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A Set of End Weights to End All End Weights

Given a moving average used to calculate a seasonal factor curve or a trend-cycle curve, the problem is to derive a set of moving average weights for the end terms which will minimize revisions between preliminary and final estimates of the seasonal or trend-cycle. As an example, consider the end weights for the 5-term moving average as a seasonal factor curve. End weights for other seasonal curves of different length and trend-cycle curves may be derived in a similar manner.

Let $X_1, \ldots, X_5$ represent the latest five S-I ratios (where X_5 represents the end year value). Similarly, let $S_1, \ldots, S_5$ represent the "true" seasonal factors corresponding to $X_1, \ldots, X_5$. Assume that $X_i = S_i + I_i$, where I_i is the irregular corresponding to X_i. Further assume that the I_i's are independent with equal variance σ_I^2.

Define:

$$S_4' = W_1 X_2 + W_2 X_3 + W_3 X_4 + W_2 X_5 + W_1 X_6,$$
$$S_5' = W_1 X_3 + W_2 X_4 + W_3 X_5 + W_2 X_6 + W_1 X_7,$$

where S_4' and S_5' are the ultimate estimates of S_4 and S_5. As the preliminary estimates of S_4 and S_5 when data is available through X_5, define:

$$S_4'' = U_1 X_2 + U_2 X_3 + U_3 X_4 + U_4 X_5,$$
$$S_5'' = V_1 X_3 + V_2 X_4 + V_3 X_5.$$

We want to minimize

$$R_1 = E(S_4'' - S_4')^2 \text{ and } R_2 = E(S_5'' - S_5')^2$$

with respect to the U_i and V_i subject to the constraints

$$\sum_1^4 U_i = \sum_1^3 V_i = 1. \quad (\text{Also, } \sum_1^5 W_i = 1.)$$

Now $R_1 = \sigma^2_{S''_4 - S'_4} + B^2_{S''_4 - S'_4}$

$$= \left[(U_1 - W_1)^2 + (U_2 - W_2)^2 + (U_3 - W_3)^2 + (U_4 - W_2)^2 + W_1^2\right] \sigma_I^2$$

$$+ \left[(U_1 - W_1)S_2 + (U_2 - W_2)S_3 + (U_3 - W_3)S_4 + (U_4 - W_2)S_5 - W_1 S_6\right]^2,$$

where $B_{S''_4 - S'_4} = E(S''_4) - E(S'_4)$.

Clearly, it is an easy matter to minimize $\sigma^2_{S''_4 - S'_4}$ with respect to the U_i and obtain estimates of the U_i in terms of the W_i. To minimize $B^2_{S''_4 - S'_4}$, however, it is necessary to make the further assumption that $S_{t+1} - S_t = \Delta_S$, where Δ_S is a constant. Hence, define $S_2 = S$, $S_3 = S + \Delta_S$, $\ldots$, $S_6 = S + 4\Delta_S$.

To minimize R_1, form $F = R_1 - \lambda\left[\sum_1^4 U_i - 1\right]$, where λ is a Lagrange multiplier. Differentiating, we have:

$$(1)\quad \frac{\partial F}{\partial U_1} = 2(U_1 - W_1)\,\sigma_I^2 - \lambda + 2S_2\,B = 0,$$

$$(2)\quad \frac{\partial F}{\partial U_2} = 2(U_2 - W_2)\,\sigma_I^2 - \lambda + 2S_3\,B = 0,$$

$$(3)\quad \frac{\partial F}{\partial U_3} = 2(U_3 - W_3)\,\sigma_I^2 - \lambda + 2S_4\,B = 0,$$

$$(4)\quad \frac{\partial F}{\partial U_4} = 2(U_4 - W_2)\,\sigma_I^2 - \lambda + 2S_5\,B = 0.$$

Summing: $2\left[1 - (1 - W_1)\right]\sigma_I^2 - 4\lambda + 2B \sum_2^5 S_i = 0$;

$$\lambda = \frac{W_1 \sigma_I^2}{2} + \frac{B \sum_2^5 S_i}{2}.$$

$$\text{Now}\quad U_1 = W_1 + \frac{\lambda}{2\sigma_I^2} - \frac{S_2\,B}{\sigma_I^2}$$

$$= W_1 + \frac{W_1}{4} + \frac{B \sum_2^5 S_i}{4\sigma_I^2} - \frac{B\,S_2}{\sigma_I^2}$$

$$= W_1 + \frac{W_1}{4} + \frac{B}{4\sigma_I^2}\left[4S + 6\Delta_S - 4S\right].$$

$$(1') \quad U_1 = W_1 + \frac{W_1}{4} + \frac{3\Delta_S}{2\sigma I^2} \quad B$$

$$= W_1 + \frac{W_1}{4} + \frac{3\Delta_S}{2\sigma I^2} \left[S \left(\sum_1^4 U_i - \sum_1^5 W_i \right) + \Delta_S \left[U_2 + 2U_3 + 3U_4 - (W_1 + 2W_3 + 3W_2 + 4W_1) \right] \right]$$

