is to recover the stochastic seasonal component s_{2t} for the levels from the component s_{2t}^* for the series changes. The procedure is the same as for the fixed deterministic seasonal, theorem 1, except that it is no longer true that the seasonal component sums to zero over a 12-month period. Instead, following usual practice, we assume that the seasonal component sums to zero over a calendar year. This constraint suffices to determine s_{2t} uniquely from s_{2t}^* (hence, s_t from s_t^*), as in (27), provided that s_{2t}^* is first adjusted by a constant each year in order to sum to zero over that year. Alternatively, if t indexes December of any given year, one may simply compute preliminary quantities

$$s^o_{t+1}=0,\ s^o_{t+j}=s^o_{t+j-1}+s^*_{2,\ t+j}(2\leq j\leq 12)$$

whereupon

$$s_{2,t+j}=s^o_{t+j}-\frac{1}{12}\sum_{i=1}^{12} s^o_{t+i}(1\leq j\leq 12) \qquad (33)$$

is the seasonal component for the series levels. From equations (27) and (33), the combined seasonal component of y_t is

$$s_{t+j}=\beta_j+s_{2,\ t+j} \qquad (34)$$

i.e., for any t, $s_t=\beta_t+s_{2t}$.

Detrending Via Symmetric Filters

We have seen that differencing in order to detrend a series introduces changes in the seasonal component so that further work is needed to recover the seasonal for the series levels. By contrast, the usual approach to detrending has been to seek symmetric filters of the form

$$\mu(B)=\sum \mu_j B^j \qquad (35)$$

such that

$$\mu(B)s_t=0,\ \mu(B)p_t=p_t \qquad (36)$$

so that

$$(1-\mu(B))y_t=D(B)y_t=s_t+e'_t \qquad (37)$$

giving a simpler decomposition of the series (compare with equation (29)), thus, facilitating the seasonal adjustment process. Most frequently (35) is a centered moving average

$$\mu(B)=\frac{1}{M}\sum_{j=-k}^{k} B^j,\ M=2k+1 \qquad (38)$$

of its operand (appropriate modifications are required if the period M is even, as it is for monthly data).

The problem with such approaches is that the filters $\mu(B)$ have been chosen with deterministic effects in mind, e.g., (38) produces (36) if s_t is a periodic mean and y_t is a linear or quadratic time trend; whereas the second section has shown that stochastic effects are best handled by symmetric filters. Depending on the stochastic structure of the series, it is entirely possible that $e^*_t=D(B)e_t$ in (37) will be more trendlike (very highly autocorrelated) than the original trend p_t itself. To see how this can arise for (38) (other detrending filters, such as described and used in [36], can be expected to exert similar effects), note that

$$D(B)y_t=(1-\frac{1}{2k+1}\sum_{-k}^{k} B^j)y_t$$

$$=\frac{1}{2k+1}\sum_{j=1}^{k}[(y_t-y_{t-j})+(y_t-y_{t+j})]$$

$$=\frac{1}{2k+1}\sum_{j=1}^{k}[(1-B^j)+(1-F^j)]y_t$$

$$=\frac{1}{2k+1}[(1-B)(1-F)\sum_{j=1}^{k} S_j(B)S_j(F)]y_t$$

$$=\frac{-1}{2k+1}\{\Delta^2\sum_{j=1}^{k} F^j[S_j(B)]^2\}y_t \qquad (39)$$

where $S_j(z)=\sum_{i=0}^{j-1} z^i$ and $F=B^{-1}$. Thus, it is interesting that $D(B)$, as it contains the factor Δ^2, or equivalently $(1-B)(1-F)$, eliminates homogeneous (stochastic) nonstationarity of order 2, which includes the type of nonstationarity (of order 1) that is evidently displayed by many economic time series. It does what it should in this respect; but it does not stop here. The remaining factor of $D(B)$ (other than Δ^2) is a complicated polynomial of high degree in B; thus, for the many series that are not strongly autocorrelated (except for seasonality) after differencing, a great deal of inappropriate smoothing is done by this operation. The result is the injection, rather than the elimination, of trend in the form of low-order autocorrelation. Additionally, the result of applying $D(B)$ is generally a noninvertible series (a series without an autoregressive representation or a positive spectrum).

In view of this drawback, and of the demonstrated recoverability of seasonally adjusted levels from seasonally adjusted changes of a series, it is recommended that differences replace symmetric moving averages as the primary detrending device in symmetric filtering procedures, such as X–11. As stated at the beginning of this section, there is no necessary reason to separate trend and irregular elements if the purpose is seasonal adjustment.

ADJUSTMENT FOR DETERMINISTIC SEASONALITY

In this section and the next, the seasonal adjustment procedure per se is presented, including the estimation of the models for seasonality and tests for deterministic or stochastic seasonality.

For deterministic seasonality, the assumptions in the third section essentially restricted the component s_{1t} in the general model (20) to a fixed seasonal component, so that the $\{d_{jt}\}$ in this equation can be monthly dummy variables, as in (26). In the fourth section, relationships between the periodic components

$$s_{1t}=\sum \beta_j d_{jt}=\beta_t$$

and

$$s_{1t}^{*}=\sum \beta_j d_{jt}^{*}=\sum \delta_j d_{jt}=\delta_t$$

were established. (See theorem (1).) What remains is estimating s_{1t} and testing for its presence.

Estimation of the Deterministic Component

The equation for a fixed seasonal component (plus trend), e.g., equation (24) or (26), is a linear model for which regression procedures are generally appropriate. Usually efficient estimates of the coefficients of the model require that the stochastic component of the series be serially independent or that an appropriate version of generalized least squares be employed. However, for certain regression functions—including fixed periodic means polynomial time trends, and interactions of the two—ordinary least squares is asymptotically efficient, provided that the regression residual is stationary; e.g., see [1]. Thus, while it is often important to compute series first differences to ensure stationarity of the stochastic component, having done this ordinary least squares generally suffices.

In practice, therefore, often the procedure is simply to regress the logarithmic first differences on a set of 12 seasonal dummy variables, thus, estimating the quantities

$$\alpha+\delta_j\equiv\lambda_j \tag{40}$$

in equation (24). Then,

$$\hat{\alpha}=\frac{1}{12}\sum \hat{\lambda}_j,\quad \hat{\delta}_j=\hat{\lambda}_j-\hat{\alpha} \tag{41}$$

are estimates of the constant term and seasonal component, and theorem (23) is used to get $\{\hat{\beta}_j\}$ from $\{\hat{\delta}_j\}$. If the series does not need differencing, then the seasonal component is estimated directly from a regression of the (logged) series levels. That is, referring to equation (26), if $\{\lambda_j, j=1, \ldots, 12\}$ are again used to denote the coefficients of the seasonal dummies, (41) is employed with β_j replacing δ_j and the constant term u_o (now a parameter rather than an initial condition) replacing α. Procedures for other patterns of differencing can also be formulated. The series adjusted for deterministic seasonality is $y_t-\hat{\beta}_t$ or $Y_t/e^{\hat{\beta}_t}$ if $y_t=\log Y_t$.

Tests of Deterministic Seasonality

Given model (20) or (26), deterministic seasonality exists if not all the β_j are zero, which is equivalent to not all the δ_j in (22) or (24) being zero. Thus, if the hypothesis

$$H_o:\beta_1=\ldots=\beta_{12}=0 \tag{42}$$

cannot be rejected at a suitable significance level, one could conclude that adjustment for deterministic seasonality is unnecessary.

While the deterministic trend and seasonality are very often efficiently estimated by ordinary least squares, the sampling distribution of the estimated coefficients depends on the covariance matrix of the regression residual, i.e., on the autocovariance function of the stochastic component of the series. There are several approaches to this problem that are all asymptotically equivalent to generalized least squares. The simplest one is to note that if the stochastic model for the residual process u_t is

$$\pi(B)u_t=a_t \tag{43}$$

then one may regress $y_t'=\pi(B)y_t$ on seasonal dummies, and the coefficients $\lambda_1', \ldots, \lambda_{12}'$ in this equation are equal to each other under (and only under) the null hypothesis (42). Moreover, the sample autocovariances of the regression residuals $\{\hat{u}_t\}$ are known [27] to have the same asymptotic distribution as those of the true stochastic component $\{u_t\}$; thus, $\pi(B)$ in (43) may be effectively estimated after the regression of the untransformed variables.

After transformation, the regression residual is approximately white noise; thus, the general linear hypothesis tests are asymptotically valid. Thus, the test of H_o in (42) may be carried out by—

1. Estimating $\lambda_1, \ldots, \lambda_{12}$, as in the fifth section.
2. Forming estimates $\hat{\pi}(B)$ and $\hat{a}_t$ of the quantities in equation (43).
3. Forming

$$\hat{b}_t=\hat{\pi}(B)(y_t-\bar{y}) \tag{44}$$

where b_t is a_t under H_o, since $\bar{y}$ is the estimate of the common value α (more generally, $\hat{b}_t$ are the residuals from the regression under H_o), and

4. Computing

$$F=\frac{(\sum \hat{b}_t^2-\sum \hat{a}_t^2)/11}{\sum \hat{a}_t^2/(n-12)} \tag{45}$$

that, under H_o, has an F-distribution with 11 and n–12 degrees of freedom.

Several variants of this, which have the same asymptotic properties, are possible; e.g., after (43), the regressions may be rerun. Whatever is done, however, it is important to take account of autocorrelation in these tests. Particularly if the fixed seasonal component is estimated from the undifferenced series, a strong bias in purported significance levels could result. In this regard, the test carried out in X–11 [36, p. 59] could be improved by procedures such as those given in this paper. The current X–11 test is carried out with series detrended by symmetric filters, virtually guaranteeing a highly autocorrelated series. (See the subsection on detrending via symmetric filters in the fourth section.)

ADJUSTMENT FOR STOCHASTIC SEASONALITY

Having estimated the deterministic components (trend and seasonal) of the time series, the remaining problem for seasonal adjustment is the estimation of the stochastic seasonal component s_{2t} in

$$u_t = p_{2t} + s_{2t} + e_t \tag{46}$$

Additionally, this section presents tests for the existence of stochastic seasonality, i.e., of the hypothesis $\mathrm{Var}(s_{2t}) = 0$ in (46). If such a component is absent, the procedure in the previous section is all that is required.

Decomposition of Stochastic Seasonal Series: The Minimal Extraction Principle

In the fourth section, it was seen that the decomposition problem was simplified by the filter-preserving property, so that we could essentially reduce (46) to

$$\tilde{u}_t = \tilde{s}_{2t} + \tilde{n}_t \tag{47}$$

in which $\tilde{n}_t$ is white noise and $\tilde{s}_{2t}$ has no autocorrelation at lags other than seasonal. (In the terminology of Granger [14], $\tilde{s}_{2t}$ is a "strongly seasonal" process.)

But, even with the simplification of (46) into the form (47), the procedure for estimating the seasonal component can be rather complicated, and, moreover, a decomposition, such as (47), is not, in general, uniquely determined from the overall model, i.e., the model for $\tilde{u}_t$. While a completely general treatment still awaits development, we have used a principle of minimal extraction, i.e., assumption 4 in the third section, to solve the problem for a group of models that, while simple, seem to cover the great majority of cases encountered.[4] These are the 12th-order ARMA models

$$(1-\phi B^{12})\tilde{u}_t = (1-\theta B^{12})a_t \tag{48}$$

for which the admissible models for $\tilde{s}_{2t}$ are of the form

$$(1-\phi B^{12})\tilde{s}_{2t} = (1-\Theta B^{12})\epsilon_t, \quad \mathrm{Var}(\epsilon_t) = \sigma_\epsilon^2 \tag{49}$$

with $\tilde{n}_t$ white noise with variance $\sigma_{\tilde{n}}^2$. (We will discuss a quarterly-annual multiplicative model in the ninth section.) The relationship between the parameters of the overall model and the components model may be seen by multiplying $\tilde{u}_t = \tilde{s}_{2t} + \tilde{n}_t$ by $1-\phi B^{12}$, obtaining

$$(1-\theta B^{12})a_t = (1-\Theta B^{12})\epsilon_t + (1-\phi B^{12})\tilde{n}_t \tag{50}$$

and noting that the variance and lag 1 autocovariance of each side of (50) must be equal. Thus,

$$(1+\theta^2)\sigma_a^2 = (1+\Theta^2)\sigma_\epsilon^2 + (1+\phi^2)\sigma_{\tilde{n}}^2$$
$$\theta\sigma_a^2 = \Theta\sigma_\epsilon^2 + \phi\sigma_{\tilde{n}}^2 \tag{51}$$

Since the model for $\tilde{u}_t$ is determined by the three parameters $(\phi, \theta, \sigma_a^2)$, whereas the components model involves four parameters $(\phi, \Theta, \sigma_\epsilon^2, \sigma_{\tilde{n}}^2)$, the model for $\tilde{s}_t$ and $\tilde{n}_t$ is unidentified.

Given a value of, e.g., Θ, the equations (51) could be used to determine σ_ϵ^2 and $\sigma_{\tilde{n}}^2$ from σ_a^2 and θ (and ϕ, common to both models). Frequently, it is simply assumed that $\Theta=0$, since this model is one of autoregressive signal (seasonal) plus white noise, which is uniquely determined from (48). Certainly, if there is a theory to suggest that $\Theta=0$, i.e., that the seasonal is a pure autoregression, this is what should be done. (For an example of this approach, see [23].)

However, for most situations the assumption $\Theta=0$ in (50) is probably arbitrary, and there is a need for resolving the identification problem when no such a priori information exists. The approach taken here is to leave the series intact, insofar as possible, to remove as little as possible and still remove the series' seasonality. This would imply that the variance of the seasonal component be chosen as small as possible, consistent with the equation system (51). Ordinarily, $\phi > \Theta > 0$ in (48), and, in this case, it can be shown that choosing $\Theta = -1$ minimizes the variance of the seasonal $\tilde{s}_t$. Any other value results in more than necessary being removed from the series in the seasonal adjustment process.

Estimation of the Model Parameters and the Seasonal Component

Having identified the model for $\tilde{s}_t$ and $\tilde{n}_t$ by assigning a value to Θ, asymptotically efficient estimates are obtained for the remaining parameters $(\phi, \sigma_\epsilon^2, \sigma_{\tilde{n}}^2)$ in this model by substituting estimates for $(\phi, \theta, \sigma_a^2)$ into (51), since this equation system defines a nonsingular linear transformation between (θ, σ_a^2) and $(\sigma_\epsilon^2, \sigma_{\tilde{n}}^2)$, ϕ being common to both. Estimation of parameters for ARMA models is discussed in numerous sources, e.g., [3; 17; 26], and all such procedures are asymptotically equivalent to maximum likelihood estimates, as are, therefore, the estimates of σ_ϵ^2 and $\sigma_{\tilde{n}}^2$ obtained from (51). Conditional on these values,[5] the minimum mean square estimate of the stochastic seasonal component is, from (2),

$$\hat{s}_{2t}^* = \nu(B)u_t^* = \frac{\hat{\sigma}_\epsilon^2 \hat{\psi}_{\tilde{s}}(B)\hat{\psi}_{\tilde{s}}(F)}{\hat{\sigma}_a^2 \hat{\psi}_{\tilde{u}}(B)\hat{\psi}_{\tilde{u}}(F)} u_t^* \tag{52}$$

$$= \frac{\hat{\sigma}_\epsilon^2 (1+B^{12})(1+F^{12})}{\hat{\sigma}_a^2 (1-\hat{\theta}B^{12})(1-\hat{\theta}F^{12})} u_t^* \tag{53}$$

equation (53) reflecting the cancellation of the common autoregressive operator in the numerator and denominator of (52). The procedure in the subsection on stochastic

[4] A valuable extension of this procedure and class of models is provided by Wecker in the accompanying discussion.

[5] This also refers to conditional on the parameter estimates in the ARMA model used in the stochastic detrending. When these two sets of estimates are not independent, one should properly estimate both sets jointly in a multiplicative ARMA model, a point brought to my attention by Engle.

detrending in the fourth section is then used to estimate s_{2t}, the stochastic seasonal component of the undifferenced time series.

Tests of Stochastic Seasonality

Given the relatively greater difficulty in determining the stochastic seasonal model specification and in estimating the resulting seasonal component, it would be of particular importance to identify those series for which little or no evidence exists for such seasonality (i.e., for a changing seasonal pattern); the seasonal adjustment procedure would then involve only a regression on seasonal dummies and the recovery of the seasonal component or factor for the observed series from this regression.

The test proposed for the presence of seasonality is based on the autocorrelations of the residuals from the seasonal dummy regression, after detrending, as in the fourth section, i.e., of $\{\tilde{u}_t\}$. If the detrending is adequate and if there is no (remaining) seasonality, then the $\{\tilde{u}_t\}$ should be a white-noise series. Thus, evidence of seasonality is revealed by (1) large individual autocorrelations at the seasonal lags, e.g., lags 12, 24, ..., and possibly 3, 6, ..., or (2) by unusually large values of such statistics as

$$Q_1=n\sum_{k=1}^{m}\hat{r}_k^2,\ Q_3=n\sum_{k=1}^{m/3}\hat{r}_{3k}^2,\ Q_{12}=n\sum_{k=1}^{m/12}\hat{r}_{12k}^2 \qquad (54)$$

where m is generally a low-order multiple of 12. The asymptotic distribution of Q_i, under the null hypothesis that $\tilde{u}_t$ is white noise, is given for some cases in [27]. If no stochastic detrending is performed ($\tilde{u}_t = u_t^*$), then Q_i is χ^2 with m/i degrees of freedom; otherwise, while Q_1 is asymptotically $\chi^2(m-r)$, where r is the number of estimated ARMA parameters in the detrending filter, the precise large-sample distribution of Q_3 and Q_{12} is not known, though, on the assumption of a low-order detrending filter, it can be shown to lie between the $\chi^2(m/i)$ and $\chi^2(m/i-r)$ distributions, probably closer to $\chi^2(m/i)$, since the effects of the ARMA estimation are concentrated on the low-order sample autocorrelations.

It should be noted that, for stationary series, frequency-domain criteria [16; 24] for seasonality have also been put forth, which generally relate to the occurrence of average power (rather than autocorrelation) at the seasonal frequencies $2\pi\frac{j}{12}$, $j=1, \ldots, 6$ (rather than at the seasonal lags). The two approaches are closely related, as, of course, they should be; e.g., the seasonal autoregressive model

$$s_t=\phi s_{t-12}+\epsilon_t \qquad (55)$$

has no autocorrelation other than at lags 12, 24, ..., and strong peaks at all six seasonal frequencies. And, the distinction between (55) and seasonal models with a first-order MA component appended, considered by [14; 24] and others, is immaterial in view of the filter-preservation property. (See assumption 3 in the third section.)

CONSUMER PRICE INDEX AND HOUSING STARTS

In the next three sections, we exemplify the procedures developed in this paper using several monthly economic time series. The analysis presented here is based on computations performed in [29], in which empirical comparisons were made between several adjustment procedures, including X–11 and the present one. This section examines two series that were found in [29] to require at most a deterministic seasonal adjustment, as described in the fifth section, and the following two sections analyze series found to possess stochastic seasonality as well.

Tests of Seasonality in the CPI

Table 1 shows the autocorrelations of various functions of the Consumer Price Index from 1947 to 1975, as in the *Survey of Current Business,* U.S. Department of Commerce. It is a highly trending and autocorrelated series, and both first and second differences of the logarithms (the first of which is essentially the monthly inflation-rate series) are shown in parts (a) and (b) of this table. It is clear, from examining these autocorrelations, that any seasonality in this series is quite mild; e.g., the lag-12 autocorrelation r_{12} is scarcely larger than r_{11} or r_{13} for either series difference.

To estimate a fixed deterministic seasonal component, a regression of $y_t = \Delta\log \text{CPI}_t$ on seasonal dummies was run, as in equation (24). The value of R^2 was only 0.03 (0.02 adjusted), confirming the above observation concerning weak seasonality. The autocorrelations of the residuals from the regression are shown in part (c) of table 1; except for some small changes, e.g., r_{12} is reduced from 0.18 to 0.15, they are virtually indistinguishable from those in part (a).

We wish to exemplify the seasonality tests in the fifth and sixth sections to see if there is in fact statistically significant seasonality in the CPI. While the pronounced low-order autocorrelation in the residuals from the regression (table 1c) does not impair the asymptotic efficiency of the least squares estimates (see the fifth section), it does affect both the F-test and the autocorrelation tests for seasonality. Thus, the first step was to detrend this residual series. The autocorrelation pattern suggests an ARMA (1, 1) model, and the fitted model

$$(1-0.81B)u_t^*=(1-0.47B)\hat{a}_t \qquad (56)$$

was obtained. The autocorrelations of the residuals $\{\hat{a}_t\}$ are in table 1e.

To see if the fixed monthly seasonal components are significantly different from each other, the procedure

Table 1. AUTOCORRELATIONS, CONSUMER PRICE INDEX

Lags	1	2	3	4	5	6	7	8	9	0
	a. Δlog (NSA) *(Variance = $0.0^4 29$)*									
1-10	0.43	0.32	0.36	0.35	0.08	0.12	0.13	0.24	0.20	0.18
11-20	.15	.18	.15	.18	.08	.08	.09	.02	-.07	.03
21-30	.10	.04	.03	.01	-.03	-.05	-.08	-.10	-.08	.01
31-40	.05	.05	.04	.00	.00	.02	.04	.04	.00	.07
41-50	.01	.00	.01	.03	.07	.09	.12	.16	.13	.14
51-60	.13	.09	.09	.13	.10	.05	.06	.07	.02	.06
	b. Δ^2 log (NSA) *(Variance = $0.0^4 33$)*									
1-10	-0.41	-0.12	0.04	0.13	-0.08	-0.05	-0.09	0.12	-0.02	0.03
11-20	-.06	.05	.05	.12	.09	.02	.03	.12	-.22	.03
21-30	.11	-.05	.01	.02	-.03	.02	-.01	-.04	-.03	.09
31-40	-.13	.11	.01	-.03	-.02	.00	.02	.03	-.08	.11
41-50	-.05	.01	-.05	.01	.02	-.02	-.00	.07	-.04	.03
51-60	.02	-.03	-.04	.06	.02	-.06	.00	.05	-.08	.07
	c. Δlog (Regression SA)—Regression Residuals *(Variance = $0.0^4 28$)*									
1-10	0.43	0.33	0.37	0.37	0.23	0.15	0.17	0.25	0.20	0.19
11-20	.14	.15	.15	.18	.08	.09	.12	.10	-.05	.04
21-30	.09	.03	.02	-.02	-.04	-.05	-.09	-.10	-.06	-.00
31-40	-.04	.06	.03	-.00	-.01	-.00	.04	.04	.00	.08
41-50	.03	.01	-.00	.03	.07	.09	.12	.14	.13	.15
51-60	.13	.10	.12	.14	.12	.05	.06	.06	.01	.04
	d. Δlog (X-11 SA) *(Variance = $0.0^4 14$)*									
1-10	0.53	0.42	0.41	0.35	0.36	0.36	0.35	0.32	0.35	0.35
11-20	.20	.06	.11	.10	.09	.14	.10	.03	.01	.02
21-30	.02	.01	-.02	-.07	-.02	-.00	.01	-.01	.08	.10
31-40	.13	.19	.11	.06	.06	.10	.15	.15	.16	.19
41-50	.17	.10	.05	.02	.06	.12	.16	.16	.09	.09
51-60	.09	.03	.06	.09	.13	.10	.11	.09	.06	.06
	e. Detrended Regression Residuals *(Variance = $0.0^4 19$)*									
1-10	-0.02	0.08	0.01	-0.04	-0.02	-0.12	-0.05	0.09	0.04	0.05
11-20	.00	-.01	.02	.08	-.01	.03	.05	.05	-.13	-.00
21-30	.09	.00	.09	-.03	-.01	-.04	-.07	-.09	-.03	.05
31-40	-.02	.13	.04	-.03	.01	-.07	.05	.04	-.02	.11
41-50	-.01	-.03	-.05	-.05	.03	-.00	.07	.06	.06	.06
51-60	.03	-.01	.06	.05	.09	-.04	.00	-.02	-.07	-.01

developed in the fifth section is to compute a series $\hat{b}_t$ given by

$$(1-0.81B)(y_t-\bar{y})=(1-0.47B)\hat{b}_t \tag{57}$$

where $y_t = \Delta\log \text{CPI}_t$. Note that, under the null hypothesis, $\{y_t-\bar{y}\}$ has the same asymptotic distribution as u_t^*; their innovations (estimated by $\hat{a}_t$ and $\hat{b}_t$) are both white noise with the same variance. The result of the calculation (45) gave

$$F=2.92$$

which is significant at the 1-percent level. Thus, although the seasonal variables explain less than 3 percent of the variation in the inflation-rate series, they are significantly nonzero.

The second test is for the existence of remaining seasonality after the regression, i.e., of seasonality in u_t^*. From the filter-preserving property, we may equivalently examine the series $\hat{a}_t=\tilde{u}_t$ for seasonality. From table 1c, there is very little evidence of seasonality in this series. For example, in (54),

$$Q_3=336(0.01^2+0.12^2+0.04^2)=5.4$$

$$Q_{12}=336(0.01^2+0.03^2+0.07^2)=2.0$$

whereas the upper 10-percent point of the χ^2 (3) distribution is 6.2. In fact, the value

$$Q=336\sum_{k=1}^{20} r_k^2=24.5$$

compared with $\chi^2_{0.10}$ (18) = 26.0, does not allow rejection of the hypothesis that this is a white-noise series.

The conclusion is that the fixed seasonal model is evidently adequate, and, as noted, even this seasonal effect is slight. By contrast, the officially adjusted CPI series is a series adjusted by changing seasonal factors, which are displayed in table A-1 in the appendix. Two main effects of this seasonal adjustment can be noted from the autocorrelations of the seasonally adjusted inflation rates in table 1d: There is overadjustment, as evidenced by the dip in r_{12} (the twice-differenced series gave $r_{12}=-0.19$, $r_{11}=-0.02$, and $r_{13}=0.07$), and the variance of the series is cut in half, suggesting an unnecessary amount of smoothing of the series.

Seasonal Adjustment of Housing Starts

A series that exhibits stronger seasonality than the Consumer Price Index is the series of total new housing units started, taken from 1959 to 1975. The autocorrelations in table 2a clearly reveal seasonality in this series, and as the Δlog series appears stationary in other respects, the regression (24) was run to extract that part of the series explainable by a fixed periodic effect. The autocorrelations of the residual series $\{\hat{u}_t\}$ are given in table 2d; the regression R^2 was 0.72.

The F-test described in the fifth section showed very strong evidence of seasonality, and no further seasonality was revealed by tests on the autocorrelation of u_t (or the detrended series $\tilde{u}_t$, which was estimated by $\hat{u}_t=(1+0.31B)\tilde{u}_t$). Consequently, a fixed seasonal adjustment of housing starts is evidently adequate, and the adjusted series and seasonal factors are shown in appendix table A-2. By contrast, the seasonal factors for the published X–11 adjusted series, also shown in this table, are quite variable. (Note, however, that the variances of the two seasonally adjusted housing starts series, in Δlog form, are comparable, in contrast, with the situation for the CPI.) The housing starts series, thus, provides a good illustration of the identifiability problems discussed in the third section. (See [29] for other illustrations as well.) Evidently, the data often do not contain sufficient information to enable a discrimination between theoretically incompatible seasonal adjustment models.[6]

UNEMPLOYMENT RATE

Table 3 presents autocorrelation data on the series of U.S. unemployment for 1947–75. As with the CPI and housing start series, the Δlog-unemployment series appears stationary, except for seasonality; thus, this series was regressed on seasonal dummies as described in the fifth section. Unlike the two previous series, however, there is in table 3d clear evidence of remaining seasonality. The autocorrelations of the regression residuals $\hat{u}_t$ at lags 12, 24, . . ., while small, are very persistent; they remained unchanged (to the second decimal place) as a result of the estimated detrending transformation

$$\tilde{u}_t=(1-0.03B-0.18B^2)\hat{u}_t \tag{58}$$

after which the statistic

$$Q_{12}=312(0.18^2+0.18^2+0.17^2)=29.2$$

is off the χ^2 (3) tables. Clearly, stochastic seasonality is also present in the unemployment series.

The second autocorrelation pattern for $\hat{u}_t$, in table 3d, that, as noted, is also the pattern exhibited by $\tilde{u}_t$, is suggestive of the seasonal ARMA process in the subsection on tests of stochastic seasonality in the sixth section, with $0<\theta<\phi<1$ but ϕ and θ both close to 1. Two approaches to the estimation of ϕ and θ were attempted. The first was nonlinear least squares, which is asymptoti-

[6] The situation regarding seasonality is, in many respects, analogous to a phenomenon noted in [28] for econometric models. In that study, it was found that, while such models are often centered around structural relationships with strong apparent associations, essentially equal explanatory power was very often obtainable by modeling an endogenous variable in terms of its past history alone. Thus, in either situation, the data provide some, but not enough, information. Certain obviously inappropriate models can always be rejected; there remain logically incompatible or contradictory sets of models, whether econometric models or seasonal adjustment models, all empirically compatible with the same data.