$$= W_1 + \frac{W_1}{4} + \frac{3\Delta_S^2}{2\sigma I^2} \left[2 - 2U_1 - U_2 + U_4 - 2 \right]$$

$$= W_1 + \frac{W_1}{4} + \frac{3\Delta_S^2}{2\sigma I^2} \left[- 2U_1 - U_2 + U_4 \right]$$

$$(1'') \quad \left(1 + 3\frac{\Delta_S^2}{\sigma I^2}\right) U_1 + \frac{3}{2} \frac{\Delta_S^2}{\sigma I^2} U_2 + 0 \cdot U_3 - \frac{3}{2} \frac{\Delta_S^2}{\sigma I^2} U_4 = W_1 + \frac{W_1}{4}.$$

It is necessary at this point to derive a functional relationship between $\Delta_S^2/\sigma I^2$ and $\bar{I}'/\bar{S}'$, since the W_i are based on the size of $\bar{I}'/\bar{S}'$.

$$\bar{I}'^2 = \frac{2}{\pi} \sigma_{\delta I}^2 \quad \left(\delta I = \frac{I_{t+1} - I_t}{I_*} \right.$$ -- this approximation due to Rosenblatt)

$$= \frac{2}{\pi} \left[\frac{\sigma I_{t+1}^2}{[E(I_{t+1})]^2} + \frac{\sigma I_t^2}{[E(I_t)]^2} - \frac{2\,\sigma I_{t+1} I_t}{E(I_{t+1})E(I_t)} \right]$$

$$= \frac{2}{\pi} \left[\frac{2\,\sigma I^2}{(100)^2} \right]$$

$$= \frac{4}{\pi} \left[\frac{\sigma I^2}{(100)^2} \right].$$

$$\sigma I^2 = \frac{\pi}{4} (100)^2 \; \bar{I}'^2.$$

$$\bar{S}' = \frac{1}{n-1} \sum_1^{n-1} \left| \frac{S_{t+1} - S_t}{S_t} \right| \qquad \text{(n = number of years)}$$

$$= \frac{\Delta S}{n-1} \sum_1^{n-1} \frac{1}{S_t}.$$

$$\bar{S}'^2 = \Delta_S^2 \left[\frac{1}{n-1} \sum_1^{n-1} \frac{1}{S_t} \right]^2.$$

Clearly, the above analysis holds only if Δ_S is constant for the entire historical period. Let $\tilde{S} = \sum_{1}^{n-1} \frac{1}{S_t}$. If all $S_t = \hat{S}$, $\tilde{S} = \frac{n-1}{\hat{S}}$.

Hence, $\bar{S}'^2 = \Delta_S^2 \left(\frac{\tilde{S}}{n-1}\right)^2$ and $\Delta_S^2 = \frac{(n-1)^2 \, \bar{S}'^2}{\tilde{S}^2}$. When all $S_t = \hat{S}$, $\Delta^2 S = \bar{S}'^2 \cdot \hat{S}^2$.

$$\text{Then} \quad \frac{\Delta^2_S}{\sigma_I^2} = \frac{(n-1)^2 \ \bar{S}'^2 / \tilde{S}^2}{\pi/4 \ (100)^2 \ \bar{I}'^2}$$

$$= \frac{4}{\pi} \ \frac{(n-1)^2}{(100)^2 \tilde{S}^2} \left(\frac{\bar{S}'}{\bar{I}'}\right)^2 = D.$$

In the special case where all $S_t = \hat{S}$,

$$D = \frac{\Delta_S^2}{I^2} = \frac{4}{\pi} \ \frac{\hat{S}^2}{(100)^2} \left(\frac{\bar{S}'}{\bar{I}'}\right)^2$$

In the special case where $\frac{\tilde{S}}{n-1} = \frac{1}{100}$,

$$D = \frac{\Delta_S^2}{I^2} = \frac{4}{\pi} \left(\frac{\bar{S}'}{\bar{I}'}\right)^2 .$$

Then we have:

$$(1') \quad U_1 = W_1 + \frac{W_1}{4} + \frac{3}{2} \ \frac{D}{\Delta_S} \cdot B,$$

$$(2') \quad U_2 = W_2 + \frac{W_1}{4} + \frac{1}{2} \ \frac{D}{\Delta_S} \ B,$$

$$(3') \quad U_3 = W_3 + \frac{W_1}{4} - \frac{1}{2} \ \frac{D}{\Delta_S} \cdot B,$$

$$(4') \quad U_4 = W_4 + \frac{W_1}{4} - \frac{3}{2} \ \frac{D}{\Delta_S} \cdot B;$$

or

$$(1'') \quad (1 + 3D) \ U_1 + \frac{3}{2} D \ U_2 - \frac{3}{2} D \ U_4 = W_1 + \frac{W_1}{4},$$

$$(2'') \quad D \ U_1 + (1 + \frac{D}{2}) U_2 - \frac{D}{2} \ U_4 = W_2 + \frac{W_1}{4},$$

$$(3'') \quad -D\,U_1 - \frac{D}{2}\,U_2 + U_3 + \frac{D}{2}\,U_4 = W_2 + \frac{W_1}{4},$$

$$(4'') \quad -3D\,U_1 - \frac{3}{2}D\,U_2 + (1 + \frac{3}{2}D)\,U_4 = W_2 + \frac{W_1}{4}.$$