Table 2. **AUTOCORRELATIONS, HOUSING STARTS**

Lags	1	2	3	4	5	6	7	8	9	0
	a. Δlog (NSA) *(Variance = 0.0277)*									
1-10	0.25	-0.01	-0.21	-0.29	-0.06	-0.09	-0.09	-0.33	-0.24	-0.05
11-20	.34	.66	.32	.04	-.24	-.32	-.02	-.08	-.08	-.26
21-30	-.28	-.02	.32	.65	.24	.01	-.22	-.26	-.05	-.09
31-40	-.04	-.26	-.26	-.01	.31	.57	.27	-.02	-.24	-.22
41-50	-.03	-.10	-.04	-.24	-.22	-.00	.27	.56	.24	-.01
51-60	-.22	-.21	-.06	-.04	-.04	-.26	-.20	.02	.26	.51
	b. $\Delta\Delta_{12}$ log (NSA) *(Variance = 0.0174)*									
1-10	-0.32	-0.04	0.10	0.11	-0.01	0.02	-0.01	-0.09	0.08	-0.06
11-20	.19	-.50	.18	.12	-.09	-.16	.11	.02	-.06	.09
21-30	-.08	.00	-.02	.07	-.10	-.02	.04	.04	-.10	-.00
31-40	.03	-.02	-.08	.00	.05	-.09	.08	-.03	-.02	.03
41-50	.09	-.10	-.00	.03	.06	-.01	-.06	.04	-.05	.06
51-60	-.05	.02	-.08	.10	-.01	-.06	.05	-.01	.05	-.01
	c. Δlog (X-11 SA) *(Variance = 0.0068)*									
1-10	-0.28	-0.02	0.05	0.10	0.05	0.01	0.00	-0.07	0.08	-0.04
11-20	.08	-.22	.08	.08	-.04	-.09	.06	.02	-.06	.07
21-30	-.08	.03	-.03	-.07	-.01	-.01	-.03	.06	-.10	-.02
31-40	.01	-.01	-.08	.05	.01	-.06	.01	-.04	.04	-.00
41-50	.00	-.02	-.05	-.01	.17	-.06	-.08	.07	-.08	.08
51-60	-.03	-.02	-.04	.08	.01	-.06	.03	-.00	.04	.02
	d. Δlog (Regression SA) *(Variance = 0.0077)*									
1-10	-0.28	-0.06	0.14	0.00	0.02	0.09	-0.06	-0.05	0.10	-0.15
11-20	.12	-.10	.02	.10	-.01	-.18	.09	.09	-.13	.10
21-30	-.08	-.07	0.6	-.02	-.11	.04	-.00	-.00	-.04	-.03
31-40	-.01	.04	-.09	-.02	.10	-.15	.04	-.01	-.04	.05
41-50	.06	-.09	-.01	.02	.05	.02	-.03	-.03	-.01	.02
51-60	-.05	.07	-.07	.07	-.00	-.05	.03	.03	.04	-.03

Table 3. **AUTOCORRELATIONS, UNEMPLOYMENT RATE**

Lags	1	2	3	4	5	6	7	8	9	0
	a. Δlog (NSA) *(Variance = 0.0130)*									
1-10	0.03	-0.07	-0.23	-0.13	0.23	-0.15	0.20	-0.13	-0.26	-0.12
11-20	.01	.73	-.03	-.13	-.26	-.16	.17	-.17	.19	-.14
21-30	-.25	-.10	-.02	.71	.01	-.10	-.22	-.13	.17	-.13
31-40	.18	-.17	-.24	-.10	-.00	.68	.00	-.09	-.23	-.15
41-50	.15	-.13	.14	-.14	-.22	-.07	.04	.65	.03	-.07
51-60	-.20	-.11	.20	-.13	.19	-.14	-.20	-.07	.00	.58
	b. $\Delta\Delta_{12}$log (NSA) *(Variance = 0.0062)*									
1-10	0.17	0.29	0.18	0.10	0.17	0.06	-0.04	0.01	-0.03	-0.16
11-20	-.00	-.46	-.17	-.19	-.15	-.12	-.13	-.12	.02	.04
21-30	-.04	.04	-.06	.01	.08	.08	.07	.08	.05	.06
31-40	.03	-.09	-.00	-.05	-.08	-.03	-.07	-.07	-.09	-10
41-50	-.12	-.02	-.18	.01	-.01	.04	.12	.07	.10	.11
51-60	.07	.15	.18	-.03	.15	-.00	.05	.02	-.06	-.05
	c. Δlog (X-11 SA) *(Variance = 0.0025)*									
1-10	0.18	0.27	0.19	0.15	0.19	0.08	0.02	0.05	-0.01	-0.13
11-20	.00	-.21	-.10	-.05	-.11	-.09	-.11	-.09	-.02	.02
21-30	-.12	-.03	-.08	-.13	.02	-.04	.05	.04	-.04	-.01
31-40	-.04	-.07	.00	-.02	-.10	-.06	-.06	-.06	-.02	-.08
41-50	-.05	.00	-.09	-.02	-.02	.05	.08	.05	.05	1.0
51-60	.07	.14	.18	.07	.11	.01	.08	.06	.01	.02
	d. Δlog (Regression SA) *(Variance = 0.0039)*									
1-10	0.03	0.18	0.10	0.10	0.04	-0.08	-0.07	0.09	-0.04	-0.06
11-20	-.04	.18	-.13	-.05	-.10	-.00	-.16	-.16	-.06	.04
21-30	-.08	.00	-.13	.18	.01	.03	.01	.07	-.11	-.03
31-40	-.08	-.04	-.05	.02	-.11	.17	-.03	-.01	-.02	-.01
41-50	-.12	-.04	-.16	.02	-.05	.09	.03	.16	.04	.08
51-60	.05	.10	.05	-.04	.07	.03	-.03	.12	-.06	.07
	e. Δlog (Regression-Moving Average SA) *(Variance = 0.0030)*									
1-10	0.10	0.20	0.15	0.10	0.12	-0.02	-0.01	0.08	-0.02	-0.11
11-20	-.01	-.04	-.11	-.10	-.12	-.04	-.13	-.13	-.01	.02
21-30	-.06	-.03	-.13	-.00	.04	.02	.01	.06	-.06	.01
31-40	-.04	-.08	-.03	-.01	-.10	.02	-.01	-.01	-.04	-.04
41-50	-.08	-.01	-.16	-.00	-.01	.08	.07	.05	.07	.12

cally equivalent to maximum likelihood. (See the sixth section.) The fitted model

$$(1-0.747B^{12})\tilde{u}_t=(1-0.546B^{12})\hat{a}_t \tag{59}$$

was obtained.

As noted in the sixth section, the models for decomposing $\tilde{u}_t=\tilde{s}_t+n_t$ into seasonal and white-noise components are those for which $\tilde{s}_t$ is the seasonal ARMA process given by equation (49), and our principle of removing no more than necessary from the series implies taking the MA parameter in this model to be -1. In terms of the estimated coefficients in (59); therefore, the equation system (51) becomes

$$1.298\hat{\sigma}_a^2=2\hat{\sigma}_\epsilon^2+1.557\hat{\sigma}_n^2$$

$$0.546\sigma_a^2=-\hat{\sigma}_\epsilon^2+0.746\hat{\sigma}_n^2 \tag{60}$$

For the symmetric filter in (53), the ratio of variances of the $\tilde{s}$ and $\tilde{u}$ innovations is needed, for which the solution of (60) is

$$\frac{\hat{\sigma}_\epsilon^2}{\hat{\sigma}_a^2}=0.039$$

Thus the filter in (53) becomes

$$\nu(B)=0.039\frac{(1+B^{12})(1+F^{12})}{(1-0.547B^{12})(1-0.547F^{12})} \tag{61}$$

Writing this as

$$\nu(B)=0.039(B+2+F)\omega(B)$$

table 4 illustrates the calculation of the $\{\nu_k\}$, utilizing the fact that, for $\omega(B)=(1-\theta B)^{-1}(1-\theta F)^{-1}$,

$$\omega_o=\frac{1}{1-\theta^2};\ \omega|k|=\theta\omega|k-1|,\ k\neq 0$$

Table 4. **CALCULATION OF FILTER WEIGHTS FOR UNEMPLOYMENT STOCHASTIC SEASONAL COMPONENT**

K	ω_K	$0.039\omega_K$	ν_K	ν_K [1]
0.	1.424	0.0554	0.171	0.067
12.	(X)	.0303	.133	.062
24.	(X)	.0166	.073	.053
36.	(X)	.0090	.040	.046
48.	(X)	.0049	.022	.038
60.	(X)	.0027	.012	.033
72.	(X)	.0015	.006	.027

X Not applicable.

[1] Based on overall model ϕ=0.95, θ=0.85.

The resulting estimate of the stochastic seasonal component for $\Delta\log U_t$ is, therefore,

$$s_{2t}^*=0.171u_t^*+0.133(u_{t-12}^*+u_{t+12}^*)$$
$$+\ldots+0.006(u_{t-72}^*+u_{t+72}^*) \tag{62}$$

To obtain s_{2t}^* near the beginning and end of the sample period, the u_t^*-series was forecasted and backcasted 6 years according to the combined trend/seasonal model (58) and (59), a procedure that, incidentally, would probably benefit any moving-average seasonal adjustment procedure.[7]

The combined seasonal component

$$s_t^*=s_{1t}^*+s_{2t}^*=\hat{\delta}_t+s_{2t}^*$$

for the Δlog series is then obtained by adding $\hat{s}_{2t}^*$ in (62) to the appropriate deterministic coefficient. The seasonal adjustment procedure is completed by obtaining $\hat{s}_t$ for the log U_t series, as described in the subsection on stochastic detrending in the fourth section, the seasonal factors $\hat{S}_t = e^{s_t}$, and the seasonally adjusted series

$$U_t^{(SA)}=U_t/\hat{S}_t \tag{63}$$

However, before this procedure was carried out, it was noticed that the series $\hat{n}_t=\hat{u}_t^*-\hat{s}_{2t}^*$, which is the same as $\Delta\log U_t^{(SA)}$, contained a negative seasonal autocorrelation $r_{12}=-0.23$, reflecting an overadjustment of the series of the sort also done by X-11 for this series ($r_{12}=-0.21$ in part (c) of table 3). At least three explanations for this seem possible: (1) Almost half of the seasonal variation in u_t^* can be attributed to a sequence of high-positive June values, spanning from 1968 to 1971, so that a fixed-parameter ARMA model for u_t is perhaps inappropriate; (2) the properties of the estimated seasonal $\hat{s}_{2t}^*$ necessarily differ from those of the true seasonal component s_{2t}^* [16; 43]; and (3) the nonlinear least squares estimates (59) of the ARMA model may be suspect. We examine the third of these here, saving the first two for the tenth section.

One reason to suspect the fitted model (59) is that the autocorrelations r_{12}, r_{24}, ..., in table 4d, are extremely slow to die out, and, since the theoretical autocorrelations for an ARMA model of this type are

$$\rho_{12k}=\phi\rho_{12,k-1},\ k\geq 2$$

we should expect a value $\hat{\phi}$ much closer to 1 than the value 0.747. If, indeed, ϕ and θ are both quite close to 1 the finite-sample properties of the least squares procedure may be poor [32]. Thus, while alternative methods may be asymptotically inefficient, they are perhaps preferable here. We thus chose values of ϕ near 1 (ϕ=0.9, 0.95, 0.98), computed the resulting $\hat{\theta}$ required to give r_{12}=0.18, solved the equation system (51), and determined $\nu(B)$ in (53) and the resulting seasonal component series. The

[7] This also has recently been found by Dagum [8] for the case of X-11 applied to many Canadian time series.

value $\hat{\phi}=0.95$ was the lowest value for which this problem did not occur; for $\hat{\phi}=0.95$, the implied value of θ is 0.85. The set of equations

$$1.722\sigma_a^2=2\sigma_\epsilon^2+1.9025\sigma_n^2$$
$$0.85\sigma_a^2=-\sigma_\epsilon^2+0.95\sigma_n^2 \qquad (64)$$

gave $\hat{\sigma}_\epsilon^2/\hat{\sigma}_a^2=0.00506$. The resulting values of ν_k, previously computed, are displayed in the right-hand column of table 4. Beyond lag 72 (6 years), $\nu(B)$ was truncated.

Table 3e shows the autocorrelations of $\Delta\log u_t$ seasonally adjusted as before except using ν'_k in place of ν_k. They give no reason to doubt the adequacy of the seasonal adjustment. Table A-3 in the appendix shows the resulting seasonally adjusted unemployment series and the seasonal factors.

DEMAND DEPOSITS

The essential features of (and some of the problems with) the deterministic-stochastic seasonal adjustment procedure of this paper were illustrated with the unemployment rate. In analyzing the demand deposit component of the money supply (published in the *Federal Reserve Bulletin*), we will concentrate on two additional aspects of the procedure, the possible effects of sample-period choice and the simultaneous presence of stochastic quarterly and annual patterns.

In [30] and [40] the demand deposit series for periods since 1968 is analyzed, and it is found that the model (24), with $\mu_t^*=a_t$, i.e.,

$$\Delta log\ DD_t=\alpha+\sum\delta_j d_{jt}+a_t \qquad (65)$$

is an adequate characterization of the series; thus, not only is the assumption of fixed deterministic seasonality adequate, but also the trend is a pure random walk with drift. This is, in part, the basis for a recommendation in [46] that the daily seasonal adjustment procedure, set forth in [40], based on fixed factors apart from trading-day variation, be given consideration as an alternative to the Federal Reserve's current procedure (X–11 plus judgmental review).

However, over earlier periods, particularly the 1960's, there are known to have occurred shifts in tax dates and other events, and, indeed, the published seasonally adjusted series have reflected pronounced changes in the estimated seasonal factors. It is, therefore, of interest to examine how the present procedure performed from 1961 to 1975, including a period where changes are known to have occurred.

Table 5 shows the autocorrelations of various series related to demand deposits; note in part (a) of this table, that quarterly, as well as annual, patterns are evident. Moreover, after the regression on seasonal dummies both quarterly and annual patterns, while reduced, are still present. (See table 5d.) Since there is no further autocorrelation evident in this residual series, apart from these stochastic seasonal affects, the tests for stochastic seasonality can be made directly on this series (i.e., $\tilde{u}_t=\hat{u}_t^*$). We find

$$Q_3=168(0.22^2+0.11^2+0.11^2)=12.2$$
$$Q_{12}=168(0.31^2+0.10^2+0.00^2)=17.8$$

both highly significant.

There are perhaps several ways to approach the problem of decomposing u_t into seasonal and nonseasonal components, as the seasonal pattern is more complex than the ARMA model (48) that adequately represented the stochastic component of the unemployment rate. The method employed here is to treat the problem iteratively, i.e., to assume that the annual and quarterly effects are multiplicative (in the sense of [4]) so that the overall model for u_t^* can be written

$$q(B)a(B)u_t^*=a_t \qquad (66)$$

where $q(B)$ represents the quarterly pattern and $a(B)$ the annual pattern. In fact the usual identification procedure [4], based on the autocorrelations in Table 5d, suggests a model of the form (66) with

$$q(B)=(1-\theta_3B^3)^{-1}(1-\theta_3B^3)$$
$$a(B)=(1-\theta_{12}B^{12})^{-1}(1-\phi_{12}B^{12}) \qquad (67)$$

or a simplification thereof. Let

$$f_t=a(B)u_t^* \qquad (68)$$

so that

$$a_t=q(B)f_t \qquad (69)$$

and define

$$g_t=q(B)u_t^* \qquad (70)$$

whence also

$$a_t=a(B)g_t \qquad (71)$$

If the coefficient of B^k in $q(B)$ is essentially 0 for $k \geq 12$, then one can reasonably hope to estimate (66) by estimating certain of these equations separately. The seasonal decomposition of u_t^*, based on these models, requires further that the filter-preservation property be extended to include quarterly filters for an annual adjustment and vice versa. For example, if the series f_t, in (69), which possesses a quarterly but not an annual pattern, is represented as the sum of seasonal and noise components

$$f_t=f_t^{(q)}+f_t^{(n)} \qquad (72)$$

then the quarterly pattern in u_t^* is

$$u_t^{(q)}=a^{-1}(B)f_t^{(q)} \qquad (73)$$

Table 5. AUTOCORRELATIONS, DEMAND DEPOSITS: 1961 TO 1975

Lags	1	2	3	4	5	6	7	8	9	0
					a. Δlog (NSA) *(Variance = $0.0^{3}348$)*					
1-10	-0.15	-0.33	0.37	-0.27	-0.19	0.30	-0.22	-0.23	0.34	-0.36
11-20	-.13	.89	-.18	-.29	.35	-.27	-.18	.27	-.20	-.20
21-30	.31	-.34	-.10	.82	-.19	-.26	.33	-.26	-.17	.24
31-40	-.19	-.17	.29	-.32	-.08	.75	-.18	-.22	.31	-.24
41-50	-.15	.22	-.17	-.15	.25	-.30	-.06	.68	-.17	-.19
51-60	.23	-.24	-.13	.21	-.14	-.12	.22	-.27	-.04	.60
					b. $\Delta\Delta_{12}$ log (NSA) *(Variance = $0.0^{4}37$)*					
1-10	0.18	0.04	0.17	-0.03	0.09	0.07	-0.21	-0.07	0.01	-0.06
11-20	-.03	-.34	-.13	.11	-.02	.06	.12	-.00	.00	.01
21-30	-.10	-.15	.02	-.04	-.16	-.12	-.03	-.07	-.03	-.05
31-40	.00	.10	.08	.16	-.03	-.07	.14	.02	-.04	.06
41-50	-.10	-.07	.08	-.06	.02	.01	.06	.11	-.04	-.03
51-60	.03	-.04	-.02	.08	-.03	-.01	.05	-.08	.01	.08
					c. Δlog (X-11 SA) *(Variance = $0.0^{4}16$)*					
1-10	0.28	0.10	0.14	0.06	0.13	0.09	-0.13	0.03	-0.03	0.02
11-20	.10	-.02	-.05	.05	.02	.14	.13	.02	.00	.00
21-30	-.10	-.04	.02	-.08	-.15	-.10	-.04	-.03	-.06	-.07
31-40	-.00	.10	.05	.13	.00	-.06	.07	.06	.04	.02
41-50	-.17	-.07	.03	-.07	-.02	.04	.06	.15	.06	.08
51-60	.12	-.01	-.07	.06	.02	-.01	-.05	-.08	.02	.04
					d. Δlog (Regression SA) *(Variance = $0.0^{4}31$)*					
1-10	0.07	-0.07	0.22	-0.10	-0.01	0.11	-0.17	-0.04	0.11	-0.19
11-20	.09	.31	-.13	-.01	.09	-.04	.06	.02	-.08	.01
21-30	-.01	-.16	.13	.10	-.14	-.05	-.01	-.08	-.06	-.08
31-40	.01	.09	-.00	.00	.11	.00	.04	.03	.00	.02
41-50	-.13	-.10	.06	-.02	-.04	-.00	.12	.07	.03	.04
51-60	.06	-.01	-.07	.03	.05	.00	-.04	-.06	.05	-.02
					e. Δlog (Regression-Moving Average SA) *(Variance = $0.0^{4}21$)*					
1-10	0.20	-0.03	-0.07	-0.03	0.03	-0.12	-0.15	-0.06	-0.07	-0.07
11-20	.02	-.11	-.07	.01	-.00	.13	.08	.00	.05	-.04
21-30	-.07	-.11	.10	.09	-.12	-.13	-.01	-.06	-.07	-.04
31-40	.03	.12	.08	.07	.04	-.03	.08	.05	.01	.02
41-50	-.14	-.12	.03	-.06	-.06	-.03	.13	.09	-.00	.02
51-60	.05	-.03	-.07	.04	.06	-.01	-.01	-.06	.00	.01

since $a^{-1}(B)f_t=u_t^*$ from (68). Therefore, if $\nu_q(B)$ is the filter (12) or (54) for estimating $f_t^{(q)}$ in (72), based on a components model derived from (69), i.e., $\hat{f}_t^{(q)}=\nu_q(B)f_t$, then also

$$\hat{u}_t^{(q)}=\nu_q(B)u_t^* \tag{74}$$

The stochastic series adjusted for quarterly pattern is then

$$\breve{u}_t=[1-\nu_q(B)]u_t^* \tag{75}$$

Similarly, we may derive a common filter $\nu_a(B)$ to estimate the annual seasonal pattern either in u_t^* or g_t. The seasonal adjustment process then consists of removing the quarterly and annual effects in sequence. If

$$u_t^*=s_{2t}^*+n_t$$

then

$$\hat{n}_t=[1-\nu_a(B)]\breve{u}_t$$
$$=[1-\nu_a(B)][1-\nu_q(B)]u_t^* \tag{76}$$

and the estimated seasonal component itself is

$$\hat{s}_{2t}^*=\hat{u}_t^*-\hat{n}_t=[\nu_a(B)+\nu_q(B)-\nu_a(B)\nu_q(B)]\hat{u}_t^* \tag{77}$$

The multiplicative model (66), thus, implies that the seasonal component is the sum of the quarterly and annual effects less an interaction effect (and, hence, the seasonal factor does not itself factor into annual and quarterly factors).

To apply this adjustment procedure to the residual from the demand deposit regression, estimating (66) gave

$$\hat{q}(B)=\frac{1-0.836B^3}{1-0.674B^3},\ \hat{a}(B)=(1+0.247B^{12})^{-1}$$

the pure MA specification being adopted for the annual pattern after $\hat{\phi}_{12}=0.036$ was computed in the mixed ARMA specification. The weights for the quarterly adjustment, i.e., the coefficients in $\nu_q(B)$, are derived from (69), just as the coefficients in $\nu(B)$ were derived in the eighth section for the unemployment stochastic adjustment, and the coefficients in $\nu_a(B)$ are derived from (71), also in this manner. These coefficients are displayed in table 6. When this adjustment was applied to the u_t^* series, however, the quarterly pattern appeared to be satisfactorily handled but not the annual pattern; the lag-12 autocorrelation of (76) was negative, and the lag-24 autocorrelation was positive and higher than before (evidently related to the fact that the coefficient $\nu_{24}^{(a)}$ is negative in table 6). The latter problem was handled in a purely ad hoc manner, setting $\nu_{24}^{(a)}=0$. The former problem is similar to that encountered in the unemployment series, and, also as a largely ad hoc procedure, the coefficient ν_0 was set equal to $\nu_{\pm12}$, thus, obtaining a 3 x 1 filter. (Note that most seasonal extraction filters used by X–11, e.g. the 3x5 filter, are such that the center weight is equal to several of the adjacent weights on either side.)

The amended $\nu^{(a)}$-weights, denoted ν_a', are shown in the righthand column of table 6, and the autocorrelations of the resulting $\hat{n}_t$ series (Δlog of the seasonally adjusted series) are in table 5e. The seasonally adjusted demand deposit series is given in table A-4 along with the seasonal factors both for the deterministic-stochastic adjustment from 1961 to 1975 and for the deterministic only (fixed-factor) adjustment over the more recent period.

Table 6. **FILTER WEIGHTS FOR DEMAND DEPOSIT STOCHASTIC COMPONENT**

Quarterly		Annual		
κ	$\nu_\kappa^{(q)}$	κ	$\nu_\kappa^{(a)}$	$\nu_\kappa^{(a)1}$
0......	0.128	-	0.396	0.149
3......	.108	12	.149	.149
6......	.074	24	.037	-
9......	.050	(X)	(X)	(X)
12.....	.033	(X)	(X)	(X)
15.....	.021	(X)	(X)	(X)
18.....	.013	(X)	(X)	(X)

- Entry represents zero.
X Not applicable.

DISCUSSION AND CONCLUSIONS

This paper has presented and illustrated a seasonal adjustment procedure with the following features vis-à-vis procedures currently in use:

1. It can allow for stochastic effects in regression (deterministic) procedures, and for deterministic effects in ratio-to-moving average (stochastic) procedures.
2. The symmetric filters or moving averages can be chosen in accordance with the stochastic properties of the individual series to be adjusted.
3. The detrending procedure recognizes the stochastic as well as the deterministic nature of trend, and preserves seasonal adjustment under such stochastic-detrending transformations, as series differencing.

This has been done while staying within the empirical or descriptive framework, recognizing that while structural or causal approaches undoubtedly are potentially more effective, there will remain many series for which, given the present state of knowledge and consensus concerning the required causal or structural system, univariate and unstructured procedures probably will continue to be used. It, thus, behooves us to continue the search for the best procedures that current statistical methodology can produce, the current paper being but one attempt at this.

In the remainder of the paper, we wish to cover some problems with the current procedure and some suggested improvements or extensions.
to consider

An Alternative to Overadjustment

Both X–11 and the stochastic seasonal adjustment of this paper have tended to produce adjusted series with negative autocorrelation at the seasonal lags or spectral dips at the seasonal frequencies. Both this and the failure of the orthogonality property are necessary consequences of the seasonal adjustment model and procedure employed. (See Grether and Nerlove [16].) Intuitively, one cannot hope to transform n independent random variables $\{a_t\}$ into $2n$ independent rv's $\{\epsilon_t\}$, $\{n_t\}$. Yet, what this state of affairs implies is that, e.g., a lower-than-expected unemployment rate this September will tend to be followed by a higher-than-(otherwise) expected value next September; this is precisely the sort of effect that seasonal adjustment is intended to eliminate.

Given that the procedures employed are optimal, with respect to the class of additive, stochastic, or unobserved-components models, perhaps alternatives to the overadjustment phenomenon should be sought through alternative models, i.e., alternative definitions of the stochastic seasonal component. Notice, e.g., concerning the present definition, that the innovations of s_t, i.e., the white-noise series ϵ_t in (49), as irregular a series as exists, is currently considered seasonal. This is related to the prominence given to the center weight of moving average filters of the form (12).

Having transformed the series (via estimating deterministic components and stochastic detrending) to the form (30), i.e., to a series with no autocorrelation other than seasonal, an alternative way to approach the decomposition problem would be to regard the seasonal component at time t as the part of the series that could have been predicted (and postdicted), given other series values; i.e., to consider

$$\hat{s}'_t=E(u_t|u_{t-1},\ u_{t-2},\ \ldots,\ u_{t+1},\ \ldots)$$

as an alternative to $\hat{s}_t$. This explicitly eliminates the contribution of the current value of u_t to the seasonal component, the rationale being that only that part of a series which is related to values of the series at seasonal distances (e.g., a year, a quarter ago, etc.) is to be considered as seasonal. For example, if u_t were generated by

$$u_t=\phi u_{t-12}+a_t$$

then

$$\hat{s}'_t=\frac{\phi(u_{t-12}+u_{t+12})}{1+\phi^2}$$

Moreover, $\hat{s}'_t$ and $u_t-\hat{s}'_t$ are now orthogonal; and the center weight is 0, so that pure noise (ϵ_t) is not part of the seasonal component. Additionally, the component $\hat{s}'_t$ will generally have smaller variance than $\hat{s}_t$, in line with our minimum extraction principle. It is hoped to explore similar approaches with a zero-center weight more fully in the future.

Alternative Stochastic Models

Throughout, it has been assumed that the stochastic component of the series is, after differencing, a stationary linear process. Several alternatives to this are possible; e.g., Hannan et al. [18] and others have considered seasonal models, where the coefficients are stochastic rather than fixed; Swamy and Tinsley [38] have presented a class of such models of which the ARIMA ones are a very special case.

Another generalization of stationarity concerns the class of periodically correlated processes, in which, e.g., marginally, the December values are stationary over time but other months might have different autocovariances than the Decembers. We noticed in the eighth section that the unemployment series was strongly affected by a series of Junes; similarly, the lag 12 autocorrelation in a series, such as retail sales, may be dominated by December, and, perhaps, the symmetric filter applied to other months should be different. In other words, our basic definition of (stochastic) seasonality has been connected with the lag 12 autocorrelation, which measures year over year association averaged over the 12 months and assumes, as a consequence of both within- and between-year stationarity, that the true year-over-year association is the same for each month. When this is not the case, the stochastic models of this paper may not adequately capture the seasonality.

Standard Errors of Seasonal Factors and Adjusted Data

An important subject in seasonal adjustment concerns the accuracy of the seasonally adjusted data, given that the true seasonal is never known. There are two sources of error, one from specification and estimation of the model and the other (relevant only for stochastic seasonality) from estimation of the seasonal component, given the model. The mean square error for some simple examples of the latter is given in Whittle [43]. We have not investigated the analogous situation for stochastic seasonal models, such as employed in this paper, that would be complicated further by parameter estimation error. However, some idea of the magnitudes involved may be obtained from a consideration of the simplest case, where only deterministic seasonality is present and the regression residual is white noise, the situation evidently exhibited, e.g., by demand deposits for the post-1968 period [29; 30]. For this case, the variance of the seasonal component, $\hat{\delta}_j$, in (24), is approximately σ_a^2/M, where σ_a^2 is the residual variance $(u_t^*=a_t)$ and M is the number of years in the sample. As the seasonally adjusted $\Delta\log$ series is $(y_t-\hat{\delta}_j)$, σ_a^2/M is also the MSE of a seasonally adjusted series value. Noting that $y_t=\Delta\log x_t$ is essentially the rate of change of the observed series x_t, we may derive corresponding MSE's for the seasonal factors and the seasonally adjusted series.

To illustrate, the $\Delta\log$ demand deposit regression for the sample period, 1969–75 ($M=7$), gave $\hat{\sigma}_a=0.0051$ [29]; thus, the root mean square error of a given seasonal factor is approximately

$$\frac{\sigma_a}{\sqrt{7}}=0.002$$

This translates into a RMSE for the seasonally adjusted demand deposit figure for a given month of over $400 million, since 1973 demand deposits have been over $200 billion. It is likely that a lower RMSE would be obtained for a longer series or for a procedure (if appropriate to the series) that does more averaging (smoothing); but even halving this interval would still give 95-percent confidence intervals of about $1 billion for current monthly seasonally adjusted demand deposit data. Such intervals would, for most months, contain all three seasonally adjusted figures: (1) The deterministic adjustment from 1969 to 1975, (2) the deterministic-stochastic adjustment discussed in the ninth section, and (3) the published seasonal adjustment. This situation of uncertainty is evidently characteristic of numerous series, as mentioned at the end of the seventh section, and more attention to the standard errors of estimated seasonal components and seasonally adjusted data is needed.

Incorporating Special Effects or Other Procedures

A limitation in the procedure as developed and exemplified here is that little provision was made for phenomena, such as outliers, trading-day variation, etc., whose the importance is widely recognized. The trading-day regression can probably be combined with regression on monthly dummy variables, the definition of the deterministic seasonal component being extended to include both effects. Special effects, such as outliers, strikes, changing holiday dates, etc., can be allowed for independently of the operation of the aspects for the procedure discussed.

Other forms of prior knowledge, e.g., policy effects on seasonality or even structural or causal seasonal models involving relationships with other variables, can also be employed a priori or a posteriori, and the procedure herein described applied to the series after or before such modification. Such approaches would be analogous either to fitting autoregressive or ARIMA models to residuals from structural econometric models or to building structural models, based on interrelating ARIMA residuals, both of which are frequently recommended [15; 19; 28], or, more generally, to synthesizing structural and empirical approaches [31; 32; 41; 46].

REFERENCES

1. Anderson, T. W. *An Introduction to the Statistical Analysis of Time Series*. New York: Wiley, 1971.

2. Box, George E. P., and Cox, David R. "An Analysis of Transformations." *Journal of the Royal Statistical Society*, ser. B, 26 (1964): 211–252.

3. ———, Hillmer, S. C., and Tiao, G. C. "Analysis and Modeling of Seasonal Time Series."
   Included in this report.

4. ——— and Jenkins, Gwilym M. *Time Series Analysis, Forecasting and Control*. (Rev. ed., 1976.) San Francisco: Holden-Day, 1970.

5. Cleveland, William P. *Analysis and Forecasting of Seasonal Time Series*. Ph.D. dissertation, Department of Statistics, University of Wisconsin, 1972.

6. ——— and Tiao, George C. "Decomposition of Seasonal Time Series: A Model for the Census X–11 Program." *Journal of the American Statistical Association* 71 (1976): 581–587.

7. Cleveland, William S., Dunn, D. M., and Terpenning, J. J. "Graphical Analysis of Seasonality and Nonlinear Seasonal Adjustment."
   Included in this report.

8. Dagum, Estela B. "Seasonal Factor Forecasts from ARIMA Models." Proceedings of the 40th Session of the International Statistical Institute, Warsaw: 1975, pp. 206–219.

9. Dickey, David A. "Hypothesis Testing for Nonstationary Time Series." Iowa City: Department of Statistics, Iowa State Univ., 1975. (Unpublished manuscript.)

10. Durbin, James, and Murphy, M. J. "Seasonal Adjustment Based on a Mixed Additive-Multiplicative Model." *Journal of the Royal Statistical Society*, ser. A, 138 (1975): 385–410.

11. Engle, Robert F. "Estimation of Structural Models for Seasonality."
    Included in this report.

12. Fisher, Franklin M. *The Identification Problem in Econometrics*. New York: McGraw Hill, 1966.

13. Fuller, Wayne A. *Introduction to Statistical Time Series*. New York: John Wiley and Sons, Inc., 1976.

14. Granger, Clive W. J. "Seasonality: Causation, Interpretation, and Implications."
    Included in this report.

15. ——— and Newbold, Paul. "The Time Series Approach to Econometric Model Building." Paper presented at the Conference for New Methods in Business Cycle Research, Minneapolis, Minn., Nov. 1–2, 1975.

16. Grether, David M., and Nerlove, Marc. "Some Properties of 'Optimal' Seasonal Adjustment." *Econometrica* 38 (1970): 682–703.