Using matrix notation, $AU = X;\ U = A^{-1}X$, where

$$AU = \begin{bmatrix} (1+3D) & \frac{3}{2}D & 0 & -\frac{3}{2}D \\ D & (1+\frac{D}{2}) & 0 & -\frac{D}{2} \\ -D & -\frac{D}{2} & 1 & \frac{D}{2} \\ -3D & -\frac{3}{2}D & 0 & (1+\frac{3}{2}D) \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{bmatrix} = \begin{bmatrix} W_1 + \frac{W_1}{4} \\ W_2 + \frac{W_1}{4} \\ W_3 + \frac{W_1}{4} \\ W_4 + \frac{W_1}{4} \end{bmatrix} = X.$$

To solve for A^{-1} and U, use the DAM regression program.

Suppose $\Delta_S = 0$; i.e., S_t is a stable seasonal. Then $B = S\left[\sum_1^4 U_i - \sum_1^5 W_i\right] = 0$. Hence, the U_i are unbiased estimates of the W_i and are as follows:

$$U_1 = W_1 + \frac{W_1}{4}, \qquad U_3 = W_3 + \frac{W_1}{4},$$

$$U_2 = W_2 + \frac{W_1}{4}, \qquad U_4 = W_2 + \frac{W_1}{4}.$$

The U_i derived from $(1'') - (4'')$ depend on:

(a) the central weights W_i,

(b) the $\bar{I}'/\bar{S}'$ ratio,

(c) the level of the S_t,

(d) the assumption of a seasonal with a linear trend which is the same in the historical and current period.

Similarly, minimize $R_2 = \sigma S_5''^2 - S_5' + B^2 S_5'' - S_5'$

$$= [(V_1 - W_1)^2 + (V_2 - W_2)^2 + (V_3 - W_3)^2 + W_2^2 + W_1^2] \sigma I^2$$

$$+ [(V_1 - W_1) S_3 + (V_2 - W_2) S_4 + (V_3 - W_3) S_5 - W_2 S_6 - W_1 S_7]^2.$$

Let $F = R_2 - \lambda [\sum_1^3 V_i - 1]$. Differentiating,

$$(1) \quad \frac{\partial F}{\partial V_1} = 2 (V_1 - W_1) \sigma I^2 - \lambda + 2 S_3 B = 0,$$

$$(2) \quad \frac{\partial F}{\partial V_2} = 2 (V_2 - W_2) \sigma I^2 - \lambda + 2 S_4 B = 0,$$

$$(3) \quad \frac{\partial F}{\partial V_3} = 2 (V_3 - W_3) \sigma I^2 - \lambda + 2 S_5 B = 0.$$

Then $\lambda = \dfrac{2 \sigma I^2 (W_1 + W_2)}{3} + \dfrac{2 B \sum_3^5 S_i}{3}$.

Now $V_1 = W_1 + \dfrac{\lambda}{2 \sigma I^2} - \dfrac{S_3 B}{\sigma I^2}$

$$= W_1 + \frac{W_1 + W_2}{3} + \frac{B \sum_3^5 S_i}{3 \sigma I^2} - \frac{S_3 B}{\sigma I^2}$$

$$(1') \quad V_1 = W_1 + \frac{W_1 + W_2}{3} + \frac{\Delta_{S'} B}{\sigma I^2}$$

$$= W_1 + \frac{W_1 + W_2}{3} + \frac{\Delta_S}{\sigma I^2} \Big[S \Big(\sum_1^3 V_i - \sum_1^5 W_i\Big) + \Delta_S \Big[V_2 + 2V_3 - (W_2 + 2W_3 + 3W_2 + 4W_1) \Big] \Big]$$

$$(1'') \quad V_1 = W_1 + \frac{W_1 + W_2}{2} = \frac{\Delta^2 S}{\sigma I^2} (2V_1 + V_2).$$

Then we have:

$$(1') \quad V_1 = W_1 + \frac{W_1 + W_2}{3} + \frac{D}{\Delta_S} \cdot B,$$

$$(2') \quad V_2 = W_2 + \frac{W_1 + W_2}{3},$$

$$(3') \quad V_3 = W_3 + \frac{W_1 + W_2}{3} - \frac{D}{\Delta S} \quad B;$$

or

$$(1'') \quad (1 + 2D)\, V_1 = - D\, V_2 + W_1 + \frac{W_1 + W_2}{3},$$

$$(3'') \quad V_3 = 2D\, V_1 + D\, V_2 + W_3 + \frac{W_1 + W_2}{3}$$

$(2'')$ is the same as $(2')$. Note that the middle weight in a pattern of end weights composed of an uneven number of terms depends <u>only</u> on the W_i.