17. Hannan, Edward J. *Multiple Time Series*. New York: Wiley, 1970.

18. ———, Terrell, R. D., and Tuckwell, N. W. "The Seasonal Adjustment of Economic Time Series." *International Economic Review* 11 (1970): 24–52.

19. Haugh, Larry D. "The Identification of Time Series Interrelationships with Special Reference to Dynamic Regression." Ph.D. dissertation, Department of Statistics, University of Wisconsin, 1972.

20. Jorgenson, Dale W. "Minimum Variance, Linear, Unbiased Seasonal Adjustment of Economic Time Series." *Journal of the American Statistical Association* 59 (1964): 681–687.

21. Kaufman, Herbert M., and Lombra, Raymond E. "Toward a Structural Model of Money Stock Seasonality." Paper presented at the Winter meeting of the Econometric Society, New York, Dec. 30, 1973.

22. Lovell, Michael C. "Seasonal Adjustment of Economic Time Series and Multiple Regression Analysis." *Journal of the American Statistical Association* 58 (1963): 993–1010.

23. Maravall, Agustin. "Estimation of the Permanent and Transitory Components of an Economic Variable with an Application to M1." Paper presented at the Econometric Society Annual Meeting, Atlantic City, N.J., Sept. 16–18, 1976.

24. Nerlove, Marc. "Spectral Analysis of Seasonal Adjustment Procedures." *Econometrica* 32 (1964): 241–286.

25. Pagano, Marcello. "Estimation of Models of Autoregressive Signal Plus White Noise." *Annals of Statistics* 2 (1974): 99–108.

26. Parzen, Emanuel. "Some Recent Advances in Time Series Modelling." *IEEE Transactions on Automatic Control* AC-19 (1974): 723–730.

27. Pierce, David A. "Distribution of Residual Autocorrelations in the Regression Model with ARMA Errors." *Journal of the Royal Statistical Society,* ser. B, 33 (1971): 140–146.

28. ———. "Relationships—and the Lack Thereof—Between Economic Time Series, with Special Reference to Money, Reserves, and Interest Rates." *Journal of the American Statistical Association* 72 (1977).

29. ———. "Uncertainty in Seasonal Adjustment Procedures," *Proceedings of the Business and Economic Statistics Section of the American Statistical Association*. Washington, D.C.: American Statistical Association, 1976: 528–533.

30. ——— and Porter, Richard D. "Linear Models and Linear Filters in the Analysis of Seasonal Time Series," *Proceedings of the Business and Economic Statistics Section of the American Statistical Association*. Washington, D.C.: American Statistical Association, 1973, pp. 537–542.

31. Plosser, Charles. "Time Series Analysis, Seasonality in Econometric Models."
Included in this report.

32. Prothero, David L., and Wallis, Kenneth F. "Modelling Macroeconomic Time Series." *Journal of the Royal Statistical Society,* ser. A, 139 (1976).

33. Quenouille, M. H. *The Analysis of Multiple Time Series*. London: Charles Griffin and Co., Ltd., 1957.

34. Slutsky, Eugen. "The Summation of Random Causes as the Source of Cyclic Processes." *Econometrica* 5 (1937): 105–146.

35. Stephenson, James A., and Farr, Helen T. "Seasonal Adjustment of Economic Data by Application of the General Linear Statistical Model." *Journal of the American Statistical Association* 67 (1972): 37–45.

36. U.S. Department of Commerce, Bureau of the Census. "The X-11 Variant of the Census Method II Seasonal Adjustment Program," by Julius Shiskin, Allan H. Young, and John C. Musgrave. Technical Paper 15. Washington, D.C.: Government Printing Office, 1967.

37. U.S. Federal Reserve System, Board of Governors of the Federal Reserve System. *Improving the Monetary Aggregates: Report of the Advisory Committee on Monetary Statistics*. Washington, D.C.: Government Printing Office, March 1976.

38. ———. *Linear Prediction and Estimation Methods for Regression Models with Stationary Stochastic Coefficients,* by P. A. V. B. Swamy and Peter A. Tinsley. Special Studies Paper No. 78. Washington, D.C.: Government Printing Office, March 1976.

39. ———. *Multiple Time Series Containing Unobserved Components,* by Richard D. Porter. Special Studies Paper No. 65. Washington, D.C.: Government Printing Office, September 1975.

40. ———. "Seasonal Adjustment of the Monetary Aggregates," *Improving the Monetary Aggregates: Staff Papers,* by David A. Pierce, Neva Van Peski, and Edward R. Fry. Washington, D.C.: Government Printing Office, 1976.

41. Wallis, Kenneth F. "Seasonal Adjustment and Multiple Time Series Analysis."
Included in this report.

42. Weiner, Norbert. *Extrapolation, Interpolation and Smoothing of Stationary Time Series*. New York: Wiley, 1949.

43. Whittle, Peter. *Prediction and Regulation by Linear Least Squares Methods*. London: English Universities Press, 1963.

44. Yalgom, A. M. *An Introduction to the Theory of Stationary Randoms Processes*. New York: Prentice Hall, 1962.

45. Yule, G. U. "On a Method of Investigating Periodicities in Disturbed Series, with Special Reference to Wolfer's Sunspot Numbers." *Philosophical Transactions*. (A) 226 (1927): 267–298.

46. Zellner, A., and Palm, F. "Time Series Analysis and Simultaneous Equation Econometric Models." *Journal of Econometrics* 2 (1974): 17–54.

APPENDIX

Table A-1. CONSUMER PRICE INDEX

Year	January	February	March	April	May	June	July	August	September	October	November	December
	Seasonal factors from published series											
1948	1.000846	0.993368	0.997443	0.997898	0.996664	1.004560	0.999726	1.002732	1.005755	1.003707	1.002901	1.001389
1949	1.002925	.993720	.997068	.997767	.996928	.997767	1.000141	1.002958	1.004778	1.003245	1.002958	1.001414
1950	1.001136	.994061	.997034	.997742	.997331	.998183	1.000555	1.002897	1.004253	1.003408	1.002578	1.000936
1951	1.001052	.994832	.997162	.997294	.998073	.998842	1.001160	1.003098	1.003336	1.002551	1.002411	1.000252
1952	1.000757	.995452	.997342	.997729	.998613	.999245	1.001252	1.002252	1.002758	1.002127	1.002252	1.000250
1953	1.000376	.996486	.997619	.997497	.998750	.999626	1.002244	1.002238	1.001738	1.002106	1.001616	.999876
1954	1.000124	.997155	.997645	.997639	.998761	1.000000	1.002609	1.001740	1.001370	1.001749	1.001497	.999376
1955	.999501	.997385	.997882	.997882	.998753	1.000625	1.003119	1.001749	1.001244	1.001493	1.001242	.999254
1956	.999129	.997763	.998262	.998140	.998889	.988687	1.003058	1.001223	1.001221	1.001457	1.000849	.999034
1957	.998914	.998198	.998322	.998686	.999046	1.001069	1.002842	1.000826	1.000825	1.001179	1.001058	.999179
1958	.998835	.998255	.988498	.999192	.998962	1.000808	1.002657	1.000462	1.000808	1.001270	1.001038	.999308
1959	.998849	.998388	.998733	.999539	.999080	1.000688	1.002176	1.000343	1.000799	1.001252	1.000796	.999319
1960	.998977	.998638	.998865	.999774	.998984	1.000451	1.001807	1.000338	1.001015	1.001010	1.000672	.999552
1961	.999217	.998546	.998993	.999776	.998993	1.000224	1.001561	1.000000	1.000891	1.001114	1.000668	.999889
1962	.999333	.998781	.998673	1.000000	.998896	1.000442	1.001767	.999890	1.000878	1.001099	1.000549	1.000000
1963	.999013	.999014	.998906	.999781	.999015	1.000218	1.001631	1.000217	1.000869	1.000977	1.000542	1.000000
1964	.999029	.999028	.999137	.999569	.998923	1.000000	1.001722	1.000538	1.000967	1.000751	1.000428	1.000000
1965	.999039	.998933	.999147	.999575	.999046	1.000317	1.001690	1.000635	1.000739	1.000738	1.000316	.999895
1966	.998848	.998648	.999274	.999793	.998968	1.000618	1.001748	1.000818	1.000408	1.000508	1.000203	1.000101
1967	.998987	.997978	1.000000	1.000000	1.001007	1.000000	1.002000	1.000996	1.000000	1.000000	.999014	.999017
1968	.999021	.999023	1.000000	1.000971	1.000000	1.000963	1.000958	1.000000	1.000000	1.000000	.999058	.999061
1969	.999064	.998136	1.000000	1.000921	1.000000	1.000912	1.000908	1.000904	1.000900	1.000000	1.000000	1.000000
1970	.998238	.998247	.999127	1.000000	.999137	1.001723	1.001717	1.000000	1.000852	1.000847	.999157	1.000000
1971	.998325	.999163	1.000000	1.000000	1.000000	1.000824	1.000822	1.000820	1.000819	1.000818	1.000000	1.000000
1972	.998379	.999193	.999194	1.000000	1.000000	1.000801	1.000797	1.000000	1.000793	1.000790	1.000000	1.000000
1973	.998436	.999223	1.000000	.999236	.999240	1.000756	1.000754	1.001483	1.000739	1.001466	1.000727	1.000000
1974	.997145	.998589	1.000000	.999306	.999313	1.000681	1.000676	1.000668	1.003639	1.001309	1.000648	1.000000
1975	.997444	.999364	.999367	.999370	.998746	1.000000	1.000616	1.000000	1.000612	1.000608	1.000604	(NA)
	Fixed seasonal factors											
	0.999749	0.997688	0.998164	0.998121	0.997624	0.998602	1.001482	1.001575	1.002269	1.002054	1.001769	1.000922
	Fixed-factor seasonally adjusted series											
1970	113.33	114.16	114.71	115.42	115.98	116.46	116.53	116.72	117.23	117.86	118.29	118.99
1971	119.23	119.68	120.02	120.43	121.09	121.67	121.62	121.91	121.92	122.15	122.38	122.99
1972	123.23	124.09	124.23	124.53	125.00	125.18	125.31	125.50	125.91	126.34	126.68	127.18
1973	127.73	128.90	130.04	130.95	131.81	132.59	132.50	134.89	135.19	136.32	137.36	138.37
1974	139.74	141.83	143.36	144.17	145.85	147.11	147.78	149.66	151.36	152.69	154.03	155.26
1975	156.14	157.56	158.09	158.90	159.68	160.82	162.06	162.54	163.23	164.26	165.31	166.15

NA Not available.

Table A-2. HOUSING STARTS

Year	January	February	March	April	May	June	July	August	September	October	November	December
	Deterministic (fixed-factor) seasonal adjustment											
1960	1462	1505	1072	1217	1247	1219	1188	1360	1112	1211	1200	1032
1961	1197	1229	1235	1116	1223	1338	1316	1327	1439	1369	1331	1321
1962	1380	1279	1377	1463	1487	1358	1418	1542	1311	1483	1563	1509
1963	1343	1487	1479	1625	1655	1530	1572	1503	1649	1814	1542	1527
1964	1664	1681	1530	1456	1464	1560	1461	1443	1375	1549	1430	1494
1965	1389	1342	1421	1471	1469	1506	1445	1339	1430	1460	1428	1625
1966	1349	1264	1399	1394	1246	1197	1031	1062	1022	840	941	967
1967	1004	1019	1084	1125	1265	1244	1302	1330	1399	1486	1531	1287
1968	1368	1404	1500	1603	1350	1368	1453	1426	1540	1545	1643	1549
1969	1725	1495	1563	1573	1490	1462	1301	1304	1484	1354	1223	1351
1970	1129	1233	1359	1271	1198	1342	1463	1343	1502	1546	1640	1951
1971	1880	1696	1989	1990	1902	1923	2019	2134	1994	1972	2245	2444
1972	2534	2525	2416	2094	2163	2214	2146	2386	2329	2375	2400	2419
1973	2492	2290	2370	2029	2242	2010	2105	2058	1703	1614	1723	1453
1974	1436	1815	1479	1578	1428	1465	1316	1160	1128	1061	971	885
1975	953	908	950	969	1112	1095	1240	1224	1284	1356	1253	1231
	Seasonal factors											
	0.706054	0.723233	1.012742	1.212646	1.252494	1.209259	1.154772	1.149777	1.045895	1.093693	0.928308	0.746742
	Seasonal factors from published series											
1959	0.696681	0.712657	0.945926	1.138113	1.221629	1.180040	1.148804	1.159720	1.062857	1.062731	0.887288	0.716552
1960	.706849	.724152	.979261	1.145074	1.229268	1.181716	1.145865	1.163393	1.059982	1.063242	.893740	.724741
1961	.714117	.725285	.953049	1.160892	1.246905	1.170477	1.137977	1.162500	1.053044	1.058374	.892419	.722637
1962	.715944	.723944	.966320	1.163779	1.255833	1.169231	1.129655	1.168359	1.035952	1.058317	.894451	.720460
1963	.762058	.738461	.976271	1.166607	1.262888	1.165239	1.124907	1.054301	.978105	1.115008	.882614	.764588
1964	.732876	.667912	1.021226	1.219061	1.249898	1.217032	1.080154	1.057744	.988041	1.111811	.893136	.752022
1965	.720353	.677460	1.011103	1.240056	1.244655	1.224193	1.091694	1.075140	1.008907	1.100000	.908219	.732609
1966	.695474	.663570	1.016643	1.250592	1.233202	1.212060	1.096133	1.091689	1.022180	1.090391	.909053	.729697
1967	.664667	.656100	1.039773	1.250596	1.214724	1.205769	1.102346	1.086567	1.029416	1.089738	.923797	.734862
1968	.700000	.667895	1.036289	1.250965	1.200852	1.177794	1.109524	1.096455	1.034961	1.076864	.935705	.747287
1969	.688525	.634135	1.013965	1.251969	1.178774	1.156806	1.098246	1.103682	1.029595	1.072266	.923678	.760512
1970	.734378	.683218	1.043518	1.218987	1.162791	1.171408	1.113777	1.103931	1.023990	1.070127	.924590	.769572
1971	.726039	.704423	1.054869	1.215106	1.162518	1.147878	1.119347	1.137164	1.021852	1.013346	.955270	.795294
1972	.717402	.764184	1.048329	1.129035	1.219991	1.187755	1.100355	1.151637	.981862	1.045473	.920446	.763314
1973	.709069	.723460	1.014799	1.180422	1.239188	1.176197	1.145172	1.153779	.950267	1.052594	.927842	.710878
1974	.697866	.735874	.964327	1.218332	1.263604	1.160681	1.177674	1.164367	.999661	1.054909	.876654	.703404
1975	.669851	.688772	.976065	1.196334	1.284055	1.225555	1.186081	1.113607	1.029755	1.036478	.841998	.712006

Table A-3. UNEMPLOYMENT RATE

Year	January	February	March	April	May	June	July	August	September	October	November	December
	Series adjusted by deterministic-stochastic method											
1950	0.065840	0.066266	0.063920	0.059392	0.056078	0.051727	0.050551	0.044081	0.043929	0.038612	0.040798	0.041286
1951	.037300	.035234	.034049	.031557	.030409	.031645	.031788	.030902	.033614	.033166	.034445	.030481
1952	.031546	.032110	.029588	.029733	.030439	.029933	.032137	.032926	.030633	.028261	.027495	.026172
1953	.029131	.026425	.026154	.027846	.025979	.025244	.026070	.025635	.028696	.029665	.034531	.044379
1954	.048618	.052963	.057097	.059770	.060732	.053417	.056035	.058401	.059997	.054716	.052701	.050093
1955	.049497	.047592	.046532	.048586	.043997	.041077	.039404	.041351	.039998	.041257	.041377	.041405
1956	.039934	.040040	.042226	.040635	.044252	.043473	.043226	.039578	.038349	.037419	.041852	.041862
1957	.041828	.039616	.038194	.039261	.041665	.042499	.040842	.040402	.042158	.043619	.049970	.051907
1958	.057623	.064954	.068859	.074418	.075082	.070006	.072979	.072314	.067965	.065649	.060367	.063092
1959	.059852	.059481	.056865	.051895	.051502	.049914	.051104	.052067	.052919	.055560	.057108	.053618
1960	.052566	.048788	.055032	.051944	.051315	.053329	.053859	.056083	.053458	.059552	.060601	.067076
1961	.066575	.069434	.069709	.070069	.070606	.066579	.068005	.066255	.065003	.064057	.060025	.060675
1962	.058207	.055324	.056181	.055820	.054458	.052717	.052373	.056727	.054584	.053029	.056283	.056012
1963	.057691	.059436	.057266	.056789	.059221	.055603	.055232	.054557	.053643	.054161	.057018	.055665
1964	.055718	.053622	.053754	.053440	.051995	.052963	.048075	.050401	.049378	.050053	.048551	.049475
1965	.047925	.049557	.046631	.048461	.047060	.047194	.043596	.043856	.042071	.041247	.041170	.039485
1966	.038878	.036660	.037360	.037424	.040515	.040858	.037960	.038106	.036061	.036650	.035967	.037008
1967	.037181	.036662	.036399	.036001	.035490	.040454	.039363	.038821	.040387	.042623	.039652	.037203
1968	.035668	.037026	.035276	.033129	.032554	.039427	.038213	.036088	.035707	.035784	.034621	.032589
1969	.032586	.032202	.032328	.032960	.031987	.036457	.036886	.036338	.039157	.039117	.035492	.034479
1970	.037416	.040708	.042602	.044700	.045845	.049235	.051035	.052383	.055575	.057469	.058950	.059548
1971	.058038	.057337	.058324	.058362	.058525	.057558	.059461	.061708	.061425	.060698	.060368	.058990
1972	.056917	.055563	.056799	.056704	.056147	.055176	.056085	.057506	.057383	.057735	.052288	.050328
1973	.048057	.048453	.047933	.049040	.047987	.048250	.048188	.049005	.050019	.047356	.048088	.048022
1974	.049328	.049636	.049050	.049231	.050765	.052528	.054440	.055603	.061062	.062383	.066196	.070771
	Seasonal factors											
1950	1.160403	1.185470	1.105305	1.003762	0.938764	1.080273	1.038665	0.932601	0.903523	0.847970	0.921477	0.944760
1951	1.167780	1.187879	1.105957	1.004387	.938970	1.077861	1.033988	.928644	.900694	.844017	.925051	.949628
1952	1.170689	1.191278	1.110791	1.011157	.943755	1.077211	1.032114	.927734	.896945	.838293	.920397	.947646
1953	1.175393	1.194252	1.114704	1.013219	.944537	1.073731	1.026553	.924719	.893443	.836137	.922484	.950712
1954	1.177447	1.192905	1.119508	1.015207	.945874	1.073743	1.022633	.923660	.888711	.834611	.923318	.953476
1955	1.176168	1.192151	1.122046	1.016519	.948354	1.075877	1.021115	.923448	.885471	.833472	.923378	.953692
1956	1.179105	1.190525	1.123032	1.014896	.946243	1.077189	1.016310	.922737	.883275	.831982	.928119	.958702
1957	1.175944	1.187145	1.120225	1.011685	.946069	1.079329	1.015173	.925760	.882574	.836265	.930441	.959091
1958	1.174275	1.186537	1.122011	1.012920	.947387	1.083242	1.013752	.928348	.880320	.837203	.926481	.957575
1959	1.171272	1.183930	1.121593	1.011256	.944940	1.086465	1.010664	.928344	.879992	.841321	.929290	.959385
1960	1.160970	1.175530	1.116474	1.005545	.944726	1.094763	1.013942	.935723	.883259	.847262	.930364	.955322
1961	1.159154	1.171270	1.109408	.998593	.938355	1.101235	1.015006	.937671	.885663	.853597	.934085	.957299
1962	1.153317	1.167871	1.103293	.991194	.933497	1.110444	1.019982	.941483	.890558	.857937	.935415	.954269
1963	1.149123	1.164055	1.096042	.987086	.927005	1.115865	1.021702	.945328	.897968	.866043	.935481	.950697
1964	1.142300	1.160970	1.092856	.985022	.923415	1.122519	1.025879	.949188	.903585	.871116	.933480	.944497

Table A-3. UNEMPLOYMENT RATE—Continued

Year	January	February	March	April	May	June	July	August	September	October	November	December
	Seasonal factors—Continued											
1965. . . .	1.139611	1.155424	1.085403	0.977565	0.916574	1.126146	1.027863	0.951323	0.909795	0.877793	0.938906	0.945763
1966. . . .	1.133528	1.149145	1.080338	.975282	.914154	1.132493	1.034006	.955394	.916289	.882543	.936418	.940814
1967. . . .	1.128032	1.147420	1.074721	.972275	.910155	1.134921	1.038648	.957923	.924592	.886426	.937167	.936660
1968. . . .	1.128849	1.146961	1.072102	.968520	.904631	1.133522	1.039771	.957728	.929531	.889294	.939648	.937811
1969. . . .	1.127785	1.147515	1.071980	.968669	.903722	1.132047	1.041895	.957153	.932795	.890712	.938051	.935833
1970. . . .	1.127759	1.146623	1.072375	.969527	.902763	1.128018	1.041878	.957746	.935586	.891002	.937653	.936473
1971. . . .	1.128626	1.147654	1.073134	.970225	.903624	1.122386	1.042170	.957226	.936532	.889596	.938161	.937640
1972. . . .	1.131847	1.148921	1.075191	.970822	.904161	1.116803	1.040837	.955649	.936521	.888104	.938121	.940054
1973. . . .	1.135120	1.153579	1.077929	.973028	.904162	1.111077	1.038548	.952954	.935534	.885303	.938608	.942042
1974. . . .	1.139484	1.157877	1.081532	.976201	.907736	1.106692	1.035872	.950189	.931632	.879903	.937479	.944729
	Fixed seasonal factors (deterministic adjustment only)											
	1.152155	1.169975	1.097798	0.991966	0.927301	1.100845	1.029562	0.941043	0.906681	0.860950	0.931498	0.947262

Table A-4. DEMAND DEPOSIT COMPONENT OF M1

Year	January	February	March	April	May	June	July	August	September	October	November	December
	Series adjusted by deterministic-stochastic method											
1962	118966	120334	119747	119577	120271	119179	118875	119414	119135	119722	120191	120172
1963	120732	121893	121326	121461	122496	121787	122393	123071	123047	123813	124701	123932
1964	124502	125485	124937	125047	125685	125366	126422	127483	128223	129009	129345	129474
1965	130086	130426	130289	130595	130347	130623	131289	131750	132806	134039	134355	135171
1966	136548	136918	137227	138463	137740	137914	136674	136785	137873	137452	137243	137634
1967	137613	138431	139755	139206	140386	141710	142902	144123	145110	146238	146558	147219
1968	148059	148338	148953	149962	151173	152350	153229	154090	155081	156228	157562	158878
1969	159904	160314	160813	161514	161419	161948	162331	161776	162165	162676	162635	162785
1970	164758	163103	164408	165593	165432	165656	166084	167530	168912	169366	169813	170829
1971	171416	172640	174152	175439	177259	178755	179642	180023	180196	180422	180807	181533
1972	182500	183814	186319	187830	187770	188672	190869	192322	194027	195341	196032	198813
1973	199857	199846	199882	200707	202845	205072	205838	205693	205162	205741	207552	209132
1974	208895	209874	211560	212243	211766	213224	214004	213704	213816	214136	214591	215529
1975	213494	213204	215258	215941	217337	220343	220767	220987	221256	220132	221723	221569
	Seasonal factors											
1962	1.033756	0.994786	0.991115	1.007757	0.980547	0.990180	0.991838	0.980884	0.991939	0.999949	1.007801	1.031085
1963	1.034227	.995435	.991081	1.006946	.979643	.988765	.991067	.980927	.992758	1.000889	1.008673	1.031277
1964	1.035166	.995019	.990532	1.006435	.977926	.987977	.990691	.980976	.993623	1.001774	1.009641	1.032043
1965	1.035224	.994698	.990789	1.008363	.978443	.989206	.990089	.980660	.993786	1.001045	1.008241	1.031218
1966	1.034133	.993366	.990683	1.007948	.978677	.990509	.991158	.981636	.993802	1.001054	1.007957	1.030746
1967	1.033759	.992255	.989843	1.007342	.979524	.991867	.992472	.982611	.993606	1.000174	1.007729	1.030416
1968	1.033779	.991599	.989364	1.006039	.979162	.991801	.993761	.983304	.993313	1.000105	1.008396	1.030987
1969	1.035238	.990952	.989143	1.007567	.979941	.991686	.992928	.982924	.993221	.999193	1.007933	1.030944
1970	1.033234	.989937	.989322	1.008431	.981169	.992969	.994420	.984174	.993670	.998711	1.006440	1.029009
1971	1.032810	.989779	.989929	1.008954	.980936	.992208	.994589	.984706	.993733	.998545	1.005706	1.029588
1972	1.032268	.990525	.989167	1.007720	.981004	.993838	.996853	.985105	.992093	.997300	1.005562	1.030021
1973	1.032586	.990815	.989284	1.007783	.979830	.993287	.996880	.984584	.991338	.997433	1.006554	1.031170
1974	1.030812	.988434	.986997	1.006029	.979601	.995320	.998739	.985910	.992391	.997943	1.008267	1.031053
1975	1.029926	.988008	.987507	1.006610	.979380	.994495	.997750	.985695	.993695	.999088	1.008640	1.030671
	Deterministic (fixed-factor) adjustment over recent period											
1970	165129	163779	164604	165605	165563	165051	165211	167154	168959	169699	170021	170704
1971	171731	173328	174467	175543	177357	177966	178728	179716	180257	180746	180896	181502
1972	182740	184685	186513	187711	187887	188148	190330	192072	193773	195449	196100	198863
1973	200181	200852	200113	200592	202729	204389	205263	205317	204737	205882	207829	209418
1974	208875	210424	211315	211753	211595	212949	213803	213601	213600	214393	215243	215799
1975	213289	213670	215120	215566	217113	219876	220342	220832	221323	220648	222480	221765
	Seasonal factors											
	1.030913	0.985851	0.988140	1.008358	0.980394	0.996606	0.999675	0.986388	0.993396	0.996751	1.005211	1.029762

COMMENTS ON "SEASONAL ADJUSTMENT WHEN BOTH DETERMINISTIC AND STOCHASTIC SEASONALITY ARE PRESENT" BY DAVID A. PIERCE

Larry D. Haugh
University of Vermont

INTRODUCTION

Seasonal adjustment has always appeared to me as one of the more perplexing areas of statistics, and so I was quite happy, upon reading this paper (as well as others presented herein), to see some deviation from Anderson's Law. Perhaps, not as well known as Murphy's series of laws, it nevertheless is often applicable to statistical treatments of real-world data problems. Anderson's Law (slightly revised) states:

> There is no problem, no matter how complex, which upon careful (statistical) analysis does not become more complex.

Quite to the contrary, this paper emphasizes clearly a number of points that should advance our understanding of how seasonal adjustment could be done. In fact, it sketches an outline of a method to compute seasonally adjusted series, together with case studies of the method's application. Along the way, it suggests a few improvements that could be made in the Census Bureau seasonal adjustment program and also suggests that taking moving averages is not the way to detrend series.

This paper does not propose, however, the X–1 variant of the Pierce method-I seasonal adjustment program (as an alternative to the X–11 variant of the Census method-II seasonal adjustment program). That is because—

1. A number of relatively minor details are left to the reader, since not all situations that could arise in his model's framework are discussed in detail. This would include, for example, a multiplicity of possible differencing schemes, together with a variety of possible trend models. In addition, some aspects of real series, such as trading-day variation, were not intended to be presented in this paper.
2. The method, as we see it now, is still under development. As pointed out, some issues that arose in the case studies deserve future work.
3. Most importantly, by the nature of the method proposed, it has not been programed for autômatic application to all economic time series. The strategy being proposed is much akin to a modeling strategy, since those aspects of the model that are important at various points in the procedure must be identified from the data at hand. And, these identifications are made through judgmental decisions on the part of the analyst, not by a programable algorithm.

The task set by the author is the seasonal adjustment of a given time series, using only the information contained in that series. It is made clear that if any strong a priori model conceptions are held or if the interrelationship with other series is to be used, then the proposed procedure should be modified.

The first point to be made is that if a particular model fits the data series well, then the seasonal adjustment procedure should be geared to that model. With this desirable assumption, the remaining issues to be considered are—

1. What general class of models is to be entertained?
2. How is the right model in this class to be identified (selected)?
3. What is the optimal seasonal adjustment for such a model?

The major points made in the paper are the emphasis he gives to the distinction between stochastic and deterministic models and the fact that he wants both stochastic and deterministic elements to be considered for possible inclusion in the model.[1]

The general form of the model considered is then

$$y_t=p_{1t}+s_{1t}+p_{2t}+s_{2t}+e_t$$

where y_t has been initially transformed (most often by taking logarithms and then first differencing). The deterministic components, trend and seasonal, are given by p_{1t} and s_{1t}, while the stochastic trend and seasonal components are p_{2t} and s_{2t} with e_t representing the irregular noise component. Of course, certain series may have only one deterministic or stochastic trend or seasonal component, but the choice of which components are to be considered in defining the optimal adjustment should be based on empirical model identification techniques.

A very brief outline of the methodology, after initial transformation to stationarity of any stochastic components, would be to—

1. Estimate the deterministic components by ordinary least squares regression methods. Here, the author's general experience has been that deterministic trend terms are trivial, once the usual first differencing

[1] See, e.g., [2] where Granger argues that it is essential for seasonal models to include stochastic components.

that is required has been performed. Therefore, little indication is given on how the form of trend and order of differencing are to be simultaneously identified in general. He mentions that an F-test could be performed to check the overall significance of any deterministic terms to be included.

2. Identify the stochastic trend and seasonal components from the residuals of the regression, a more difficult feat than (1). Actually, as is discussed by the author, the remaining stochastic components may be usefully represented as

$$u_t^* = s_{2t}^* + n_t^*$$

where the trend-cycle irregular components are lumped into the stochastic nonseasonal component n_t^*. The identification problems existing here with the separation of seasonal from nonseasonal components are related to the usual ones that exist whenever unobservable components have to be identified.

In the following section, attention will be directed to an implication about the form of the proposed stochastic model and to a generalization of it.

IDENTIFICATION OF STOCHASTIC COMPONENTS

Given that the stochastic residuals are to be represented as $(s_{2t}^* + n_t^*)$, the author's approach is to initially focus on the nonseasonal part n_t^*. Roughly, he suggests looking at the autocorrelation function of the residuals and using the pattern appearing at the low-order lags to identify the form of n_t^*. This pattern is identified in the form of a polynominal $h(B)$, which then represents the autoregressive part of u_t^* or n_t^*, since

$$h(B)u_t^* = \tilde{s}_{2t} + \tilde{n}_t$$

with $\tilde{n}_t$ being white noise and $\tilde{s}_{2t}$ strongly seasonal, i.e., having autocorrelation only at seasonal lags.

I think there is a useful way to look at what the author is doing at this stochastic modeling stage in a different manner. A filter $h(B)$ is desired that will give a strongly seasonal output when applied to the residuals u_t^*. Note that the $\tilde{s}_{2t}$ series will be strongly seasonal if, and only if, $(\tilde{s}_{2t} + \tilde{n}_t)$ is strongly seasonal, and, thus, the white noise component is not initially an important aspect of the problem.

It turns out that, to find such a filter $h(B)$, one must essentially model the residuals as a multiplicative seasonal model [1]

$$u_t^* = N(B)S(B)\epsilon_t$$

where $N(B)$ and $S(B)$ represent the nonseasonal and strongly seasonal parts of u_t^*, and ϵ_t is white noise. That is, one should identify from the ACF (or otherwise) and then efficiently estimate, a multiplicative seasonal model in which $N(B)$ and $S(B)$ could be rational functions of B in general. Assuming this is done, $h(B)$ may then be taken as $h(B) = N(B)^1$. Note that there would seem to be no reason why $h(B)$ could not be generalized to a rational function of B instead of being restricted to a polynominal in B. The residuals u_t^* are then filtered by $h(B)$, actually by its estimate, to obtain the strongly seasonal output $S(B)\epsilon_t$.

Next, to accomplish the additive separation into seasonal and nonseasonal components that is desired, one can think of the problem in terms of extracting as much white noise as is possible from $S(B)\epsilon_t$ to obtain

$$S'(B)\epsilon_t' + \tilde{n}_t$$

This maximum extraction of white noise corresponds to a minimum extraction of seasonal components—coined the "minimum extraction principle" by the author.[2] See the discussion by W. Wecker [5] concerning this extraction. Finally, the desired decomposition of u_t^* is achieved as

$$u_t^* = N(B)S'(B)\epsilon_t' \div N(B)\tilde{n}_t$$

Seasonal adjustment of the stochastic component is accomplished by a symmetric moving average filtering of the residuals, where the filter depends on the estimated filters $N(B)$, $S(B)$ and $S'(B)$.

SEASONALITY OF ARMA MODELS

One of the inherent problems in seasonal time series analysis is defining what "seasonality," or "strong seasonality," is, contrasted to trend, cycle, or irregular movements. (See, e.g., [2; 3].) It would seem imperative that time series analysts should be able to say whether particular time series models (as opposed to real data) are seasonal, strongly seasonal, or not seasonal. In particular, I wonder whether it is clear to everyone concerned how ARMA models should be classified?

It would seem that $(1-\phi B)$ and $(1-\phi_{12}B^{12})$ would be termed "nonseasonal" and "strongly seasonal" factors. Unfortunately, as is well known, a series with an autoregressive factor $(1-\phi_1 B-\phi_2 B^2)$, appearing to be nonseasonal on the face of it, can have a seasonal or periodic appearance for certain parameter values.

The classification confusion seems worse when seasonal and nonseasonal lags both appear, either in the same factor or in factors being multiplied together, e.g., $(1-\phi_1 B-\phi_{12}B^{12})$ or $(1-\phi B)(1-\phi_{12}B^{12})$. Note that Granger [2] classifies the combination of nonseasonal and strongly seasonal factors, e.g., $(1-\theta B)(1-\phi_{12}B^{12})^{-1}$ as strongly seasonal. Additionally, the distinction between seasonal and strongly seasonal models needs to be clarified.

[2] The phrase "minimal extraction principle" is reminiscent of the long-standing principle adopted by dentists to take a tooth out only if absolutely necessary.

Note that an implication of the author's assumption 3, i.e., the seasonal component of a filtered series is the filtered version of the original series' seasonal component, is that the seasonal components of series can easily involve factors with nonseasonal lags. For example, the seasonal component of u_t^* involves the nonseasonal factor $N(B)$.

CONCLUDING REMARKS

Realizing, from the section on the identification of stochastic components in this paper, that the stochastic component of the proposed general time series model is really of the multiplicative seasonal type, then we have the transformed version of y represented as a sum of deterministic trend and seasonal terms plus a multiplicative seasonal stochastic component. For the purpose of seasonal adjustment, the multiplicative component is additively broken into unobservable components using the minimum extraction principle.

It is important, for practical work, to have a class of models broad enough to mimic most real series well, and there is high hope that the model proposed here will be adequate in that regard.

Concerning the estimation of the model parameters, the author proposes a stagewise approach, estimating the deterministic components' parameters first and then estimating the stochastic component parameters at a later stage. I wonder if a more reasonable approach would not be to estimate the regression and time series parameters simultaneously once the components have been identified in the present stagewise manner. The properties of such parameter estimates are well known, even in the somewhat generalized model discussed in these comments [4].

REFERENCES

1. Box, George E. P., and Jenkins, G. M. *Time Series Analysis: Forecasting and Control.* (Rev. ed.) San Francisco: Holden-Day, Inc., 1976.

2. Granger, Clive W. J. "Seasonality: Causation, Interpretation, and Implications." Included in this report.

3. Nerlove, Marc. "Spectral Analysis of Seasonal Adjustment Procedures." *Econometrica* 32 (July 1964): 241–286.

4. Pierce, David A. "Least Squares Estimation in the Regression Model with Autoregressive-Moving Average Errors." *Biometrika* 58 (1971): 299–312.

5. Wecker, William E. "Comments on 'Seasonal Adjustment When Both Deterministic and Stochastic Seasonality Are Present' by David A. Pierce. Included in this report.

COMMENTS ON "SEASONAL ADJUSTMENT WHEN BOTH DETERMINISTIC AND STOCHASTIC SEASONALITY ARE PRESENT" BY DAVID A. PIERCE

William E. Wecker
University of Chicago

INTRODUCTION

The paper by Pierce offers a comprehensive approach to seasonal adjustment that considers both deterministic and stochastic seasonal components. The deterministic seasonal component is removed using ordinary least squares regression on dummy variables. Any remaining seasonality is extracted with a symmetric linear filter. The method is illustrated nicely with four examples.

While the approach is, in principle, quite general, Pierce's development of the symmetric linear filter (used to extract the stochastic seasonal component) is carried out under rather restrictive assumptions, and I would like to offer suggestions for broadening the approach while retaining, I think, the spirit and intent of Pierce's method.

STOCHASTIC SEASONALITY: A SPECIAL CASE

After the deterministic seasonality has been removed from the time series (using ordinary least squares regression on dummy variables) it will be the case, in general, that the residuals u_t of that regression will be marked by correlation at seasonal lags. To estimate this remaining seasonality, Pierce first constructs a (nonseasonal) linear filter to remove the low frequency component from the residuals. The filtered residuals $\tilde{u}_t$ are then represented as

$$\tilde{u}_t=\tilde{s}_t+\tilde{n}_t$$

where $\tilde{s}_t$ is a purely seasonal stationary random sequence (having correlation only at seasonal lags) and $\tilde{n}_t$ is a white-noise sequence. The components $\tilde{s}_t$ and $\tilde{n}_t$ are taken to be orthogonal. It is at this point that Pierce imposes the restriction that $\tilde{u}_t$ has the univariate representation

$$\tilde{u}_t=\frac{(1-\Theta B^{12})}{(1-\phi B^{12})}a_t$$

where B is the backshift operator, a_t is a white-noise sequence, and the parameters ϕ and Θ satisfy the relation $0<\Theta<\phi<1$.

For this special case, Pierce develops the signal extraction filter, his equation (53), for estimating the seasonal component of $\tilde{u}_t$

$$\hat{s}_t=\frac{\sigma_\epsilon^2(1-B^{12})(1-B^{-12})}{\sigma_n^2(1-\Theta B^{12})(1-\Theta B^{-12})}u_t$$

STOCHASTIC SEASONALITY: A MORE GENERAL APPROACH

When the filtered residuals $\tilde{u}_t$ are not well represented by the simple model

$$\tilde{u}_t=\frac{(1-\Theta B^{12})}{(1-\phi B^{12})}a_t$$

a more general approach to seasonal adjustment is necessary. Here, we extend Pierce's work by allowing $\tilde{u}_t$ to be represented by the more general expression

$$\tilde{u}_t=\Psi(B^{12})a_t \tag{1}$$

where $\Psi(B^{12})$ is a rational function of the seasonal lag operator B^{12} and a_t is a white-noise sequence. We denote the spectrum of $\tilde{u}_t$ by

$$f(\lambda)=\sigma_a^2\,\Psi(Z^{12})\Psi(Z^{-12}) \qquad -\pi\le\lambda\le\pi$$

where $Z=e^{i\lambda}$.

To determine the filter for estimating $\tilde{s}_t$ in the components model

$$\tilde{u}_t=\tilde{s}_t+\tilde{n}_t$$

we require $f_s(\lambda)$, the spectrum of $\tilde{s}_t$. Unfortunately, the seasonal component spectrum $f_s(\lambda)$ cannot be uniquely determined from the overall model (1) alone. We resolve this ambiguity using Pierce's principle of "minimal extraction" that says "to remove as little as possible yet still remove the series seasonality" [3]. More precisely, the principle of minimal extraction requires that the variance of the seasonal component be chosen as small as possible, or equivalently, that the variance of $\tilde{n}_t$ be as large as possible. The magnitude of σ_n^2 is limited by the restriction that the seasonal component spectrum (as any spectrum) must be nonnegative for all $\lambda(-\pi\le\lambda\le\pi)$. Since the components $\tilde{s}_t$ and $\tilde{n}_t$ are taken to be orthogonal, we have $f_s(\lambda)=f(\lambda)-f_n(\lambda)$, which fixes the (constant) value of $f_n(\lambda)$ at $f_n(\lambda)=\min_\lambda f(\lambda)$ and the seasonal component spectrum at

$$f_s(\lambda)=f(\lambda)-\min_\lambda f(\lambda)$$

Once the spectrum of $\tilde{s}_t$ has been determined, the symmetric filter

$$\frac{f_s(\lambda)}{f(\lambda)}$$

can be constructed and the stochastic seasonal component estimated [5; 6]

$$\hat{s}_t = \frac{f_s(\lambda)}{f(\lambda)} u_t$$

We illustrate the more general approach by applying it to Pierce's special case where the overall model is taken to be

$$\tilde{u}_t = \frac{(1-\Theta B^{12})}{(1-\phi B^{12})} a_t$$

We first form the spectrum of $\tilde{u}_t$, which is

$$f(\lambda) = \sigma_a^2 \frac{(1-\Theta Z^{12})(1-\Theta Z^{-12})}{(1-\phi Z^{12})(1-\phi Z^{-12})} Z = e^{i l}$$

$$= \sigma_a^2 \frac{1-\Theta[2\cos(12\lambda)]+\Theta^2}{1-\phi[2\cos(12\lambda)]+\phi^2}$$

The spectrum of $\tilde{u}_t$ is shown graphically in figure 1.[1] In the case studied by Pierce where $0 < \Theta < \phi < 1$, $f(\lambda)$ will have a minimum of

$$\min_\lambda f(\lambda) = \left(\frac{1+\Theta}{1+\phi}\right)^2 \sigma_a^2$$

so that, by the minimal extraction principle, the spectrum of the white-noise sequence $\tilde{n}_t$ can be taken to be

$$f_n(\lambda) = \left(\frac{1+\Theta}{1+\phi}\right)^2 \sigma_a^2$$

The spectrum of the seasonal component $\tilde{s}_t$ (shown in fig. 2) is, therefore,

$$f_s(\lambda) = f(\lambda) - f_n(\lambda)$$

$$= \sigma_a^2 \left\{ \frac{(1-\Theta Z^{12})(1-\Theta Z^{-12})}{(1-\phi Z^{12})(1-\phi Z^{-12})} - \left(\frac{1+\Theta}{1+\phi}\right)^2 \right\}$$

This results in the signal extraction filter

$$\frac{f_s(\lambda)}{f(\lambda)} = \frac{(1-\Theta Z^{12})(1-\Theta Z^{-12}) - \left(\frac{1+\Theta}{1+\phi}\right)^2 (1-\phi Z^{12})(1-\phi Z^{-12})}{(1-\Theta Z^{12})(1-\Theta Z^{-12})}$$

$$= \frac{\left[\phi\left(\frac{1+\Theta}{1+\phi}\right)^2 - \Theta\right]\left[(1+Z^{12})(1+Z^{-12})\right]}{(1-\Theta Z^{12})(1-\Theta Z^{-12})} \qquad (2)$$

where Z is interpreted as the backward shift operator.

For this same case using a different approach Pierce obtains the filter

$$\frac{\sigma_\epsilon^2(1+Z^{12})(1+Z^{-12})}{\sigma_a^2(1-\Theta Z^{12})(1-\Theta Z^{-12})} \qquad (3)$$

[1] Parameter values for the figs. are taken from Pierce's CPI example.

where σ_ϵ^2 is determined from his equation (51)

$$(1+\Theta^2)\sigma_a^2 = (1+\Theta^2)\sigma_\epsilon^2 + (1+\phi^2)\sigma_n^2$$

$$\Theta\sigma_a^2 = \Theta\sigma_\epsilon^2 + \phi\sigma_n^2$$

$$\Theta = -1 \qquad (4)$$

To show that Pierce's filter (3) is the same as the filter (2), we first solve equation (4) for σ_ϵ^2, yielding

$$\sigma_\epsilon^2 = \frac{\sigma_a^2}{2}\left\{(1+\Theta^2) - \left(\frac{1+\Theta}{1+\phi}\right)^2 (1+\phi^2)\right\} \qquad (5)$$

Pierce's filter (3), stated in terms of the parameters ϕ, Θ, and σ_a^2, then becomes, by substituting (5) in (3),

$$\frac{\frac{1}{2}\left[(1+\Theta^2) - \left(\frac{1+\Theta}{1+\phi}\right)^2 (1+\phi^2)\right](1+Z^{12})(1+Z^{-12})}{(1-\Theta Z^{12})(1-\Theta Z^{-12})}$$

which reduces to

$$\frac{\left[\phi\left(\frac{1+\Theta}{1+\phi}\right)^2 - \Theta\right]\left[(1+Z^{12})(1+Z^{-12})\right]}{(1-\Theta Z^{12})(1-\Theta Z^{-12})}$$

showing that filters (2) and (3) are identical.

OVERADJUSTMENT

Pierce observed that, in two of his examples where stochastic seasonality was present, his method had "a tendency to overadjust" in the sense that the seasonally adjusted series had negative correlation at the seasonal lags (or dips in the spectrum at the seasonal frequencies). Ad hoc adjustments were made by Pierce to counter this tendency. We will show that these dips are not an inadequacy of this method. At least, they are not as conjectured by Pierce—an artifact of the sampling properties of his estimation procedures. The dips would persist even if the spectra, $f(\lambda)$ and $f_s(\lambda)$, were exactly known.

To show this, we denote the spectra of $\tilde{s}_t$, $\tilde{n}_t$, and $\tilde{u}_t$ by $f_s(\lambda)$, $f_n(\lambda)$, and $f(\lambda) = f_s(\lambda) + f_n(\lambda)$. Using a theorem on the spectrum of a linear transform of a stationary time series [1], the spectrum of the seasonally adjusted series is determined to be

$$f_{u-\hat{s}}(\lambda) = \left[1-\gamma(Z)\right]\left[1-\gamma(Z^{-1})\right] f(\lambda) \qquad (6)$$

where $Z = e^{i\lambda}$ and $\gamma(Z) = \frac{f_s(\lambda)}{f(\lambda)}$. Equation (6) reduces to

$$f_{u-\hat{s}}(\lambda) = f_n(\lambda)\left(\frac{f_n(\lambda)}{f(\lambda)}\right) \qquad (7)$$

Unless the white-noise component $\tilde{n}_t$ has zero power at the seasonal frequencies, the spectrum of the seasonally adjusted series will have dips at the seasonal frequencies,

Figure 1. **SPECTRUM FOR THE TIME SERIES** $\tilde{u}_t = \frac{(1-\theta B^{12})}{(1-\phi B^{12})} a_t$

$(\phi = 0.81,\ \theta = 0.47,\ \sigma_a^2 = 0.000019)$

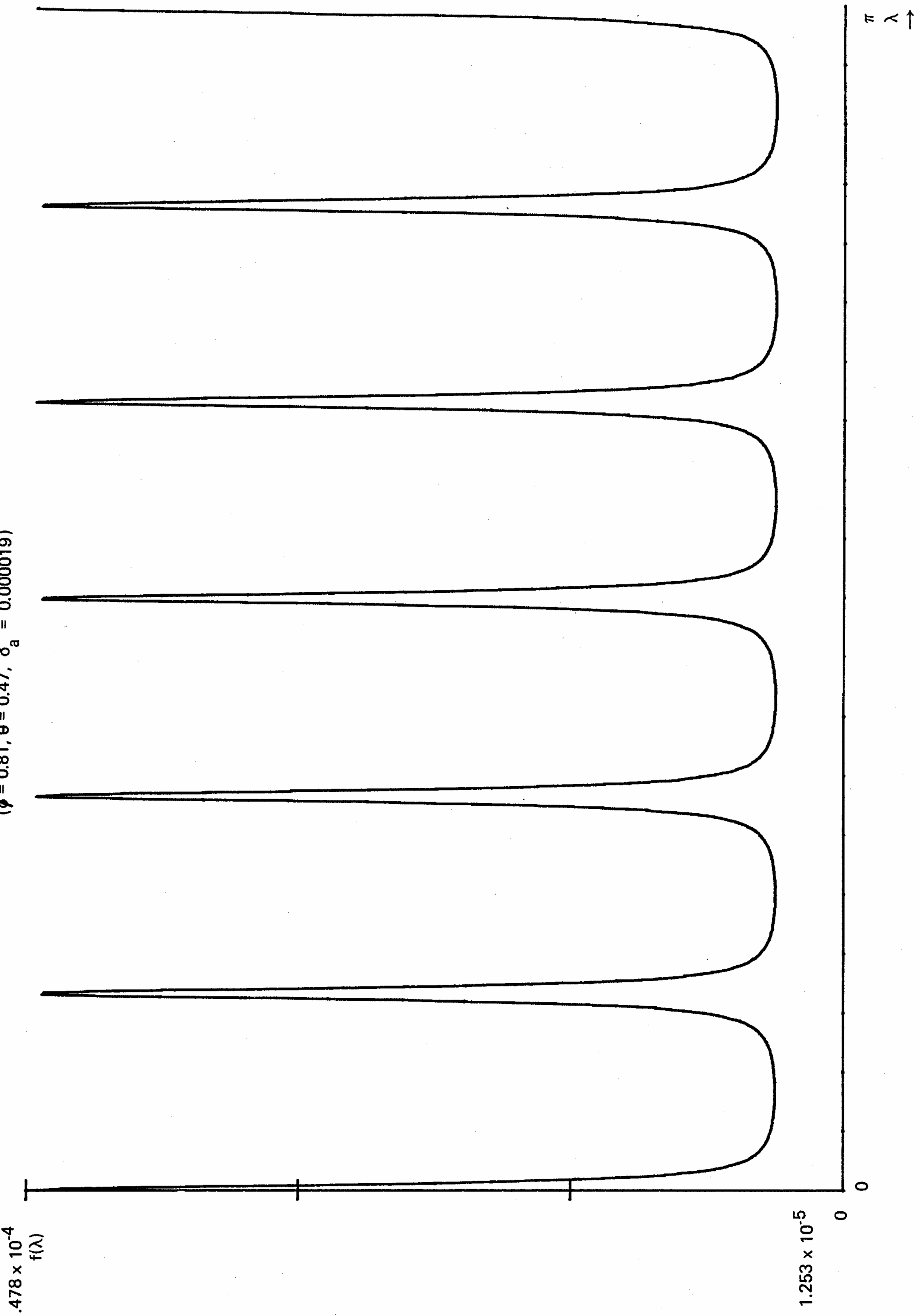


Figure 2. **SPECTRUM OF THE SEASONAL COMPONENT FOR THE TIME SERIES**

$$\left(\tilde{u}_t = \frac{(1 - \Theta B^{12})}{(1 - \phi B^{12})} a_t;\ \phi = 0.81,\ \Theta = 0.47,\ \sigma_a^2 = 0.000019\right)$$

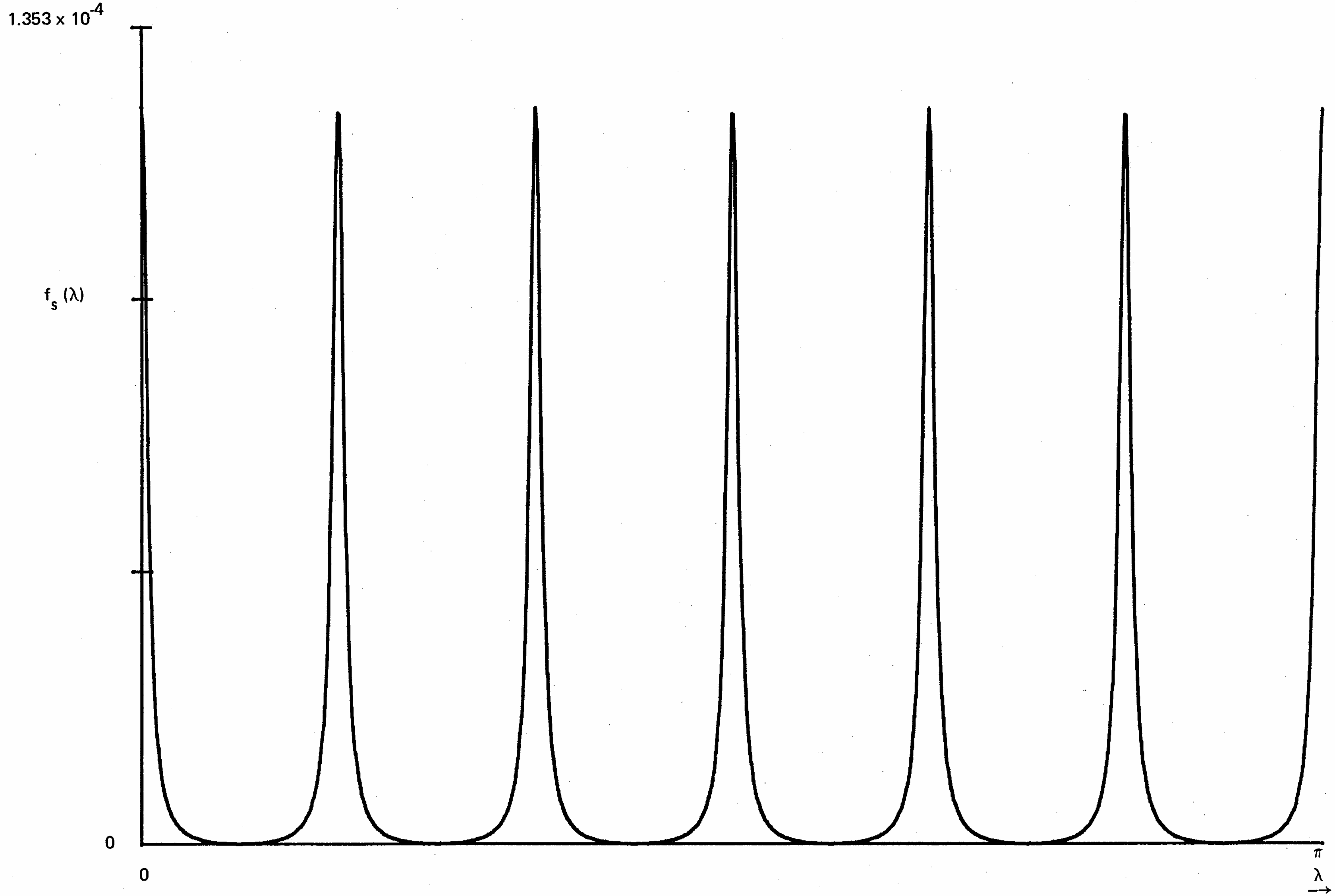
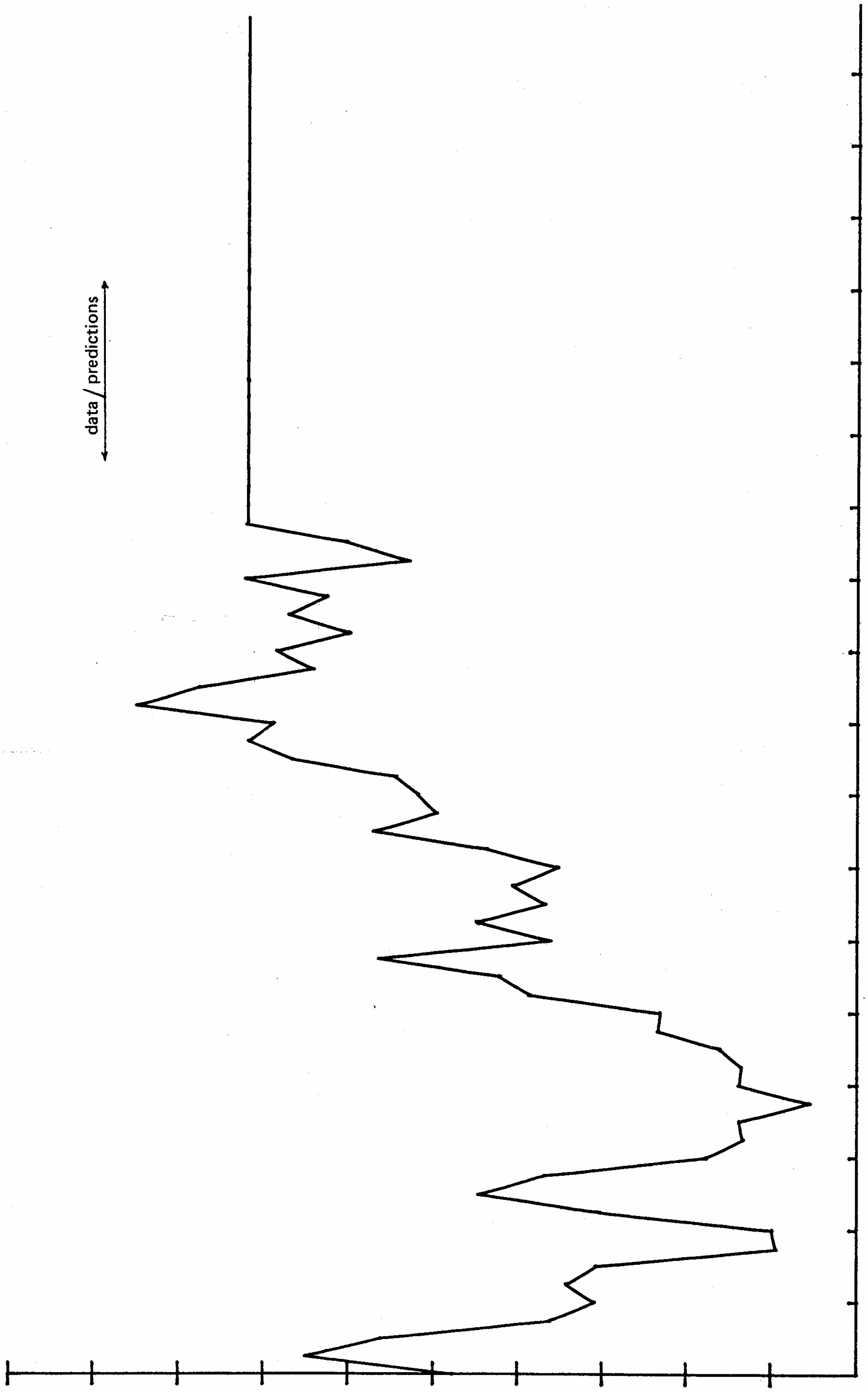


Figure 3. **AN ARTIFICIALLY GENERATED RANDOM WALK WITH OPTIMAL (MEAN SQUARE ERROR) PREDICTIONS**



due to the factor $\frac{f_n(\lambda)}{f(\lambda)}$ on the righthand side of equation (7). Should these dips be viewed as a discrepancy? Apparently not, if the mean square error criterion for signal extraction has been accepted. It may be of help to the intuition of some (and a further burden on the intuition of others) to recall that, in prediction, the sequence of (mean square error criterion) estimates of future values will ordinarily bear little resemblance to the actual sequence of values. We illustrate this point in figure 3, where an (artificially generated) random walk is shown with predictions. The autocovariance properties of data x_t and of predictions $\hat{x}_t$ are different. Are the predictions to be considered inadequate because of this difference? Certainly not. The predictions $\hat{x}_t$ are perfectly respectable mean square estimates of future values of the time series, just as the estimates $\hat{s}_t$ are perfectly respectable mean square error estimates of the values of the seasonal component (despite the fact that the autocovariance properties of s_t and of $\hat{s}_t$ are different).

Perhaps the reason Pierce finds this difference unsettling is that he has, in the back of his mind, the intent to use the seasonally adjusted series to make statements about path properties of the time series, such as turning points. If that is the objective of the analysis, an entirely different approach may be appropriate—an approach determined by explicit consideration of the requirement to estimate, e.g., the timing of a turning point. It is not at all clear, for example, that a seasonally adjusted time series constructed according to the criterion of mean square error estimation can be used (in any simple way) to estimate the timing of a turning point. (See [4]).

THE LOGARITHMIC TRANSFORMATION

Finally, I would like to brieflv comment on Pierce's use of the logarithmic transformation of the original time series as the starting point for his analyses. I am reminded of the case study reported by Chatfield and Prothero [2] where the results of an analysis of a seasonal time series were criticially dependent on the choice of transformation and, in particular, were grossly unsatisfactory when a logarithmic transformation was used. The analysis of transformations for time series is, of course, an open area, and Pierce has simply adopted an opening move that has become as common as pawn-to-king four. Still, it may be that the choice of transformation is a crucial determinant of the seasonal adjustment results (as was the case in [2]), making the subject of transformations worthy of further attention.

REFERENCES

1. Anderson, T. W. *The Statistical Analysis of Time Series*. New York: Wiley and Sons, 1971.
2. Chatfield, C., and Prothero, D. L. "Box-Jenkins Seasonal Forecasting: Problems in a Case Study." *Journal of the Royal Statistical Society,* ser. A, 136 (1973): 295–315.
3. Pierce, David A. "Seasonal Adjustment When Both Deterministic and Stochastic Seasonality Are Present." Included in this report.
4. Wecker, William E, "The Prediction of Turning Points," *Proceedings of the Business and Economic Section of the American Statistical Association.* Washington, D.C.: American Statistical Association, 1976.
5. Whittle, P. *Prediction and Regulation by Least Squares Methods*. London: English Universities Press, 1963.
6. Yalgom, A. M. *An Introduction to the Theory of Stationary Random Functions*. Englewood Cliffs, N. J.: Prentice-Hall, 1962.

COMMENTS ON "SEASONAL ADJUSTMENT WHEN BOTH DETERMINISTIC AND STOCHASTIC SEASONALITY ARE PRESENT" BY DAVID A. PIERCE

John P. Burman
Bank of England

This is a very illuminating paper, giving full practical details of seasonal signal extraction, using a particular ARIMA model.

The model $(1, 0, 1)_{12}$ used for the seasonal operator is different from the example in (1). But, both come within the framework of the partial fraction technique, described in the appendix of my discussion on Kuiper. An improvement, suggested by Pierce, is the extraction of any deterministic seasonal component (seasonal mean correction) from the differenced series.

He removes the trend by differencing and a nonseasonal autoregressive filter, instead of including this in a single seasonal ARIMA model; this makes for computational simplicity in finding the minimum of the spectrum of the seasonal component and also the weights of the signal extraction filter. But, if there is any interaction between the seasonal and nonseasonal parts of the model, this may not be the optimal procedure.

Another difference from (1) is that Pierce includes the positive real root of $(1-\Phi B^{12})$ in the seasonal model, although it is usually close to 1 and, thus, generates a spike in the spectrum at zero frequency. For example, for U.S. unemployment, Pierce finds $\Phi=0.547$, and its 12th root is 0.95.